A CHARACTERIZATION OF THE LOCAL RADON-NIKODÝM PROPERTY BY TENSOR PRODUCTS

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In this paper, results are presented that characterize the collection of all vector valued measures expressible as an indefinite Bochner integral. More precisely, if \( X \) is a Banach space, an \( X \)-valued vector measure, \( \tau \), defined on a measurable space \( (S, \Omega) \) is expressible as a Bochner integral if and only if \( \tau \) belongs to \( ca(S, \Omega) \otimes_\pi X \), where \( \otimes_\pi \) denotes the strong (or projective) tensor product of two Banach spaces. Other related results are given.

Introduction. Throughout this paper, \( (S, \Omega) \) will denote a measurable space and \( X \) a Banach space. By \( ca(\Omega) \) \( [cafv(\Omega; X)] \) we mean the Banach space of all real valued (resp., \( X \)-valued) countably additive set functions with finite variation, equipped with the total variation norm \( |\cdot| \). Generally, we use the basic notions and notation in Dunford and Schwartz [2].

A vector valued measure \( \tau \in cafv(\Omega; X) \) is said to have the Radon-Nikodym property if whenever \( \lambda \in ca(\Omega) \) is a positive measure such that \( \tau \ll \lambda \) (that is, \( |\tau(E)| \to 0 \) whenever \( \lambda(E) \to 0 \)), then there exists a Bochner integrable function \( f : S \to X \), (see pages 144–154 in [2]) such that

\[
\tau(E) = \int_E f d\lambda \quad \text{for all } E \in \Omega.
\]

In this case, \( f \) is called the Radon-Nikodym derivative of \( \tau \) with respect to \( \lambda \). The space of Bochner integrable functions from \( S \) into \( X \) with respect to a scalar measure \( \lambda \) is denoted \( B(S, \Omega, \lambda; X) \); the space of all \( X \)-valued measures on \( \Omega \) that have the RN (Radon-Nikodym) property is denoted \( RNca(\Omega; X) \), and forms a closed linear subspace of \( cafv(\Omega; X) \).

The Main Results. The RN property of a measure is important in classifying certain tensor products of spaces of measures. In preparation for this, we establish an important lemma.

**Lemma 1.** Suppose \( \tau \in cafv(\Omega; X) \) such that \( \tau \ll \lambda \) and \( \lambda \ll \nu \), for some two positive measures \( \lambda \) and \( \nu \) on \( \Omega \). If \( \tau \) has a Radon-Nikodym derivative with respect to \( \nu \), then it has a derivative with respect to \( \lambda \).
Proof. By the Lebesgue Decomposition theorem, there exists positive measures $\mu$ and $\sigma$ such that $\nu = \mu + \sigma$ and $\mu \ll \lambda$ and $\sigma \perp \lambda$. Since $\sigma \perp \lambda$, there exists a set $E_0 \in \Omega$ with $\sigma(E_0) = 0$ and $\lambda(S - E_0) = 0$. From $\mu \ll \lambda$, there exists an $h \in L^+_1(S, \Omega, \lambda)$ such that

$$\mu(E) = \int_E h d\lambda \text{ for all } E \in \Omega.$$ 

Let $f$ denote the derivative of $\tau$ with respect to $\nu$, then for $E \in \Omega$.

$$\tau(E) = \int_E f d\nu = \int_E f d\mu + \int_E f d\sigma = \int_E f h d\lambda + \int_E f d\sigma.$$ 

It is easily seen that $\int_E f d\sigma = \int_{E_0} f d\sigma + \int_{E - E_0} f d\sigma = 0$ for all $E \in \Omega$. Thus, $\tau(E) = \int_E f h d\lambda$ and, therefore, $fh$ is the Radon-Nikodym derivative of $\tau$ with respect to $\lambda$.

**Theorem 2.** Let $\{\tau_k\} \subseteq caf\nu(\Omega; X)$ be a sequence of vector measures such that $\sum_{k=1}^\infty |\tau_k|(S) < +\infty$. If $\tau_k$ has the RN property for each $k$, then so does $\tau = \sum_{k=1}^\infty \tau_k$.

**Proof.** Suppose $\lambda \in ca(\Omega)$ is a positive measure such that $\tau \ll \lambda$. Note that $\sum |\tau_k|(E)$ converges absolutely for each $E \in \Omega$, consequently, $\sum |\tau_k|$ defines a $\sigma$-additive measure on $\Omega$ such that $\tau_n \ll \sum |\tau_k|$ for each $n$.

Define $\nu = \lambda + \sum |\tau_k|$. Then $\nu$ is a positive measure on $\Omega$ such that $\lambda \ll \nu$; consequently, $\tau \ll \lambda \ll \nu$. It suffices, in view of Lemma 1 to show that $\tau$ has a derivative with respect to $\nu$.

Indeed, for each $n$, $\tau_n \ll \nu$ and $\tau_n$ has the RN property implies there exists a function $f_n \in B(S, \Omega, \nu; X)$ such that $\tau_n(E) = \int_E f_n d\nu$. It is easily seen that $\sum_{n=1}^\infty f_n$ converges in $B(S, \Omega, \nu; X)$. Therefore, if we define $f = \sum f_n$ it is seen that

$$\tau(E) = \sum \tau_n(E) = \sum \int_E f_n d\nu = \int_E \sum f_n d\nu = \int_E f d\nu.$$ 

Thus, $f$ is the derivative of $\tau$ with respect to $\nu$.

We now present the main result of this paper which constitutes a generalization of a theorem of Gil de Lamadrid (Theorem 4.2 [3]). In his paper, he identifies $C^*(H) \hat{\otimes}_\pi X$ as the class of all regular $X$-valued Radon measures of bounded variation which can be represented as an absolutely convergent series of “step measures.” In his paper, $H$ is a compact Hausdorff space, and, of course $C^*(H)$ is the space of all regular Radon measures on $H$. 
THEOREM 3. Let $(S, \Omega)$ be a measurable space and $X$ a Banach space, then $ca(\Omega) \hat{\otimes}_\pi X = RNca(\Omega; X)$ isometrically.

Indication of Proof. In [5], we show that $ca(\Omega) \hat{\otimes}_\pi X$ can be isometrically embedded in $cafv(\Omega; X)$ by the canonical isomorphism

$$\sum_{i=1}^k \mu_i \otimes x_i \rightarrow \sum_{i=1}^k x_i \mu_i(\cdot)$$

To prove $ca(\Omega) \hat{\otimes}_\pi X = RNca(\Omega; X)$, let $\tau \in RNca(\Omega; X)$. Put $\lambda = |\tau|$, then $\tau \ll \lambda$. Since $\tau$ has the RN property, there exists a function $f \in B(S, \Omega, \lambda; X)$ such that $\tau(E) = \int_E f d\lambda$ for all $E \in \Omega$.

Because $f$ is Bochner integrable, $f$ can be written in the form $f = \sum_{n=1}^\infty x_n \xi_{\lambda_n} -$ a.e, where $x_n \in X$, $E_n \in \Omega$, and $\sum_{n=1}^\infty |x_n| \lambda(E_n) < +\infty$ (see Brooks [1]). Here $\xi_E$ is the characteristic function of the set $E$.

Define $\tau_n: \Omega \rightarrow X$ for each positive integer $n$ by $\tau_n(E) = x_n \cdot \lambda(EE_n)$. $\tau_n$ is easily seen to have the RN property and $\tau_n \in ca(\Omega) \hat{\otimes}_\pi X$. Furthermore,

$$\sum_{k=1}^\infty |\tau_k|(S) = \sum_{k=1}^\infty |x_k| \lambda(E_k) < +\infty .$$

Thus, we have

$$\tau(E) = \int_E f d\lambda = \int_E \sum x_k \xi_{E_k} d\lambda = \sum x_k \lambda(EE_k) ,$$

or,

$$\tau(E) = \sum_{k=1}^\infty \tau_k(E)$$

for each $E \in \Omega$.

As remarked above $\tau_k \in ca(\Omega) \hat{\otimes}_\pi X$, hence $\sum_{k=1}^\infty \tau_k \in ca(\Omega) \hat{\otimes}_\pi X$ also. Note that (1) implies that the sequence $\{\sum_{k=1}^n \tau_k\}$ is Cauchy in $ca(\Omega) \hat{\otimes}_\pi X$, because the variation norm is the same as the $\pi$-norm. But by (2), $\sum_{k=1}^\infty \tau_k$ converges setwise to $\tau$, therefore in variation ($\pi$-norm). Thus $\tau \in ca(\Omega) \hat{\otimes}_\pi X$.

Conversely, if $\tau \in ca(\Omega) \hat{\otimes}_\pi X$, by the general theory of projective tensor products (see Trèves [6]), there exists $x_n \in X$ and $\lambda_n \in ca(\Omega)$ such that $\sum_{k=1}^\infty |x_k| |\lambda_k|(S) < +\infty$ and $\tau(E) = \sum_{k=1}^\infty x_k \lambda_k (E)$ for all $E \in \Omega$. Write $\tau_k = x_k \lambda_k$, then clearly $\tau_k$ has the RN property, $\tau = \sum \tau_k$ and $\sum |\tau_k| < +\infty$. By Theorem 2, $\tau$ has the RN property.

COROLLARY 1. A measure $\tau \in cafv(\Omega; X)$ has the RN property if and only if $\tau$ is expressible as the indefinite Bochner integral with respect to some positive measure.
Recall that a Banach space $X$ has the Radon-Nikodym property if it's true that any $X$-valued vector measure of finite variation can be expressed as an indefinite Bochner integral.

**Corollary 2.** A Banach space $X$ has the Radon-Nikodym property if and only if $ca(S, \Omega) \hat{\otimes} X = cafv(S, \Omega; X)$ for every measurable space $(S, \Omega)$.

**Remarks.** In particular, if $X$ is reflexive or a separable dual space, then $ca(\Omega) \hat{\otimes} X = cafv(S, \Omega; X)$ for every measurable space $(S, \Omega)$. It has been shown that $ca(S, \Omega) \hat{\otimes} X$ is the Banach space, with total variation norm, of all $X$-valued measures on $\Omega$ with the RN property; for sake of completeness, it has been shown in [4] and [5], that $ca(\Omega) \hat{\otimes} X$, where $\hat{\otimes}$ is the weak (or inductive) tensor product, is the Banach space of all $X$-valued vector measures with relatively norm compact range, equipped with the semi-variation norm. In conclusion, the following question is posed: can the criterion of Corollary 2 be used to give an "external" proof of the fact that reflexive Banach spaces and separable dual spaces have the Radon-Nikodym property?

**References**


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