STABLE SEQUENCES IN PRE-ABELIAN CATEGORIES

YONINA S. COOPER
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In the Pacific Journal of Mathematics, 71 (1977), Richman and Walker gave a natural definition for Ext in an arbitrary pre-abelian category. Their Theorem 4, which states that 

\((aE)\beta = \alpha(E\beta)\)

for an arbitrary sequence \(E\), is in error. We show, however, that 

\((aE)\beta = \alpha(E\beta)\)

does hold for a stable exact sequence. Without Theorem 4, the crucial step in their theory is showing that \(aE\) is stable if \(E\) is stable. We prove this. Consequently, the theory of Richman and Walker for Ext in a pre-abelian category is valid.

1. Introduction. An additive category with kernels and cokernels is called pre-abelian. Richman and Walker [3] developed an additive bifunctor Ext from an arbitrary pre-abelian category to the category of abelian groups. The Ext introduced in [3] coincides with the standard Ext (e.g., see [2]) if the category is, in fact, abelian. This theory is subsequently used by Richman and Walker [4] in the category of valuated groups. The theory of [3] is also used in [1] to examine certain relative homological algebras and to compute certain \(\text{Ext}(C, A)\) in the category of finite valuated groups.

However, Theorem 4 [3, p. 523] is incorrect. Without Theorem 4, one needs to prove that the sequence \(aE\) is stable if the sequence \(E\) is stable. This is our Theorem 2.

We use the terminology and notation of [3]. Thus, we are working working in an arbitrary pre-abelian category. If \(f: A \to B\) and \(\alpha: A \to A'\), then the pushout diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\alpha} & & \downarrow{\xi} \\
A' & \xrightarrow{\beta} & P \\
\end{array}
\]

is constructed by setting \(P = \text{coker}(f \oplus (-\alpha))\), where \(\Delta: A \to A \oplus A\) is the diagonal map. We say that \(\beta\) is the pushout of \(f\) along \(\alpha\). Pullbacks are obtained dually. A sequence \(E\) is a diagram \(A \xrightarrow{f} B \xrightarrow{g} C\) such that \(gf = 0\). \(E\) is left exact if \(f\) is the kernel of \(g\), right exact if \(g\) is the cokernel of \(f\), and exact if it is both left and right exact. If \(\alpha: A \to A'\), we pushout \(f\) along \(\alpha\) to construct the sequence \(aE\).
The pushout property gives the existence and uniqueness of $g': B' \to C$ such that $g'f' = 0$ and $g'\varphi = g$. We obtain $E\beta$ in the dual manner for $\beta: C' \to C$.

Richman and Walker [3, Theorem 4] assert that $(\alpha E)\beta = \alpha (E\beta)$ for an arbitrary sequence $E$. This is not true, even if $E$ is exact. Consider the category of abelian $p$-groups with no elements of infinite height. Let $B$ be a direct sum of cyclic groups of order $p^n$ for $n = 1, 2, 3, \ldots$, $\bar{B}$ be the torsion subgroup of the corresponding direct product, and $G[p] = \{g \in G: pg = 0\}$. Then we have

$$
\begin{array}{c}
E: B[p] \xrightarrow{i} \bar{B} \xrightarrow{\beta} B/B[p] \\
\downarrow \alpha \quad \downarrow \beta \quad \downarrow \\
\alpha E: B[p] \xrightarrow{0} \bar{B} \xrightarrow{\beta} B/B[p] \\
\downarrow \quad \downarrow \beta \quad \downarrow \\
(\alpha E)\beta: B[p] \xrightarrow{0} \bar{B} \xrightarrow{\beta} \bar{B}
\end{array}
$$

where $i$ is the injection map and $\alpha$ and $\beta$ are the coset maps. The fact that 0 is the pushout of $i$ along $\alpha$ is due to Richman and Walker [3, p. 522]. Note that $E$ is exact, but $\alpha E$ is not left exact. On the other hand, we have

$$
\begin{array}{c}
\alpha(E\beta): B[p] \xrightarrow{i} B[p] \oplus \bar{B} \xrightarrow{\beta} B \\
\downarrow \alpha \quad \downarrow \quad \downarrow \\
E\beta: B[p] \xrightarrow{0} B[p] \oplus \bar{B} \xrightarrow{\beta} \bar{B} \\
\downarrow \quad \downarrow \beta \\
E: B[p] \xrightarrow{i} \bar{B} \xrightarrow{\beta} B/B[p]
\end{array}
$$

where $\varphi: B \oplus \bar{B} \to \bar{B}$ is the codiagonal map. Hence $(\alpha E)\beta \neq \alpha (E\beta)$.

2. Stable exact sequences. The example motivates following definition.

**Definition** (Richman and Walker [3, p. 524]). An exact sequence
$E$ is said to be stable if $\alpha E$ and $E\beta$ are exact for all maps $\alpha$ and $\beta$.

**Lemma 1** (Richman and Walker [3, Theorem 5]). If $E$ is right exact, then $\alpha E$ is right exact. If $E$ is left exact, then $E\beta$ is left exact.

The objective of [3] was to define $\text{Ext}$ so that it is a functor. Thus showing that $(\alpha E)\beta = \alpha(E\beta)$ if $E$ is stable is crucial. Now there is always a morphism $\alpha(E\beta) \to (\alpha E)\beta$. The problem is to get the morphism back. We now construct the morphism $\alpha(E\beta) \to (\alpha E)\beta$. Consider the diagram

\[
\begin{array}{ccc}
E\beta: & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C' \\
\downarrow & & & \downarrow & & \\
E: & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow & & & \downarrow & & \\
\alpha E: & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C.
\end{array}
\]

Construct $(\alpha E)\beta$:

\[
(\alpha E)\beta: A' \xrightarrow{f_3} B_3 \xrightarrow{g_3} C'
\]

Then $\beta g_1 = g'\lambda\xi$ implies there exists $\delta: B_1 \to B_3$ such that $g_3\delta = g_1$ and $\varphi\delta = \lambda\xi$ (since $g_3$ is the pullback of $g'$ along $\beta$). Thus the diagram

\[
\begin{array}{ccc}
E\beta: & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C' \\
\downarrow & & & \downarrow & & \\
(\alpha E)\beta: & A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C'
\end{array}
\]

commutes since $g_3(\delta f_1 - f_3\alpha) = g_3\delta f_1 = g_1 f_1 = 0$ and $\varphi(\delta f_1 - f_3\alpha) = \varphi\delta f_1 - \varphi f_3\alpha = \lambda\xi f_1 - f'\alpha = 0$ imply $f_3\alpha = \delta f_1$ (again using the fact that $g_3$ is a pullback). Now construct $E\beta g_3$ and factor the morphism $(\alpha, \delta, 1)$ through $\alpha(E\beta)$, using the pushout property, to obtain the commutative diagram
where \( \phi \phi_1 = \delta \). We now use this diagram to prove

**Theorem 2.** Let \( E: A \rightarrow B \rightarrow C \) be stable exact. Then \( \alpha E \) and \( E\beta \) are stable exact. Furthermore, \( \alpha(E\beta) = (\alpha E)\beta \) for all \( \alpha: A \rightarrow A' \) and \( \beta: C' \rightarrow C \).

**Proof.** Now \( E\beta \) and \( \alpha E \) are exact since \( E \) is stable. And since the pull back of a pullback, is a pullback, \( (E\beta)\gamma = E(\beta\gamma) \) is exact. Dually \( \mu(\alpha E) = (\mu\alpha)E \) is exact. Thus to show that \( \alpha E \) is stable requires \( (\alpha E)\beta \) to be exact. Dually, the stability of \( E\beta \) requires the exactness of \( \alpha(E\beta) \). However, \( g_3 = \text{coker } f_2 \) and \( f_3 = \ker g_3 \) by Lemma 1. Thus, in order to show that \( \alpha E \) is stable, it only remains to show \( g_3 = \text{coker } f_3 \).

First, we show that \( \phi_2 \) is an epimorphism. From the diagram, we observe that \( g_3\phi_2\phi_3\phi_0 = g_0g_3 \), that is, \( g_3(\phi_2\phi_3\phi_0 - g_0) = 0 \). So there is \( \gamma: X \rightarrow A' \) such that \( f_3\gamma = \phi_2\phi_3\phi_0 - g_0 \) since \( f_3 = \ker g_3 \). Then \( f_3\gamma f_0 = \phi_2\phi_3\phi_0 f_0 = f_3\alpha \). So \( \gamma f_0 = \alpha \) since \( f_3 \) is a monomorphism. Now \( 0 = \phi_2\phi_3\phi_0 f_0 - f_3\alpha = \phi_2\phi_3\phi_0 f_0 - f_3\gamma f_0 = (\phi_2\phi_3\phi_0 - f_3\gamma)f_0 \). So there is \( \nu: B_3 \rightarrow B_2 \) such that \( \gamma g_0 = \phi_2\phi_3\phi_0 - f_3\gamma \) since \( g_0 = \text{coker } f_0 \). Then \( \phi_2\nu g_0 = \phi_2\phi_3\phi_0 - \phi_2 f_3\gamma = \phi_2\phi_3\phi_0 - f_3\gamma = g_0 \). But \( \phi_2\nu g_0 = g_0 \) implies \( \phi_2\nu = 1 \) since \( g_0 \) is a cokernel and hence an epimorphism. Hence, \( \phi_2 \) is an epimorphism since \( \mu\phi_2 = 0 \) implies \( \mu = \mu\phi_2\nu = 0 \).

Suppose \( \mu f_3 = 0 \). Then \( \mu\phi_3f_2 = 0 \). Since \( g_2 = \text{coker } f_2 \), there is \( \eta \) such that \( \eta g_2 = \mu \phi_2 \). And \( \eta g_3\phi_2 = \eta g_2 = \mu\phi_2 \) implies \( \eta g_3 = \mu \). And if \( \eta'g_3 = \mu \), then \( \eta'g_2 = \eta'g_3\phi_2 = \mu\phi_2 = \eta g_2 \). And \( \eta' = \eta \) since \( g_2 \) is an epimorphism. Hence \( g_3 = \text{coker } f_3 \).

Consider the dual diagram

\[
\begin{array}{c}
\alpha(E\beta) \longrightarrow (\alpha E)\beta \longrightarrow \alpha E \longrightarrow f_3\alpha E.
\end{array}
\]

Now the dual of the above argument gives \( \phi_2 \) is a monomorphism and \( f_2 = \ker g_2 \). Consequently, \( E\beta \) is stable.

Recall \( \nu\phi_2 = 1 \). Then \( \phi_2\nu\phi_2 = \phi_2 \) implies \( \nu\phi_2 = 1 \) since \( \phi_2 \) is a monomorphism. Hence \( (\alpha E)\beta = \alpha(E\beta) \) in the sense that \( \phi_2 \) is an
isomorphism.

Ext in a pre-abelian category can now be pursued as in [3].
That is, the results and proofs of Richman and Walker [3, §3ff.] hold
as stated. We conclude with the fact for an exact sequence, stability
is equivalent to associativity.

\textbf{Theorem 3.} An exact sequence $E$ is stable if and only if $(\alpha E)\beta = \alpha (E\beta)$ for all $\alpha$ and $\beta$.

\textit{Proof.} Only associativity implies stability needs to proved. (The
following argument is Fred Richman's.) Consider the diagrams

\begin{align*}
E: A & \longrightarrow B \longrightarrow C & E: A & \longrightarrow B \longrightarrow C \\
\alpha & \\ \alpha E: A' & \longrightarrow K \longrightarrow C & E0: A & \longrightarrow A \longrightarrow 0 \\
\phi & \\
(\alpha E)0: A' & \longrightarrow \ker \phi \longrightarrow 0 & \alpha(E0): A' & \longrightarrow A' \longrightarrow 0.
\end{align*}

So $(\alpha E)0 = \alpha(E0)$ implies $A'$ maps isomorphically onto $\ker \phi$. So $\alpha E$
is exact. Similarly, $E\beta$ is exact.

\textbf{References}


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