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**THE LEVI DECOMPOSITION OF A SPLIT  $(B, N)$ -PAIR**

N. B. TINBERG

# THE LEVI DECOMPOSITION OF A SPLIT $(B, N)$ -PAIR

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**Let  $p$  be a prime number. If  $G$  is a finite group with a split  $(B, N)$ -pair of characteristic  $p$  then each parabolic subgroup  $G_J$  of  $G$  can be written as a semidirect product of certain subgroups  $C_J$  and  $L_J$ . Moreover  $G_J$  is the full normalizer of  $C_J$  in  $G$ .**

**1. Introduction.** Let  $G$  be a Chevalley group. The set of its parabolic subgroups  $\{G_J | J \subseteq R\}$  is indexed by the subsets of the set  $R$  of fundamental roots of the associated Lie algebra. Each  $G_J$  admits a decomposition of the following form:

$$G_J = C_J L_J$$

where  $C_J \trianglelefteq G_J$  and  $C_J \cap L_J = \{1\}$ . Furthermore,  $G_J$  is the normalizer of  $C_J$  in  $G$ . This decomposition of  $G_J$  into the semidirect product of  $C_J$  and  $L_J$  is called the Levi decomposition and  $L_J$  and its conjugates in  $C_J$  are called the Levi subgroups of  $G_J$  (see [2, p. 118-119]). In this paper we show that if  $G$  is a finite group with a split  $(B, N)$ -pair of characteristic  $p$  then the parabolic subgroups of  $G$  admit a similar decomposition.

The difficulty in proving the existence of the Levi decomposition for an arbitrary finite group with a split  $(B, N)$ -pair is showing that  $C_J \trianglelefteq G_J$  (Lemma 1) and that  $L_J$  is itself a group with a split  $(B, N)$ -pair (Lemma A). Curtis proves these facts in [5, Proposition 1.5(a), (d), p. 669] and concludes that  $G$  admits a Levi decomposition. However, his arguments depend on the use of the commutator relations ([5, Proposition 1.4(f), p. 669]) and the proof of these relations relies heavily on the Fong-Seitz classification ([6], [7]) of split  $(B, N)$ -pairs of rank 2. It is the advantage of this present note to prove the required facts without these commutator relations, but under the assumption

$$(*) \quad U \cap U^{w_i} \trianglelefteq U \quad \text{for all } w_i \in R.$$

The reader should note that  $(*)$  appears as a hypothesis in [7, Theorem D, p. 238]. Moreover, in the case when  $p$  is an odd prime  $(*)$  follows using a very strong result on 2-transitive permutation groups due to Kantor and Seitz ([8, Theorem C', p. 131]. See [10, proof of Theorem 4.5]). That result is essential to the Fong-Seitz classification (see [6, p. 2]). By assuming  $(*)$  we too then employ the Kantor-Seitz result; however, since we do not refer to the Fong-

Seitz papers, we have achieved a substantial simplification of the existing proof.

Throughout our discussion  $G = (G, B, N, R, U)$  will denote a finite group with a split  $(B, N)$ -pair of characteristic  $p$  and rank  $n$  (see [4, Definition 2.1, p. B-8]). Hence  $G$  satisfies the following conditions:

(i)  $G$  has a  $(B, N)$ -pair ([4, Definition 2.1, p. B-8]) where  $H = B \cap N$  and the Weyl group  $W = N/H$  is generated by the set of involutions  $R = \{w_1, \dots, w_n\}$ .

(ii)  $H = \bigcap_{n \in N} n^{-1}Bn$ .

(iii)  $U$  is a normal  $p$ -subgroup of  $B$ ;  $B = U \cdot H$  is a semidirect product and  $H$  is abelian with order prime to  $p$ .

Notice that (iii) tells us that we always have a Levi decomposition in the case  $J = \Phi$ ,  $G_J = B$ .

The author wishes to thank J. A. Green for his helpful suggestions.

**2. Preliminaries.** The Weyl group of a  $(B, N)$ -pair is isomorphic to the Weyl group of a root system in Euclidean space in such a way that  $R$  corresponds to the set of fundamental reflections (see [9, p. 439]). We therefore define  $\Delta = \{a_i | w_i \in R\}$  to be the set of fundamental roots of this root system.

Let  $\nu: N \rightarrow W$  be the natural epimorphism. For each subset  $J \subseteq R$ , the parabolic subgroup  $G_J = (G_J, B, N_J, J, U)$  is an unsaturated split  $(B, N)$ -pair of characteristic  $p$  and rank  $|J|$  where  $W_J = \langle w_i | w_i \in J \rangle$  and  $N_J = \nu^{-1}(W_J)$  (see [1, Proposition 1, p. 28]). The group  $G_J = BN_JB$  is unsaturated (see [10]) since  $\bigcap_{n \in N_J} n^{-1}Bn$  may be larger than  $H$ ; that is,  $\bigcap_{n \in N_J} U^n > 1$ . Any  $w \in W$  can be written as a minimal product of the generators in  $R$ . We denote by  $l(w)$  the length of such an expression. For each  $J \subseteq R$ ,  $w_J$  will denote the unique element of maximal length in  $W_J$ . In the case  $J = R$  we write  $w_0$  for  $w_R$ . If  $X$  is any subset of  $G$  and  $g \in G$ , then  $X^g = g^{-1}Xg$ .

**DEFINITIONS.** Let  $w \in W$ . Then  ${}_wU^- = U \cap U^{w_0w}$ ,  ${}_wU^+ \cap U^w$ . Write  ${}_{w^{-1}}U^-$  as  $U_w^-$  and  ${}_{w^{-1}}U^+$  as  $U_w^+$ . In the case  $w = w_i$  we write  $U_i$  for  ${}_{w_i}U^-$ . Let  $V_i = U_i^{w_i}$  and  $V = U^{w_0}$ . Set  $G_i = \langle U_i, V_i \rangle$  and  $H_i = H \cap G_i$ .

Let  $(w_i) \in N$  be such that  $(w_i)H = w_i(w_i \in R)$ . As in [3, Lemma 2.2, p. 351] we choose

(a)  $(w_i) \in G_i$  for each  $w_i \in R$ .

In which case

(b)  $G_i = U_i H_i \cup U_i H_i (w_i) U_i$

for all  $w_i \in R$  ([3, Lemma 2.7, p. 351]).

DEFINITIONS. For each  $J \subseteq R$  let  $L_J = \langle H, (U_i)^w \mid w \in W_J, w_i \in J \rangle$ .  
 Set  $U_J = {}_{w_J}U^-, B_J = HU_J$ .

Notice that  $L_J = \langle H, (G_i)^w \mid w \in W_J, w_i \in J \rangle$  and that

$$(c) \quad L_J = \langle H, G_i \mid w_i \in J \rangle$$

by our choice of representatives (a).

LEMMA A. *If  $G$  is a finite group with a split  $(B, N)$ -pair then  $(L_J, B_J, N_J, J, U_J)$  is a split  $(B, N)$ -pair for any  $J \subseteq R$ .*

Curtis proved Lemma A using the commutator relations ([5, Proposition 1.4(f), Proposition 1.5(d), p. 669]). The following proof was suggested to the author by J. A. Green. We first require:

LEMMA B. *Let  $w \in W_J$ . Then  ${}_{w}U^- \subseteq U_J$ . In particular  $U_i \subseteq U_J$  for all  $w_i \in J$ .*

*Proof.* Write  $w_J = vw$  with  $l(v) + l(w) = l(w_J)$ . By [10, Lemma 2.2],  $U_J = {}_{w}U^-(U^-)^w$  and the result follows.

*Proof of Lemma A.* We verify the  $(B, N)$ -pair axioms as given, for example in [4, p. B-8]:

$$(i) \quad L_J = \langle B_J, N_J \rangle \text{ and } B_J \cap N_J \leq N_J.$$

Let  $w_J = w_{i_1} \cdots w_{i_q}$  be a reduced expression for  $w_J$  with all  $w_{i_s} \in J$  ( $1 \leq s \leq q$ ). Then

$$U_J = (U_{i_q})(U_{i_{q-1}})^{w_{i_q}} \cdots (U_{i_1})^{w_{i_2} \cdots w_{i_q}}$$

by [4, Proposition 3.3(vi), p. B-13]. Hence  $U_J \leq L_J$  and  $B_J \subseteq L_J$  since  $H \subseteq L_J$ . By (a) and (c)  $L_J$  contains each  $(w_i)$  where  $w_i \in J$  so that  $N_J \subseteq L_J$ . Hence  $\langle B_J, N_J \rangle \subseteq L_J$ . Conversely, if  $w_i \in J$  then  $U_i \subseteq U_J \subseteq \langle B_J, N_J \rangle$  by Lemma B and  $Hw \subseteq N_J \subseteq \langle B_J, N_J \rangle$  all  $w \in W_J$ . Therefore  $\langle U_i \rangle^w \subseteq \langle B_J, N_J \rangle$  all  $w \in W_J$  and  $L_J \subseteq \langle B_J, N_J \rangle$ . We also have that  $H \subseteq B_J \cap N_J \subseteq B \cap N = H$  and (i) is proved.

(ii) The finite group  $W_J \cong N_J/(N_J \cap B_J) = N_J/H$  is generated by the set  $J$  of involutions.

(iii) For all  $w_i \in J$  and  $w \in W_J$  there holds  $w_i B_J w \subseteq B_J w B_J \cup B_J w_i w B_J$ . To prove (iii) we need only show that

$$(1) \quad (w_i)u(w) \in B_J w B_J \cup B_J w_i w B_J$$

for any  $u \in U_J$ . By [4, Proposition 3.3(iii), p. B-13] we may write  $u = u_1 u_2$  with  $u_1 \in U_w^-, u_2 \in U_w^+$ . By Lemma B,  $u_1 \in U_J$ . So  $u_2 \in U_J$  and  $u_2 \in U_w^+ \cap U_J = U \cap U^{w^{-1}} \cap U \cap U^{w_0 w_J}$  and  $(w)^{-1} u_2 (w) \in U^w \cap U \cap U^{w_0 w_J w} \subseteq {}_{w_J}w U^- \subseteq U_J$  by Lemma B. Therefore

$$(w_i)u(w) = (w_i)u_1(w)(w)^{-1}u_2(w) \in (w_i)u_1(w)B_J .$$

It is therefore sufficient to prove (1) for any  $u(=u_1) \in U_w^+$ . We examine the following two cases:

**Case I.**  $l(w_i w) = l(w) + 1$ . By [4, Proposition 3.3(i), p. B-13] and Lemma B

$$(U_w^-)^{w_i} \subseteq (w_i U^-) \cdot (U_w^-)^{w_i} = U_{w_i w}^- \subseteq U_J .$$

If  $u \in U_w^-$  then

$$(w_i)u(w)B_J = (w_i)u(w_i)^{-1}(w_i)(w)B_J \subseteq U_J w_i w B_J .$$

Hence if

$$(2) \quad l(w_i w) = l(w) + 1 \quad \text{then} \quad (w_i)u(w) \in B_J w_i w B_J$$

for any  $u \in U_J$ .

**Case II.**  $l(w_i w) = l(w) - 1$ . Writing  $v = w_i w$  we then have  $w = w_i v$  with  $l(v) + 1 = l(w_i v)$  and as above

$$U_w^- = U_{w_i v}^- = w_i U^-(U_v^-)^{w_i} = (U_v^-)^{w_i} U_i .$$

Therefore  $(U_w^-)^{w_i} = (U_v^-)(U_i)^{w_i} \subseteq U_J(U_i)^{w_i}$  by Lemma B. If  $u \in U_w^-$  we have  $(w_i)u(w) = (w_i)u(w_i)^{-1}(w_i)(w) \in B_J g v B_J$  for some  $g \in G_i$ . From (b) either  $g$  lies in  $U_i H_i \subseteq B_J$  in which case  $(w_i)u(w) \in B_J w_i w B_J$  or  $g$  lies in  $U_i H_i(w_i)H_i \subseteq B_J w_i B_J$  in which case  $(w_i)u(w) \in B_J w_i B_J v B_J \subseteq B_J w_i v B_J = B_J w B_J$  using (2) (with  $v$  replacing  $w$ ). Thus (1) holds in Case II.

(iv) For all  $w_i \in J$ ,  $w_i B_J w_i \neq B_J$ . Now  $U_i \subseteq B_J$  so that  $w_i B_J w_i \supseteq (U_i)^{w_i}$ . If (iv) were false then  $w_i B_J w_i = B_J \supseteq (U_i)^{w_i}$  so that  $(U_i)^{w_i} \subseteq U$  contrary to [4, Proposition 3.3(v), p. B-13].

The  $(B, N)$ -pair is saturated since

$$(U_J)^{w_J} \cap U_J = U \cap U^{w_0 w_J} \cap U^{w_J} \cap U^{w_0} \subseteq U \cap V = \{1\} .$$

**3. Proof of the Theorem.** We now state our theorem and prove it by a succession of lemmas. Assume (\*) holds:

**THEOREM (Levi Decomposition).** *Let  $G = (G, B, N, R, U)$  be a finite group with a split  $(B, N)$ -pair of characteristic  $p$ . For each subset  $J \subseteq R$ , there exist subgroups  $C_J$  and  $L_J$  such that*

- (a)  $G_J = C_J L_J$  where  $C_J \trianglelefteq G_J$  and  $C_J \cap L_J = \{1\}$ .
- (b) The normalizer in  $G$  of  $C_J$  is  $G_J$ .

Fix  $J \subseteq R$ . Let  $C_J = \bigcap_{n \in N_J} U^n$ . Then  $C_J = U \cap U^{w_J}$  and  $C_J^w = C_J$  for all  $w \in W_J$  by [10, Lemma 2.1 and the subsequent remark]. It can easily be shown that  $C_J$  is generated by certain root subgroups of  $G$  as in [5, p. 669] or [2, p. 119]. In the special case  $J = R$  we know that  $C = C_R = U \cap U^{w_0} = \{1\}$  since  $G$  is saturated.

LEMMA 1.  $C_J \trianglelefteq G_J$ .

*Proof.* The result follows by [10, Lemmas 4.2 and 4.3].

LEMMA 2. Let  $L_J = \langle H, G_i \mid w_i \in J \rangle$ . Then  $G_J = \langle C_J, L_J \rangle$ .

*Proof.* Notice that  $\langle C_J, L_J \rangle = \langle C_J, B_J, N_J \rangle$  by (i) in the proof of Lemma A so that  $\langle C_J, L_J \rangle = \langle C_J, U_J, N_J \rangle = \langle B, N_J \rangle$  by [4, Proposition 3.3(ii), p. B-13] and the result follows.

Since  $C_J \trianglelefteq G_J$ ,

LEMMA 3.  $G_J = C_J L_J$ .

LEMMA 4.  $U \cap L_J \subseteq U_J$ .

*Proof.* By Lemma A, we have a Bruhat Decomposition

$$L_J = \bigcup_{w \in W_J} B_J w B_J.$$

If  $w \neq 1$ ,  $w \in W_J$  then  $B_J w B_J$  does not intersect  $B$  since  $B_J w B_J \subseteq B w B$  and  $B w B \cap B$  is empty. Hence  $B \cap L_J \subseteq B_J$  and Lemma 4 follows.

LEMMA 5.  $C_J \cap L_J = \{1\}$ .

*Proof.*  $C_J \cap L_J = C_J \cap U \cap L_J \subseteq C_J \cap U_J = \{1\}$  by [4, Proposition 3.3(iii), p. B-13].

The proof of the following lemma is based on [2, p. 120].

LEMMA 6. The normalizer,  $N_G(C_J)$ , of  $C_J$  in  $G$  is  $G_J$ .

*Proof.* We know that  $G_J \subseteq N_G(C_J)$  so that  $N_G(C_J) = G_K$  with  $J \subseteq K$ . If  $J \subset K$  take  $w_i \in K$ ,  $w_i \notin J$ . Then by [4, Proposition 3.3(v), p. B-13],  $U_i \subseteq {}_w U^+$  for any  $w \in W_J$  since  $w(a_i) > 0$  all  $w \in W_J$ . But  $C_J = \bigcap_{w \in W_J} w U^+$  so that  $U_i \subseteq C_J$ . Since  $w_i \in G_K$ ,  $U_i^{w_i} \subseteq C_J^{w_i} = C_J$ . On the other hand  $U_i^{w_i w_0} \subseteq U$  by [4, Proposition 3.3(v), p. B-13] since  $w_i(a_i) = -a_i$ . Therefore,  $U_i^{w_i} \subseteq C_J \cap V \subseteq C \cap V = \{1\}$  and  $U_i = \{1\}$ ,

contrary to the  $(B, N)$ -pair axioms since for all  $w_i \in R$ ,  $w_i U w_i \neq U$  and  $U = U_{i w_i} U^+$  (see [4, Proposition 3.3(iii), p. B-13]).

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