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STABLE SEQUENCES IN PRE-ABELIAN CATEGORIES

YONINA S. COOPER

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In the *Pacific Journal of Mathematics*, 71 (1977), Richman and Walker gave a natural definition for Ext in an arbitrary pre-abelian category. Their Theorem 4, which states that $(\alpha E)\beta = \alpha(E\beta)$ for an arbitrary sequence E , is in error. We show, however, that $(\alpha E)\beta = \alpha(E\beta)$ does hold for a stable exact sequence. Without Theorem 4, the crucial step in their theory is showing that αE is stable if E is stable. We prove this. Consequently, the theory of Richman and Walker for Ext in a pre-abelian category is valid.

1. Introduction. An additive category with kernels and cokernels is called *pre-abelian*. Richman and Walker [3] developed an additive bifunctor Ext from an arbitrary pre-abelian category to the category of abelian groups. The Ext introduced in [3] coincides with the standard Ext (e.g., see [2]) if the category is, in fact, abelian. This theory is subsequently used by Richman and Walker [4] in the category of valuated groups. The theory of [3] is also used in [1] to examine certain relative homological algebras and to compute certain $\text{Ext}(C, A)$ in the category of finite valuated groups.

However, Theorem 4 [3, p. 523] is incorrect. Without Theorem 4, one needs to prove that the sequence αE is stable if the sequence E is stable. This is our Theorem 2.

We use the terminology and notation of [3]. Thus, we are working in an arbitrary pre-abelian category. If $f: A \rightarrow B$ and $\alpha: A \rightarrow A'$, then the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & & \downarrow \xi \\ A' & \xrightarrow{\beta} & P \end{array}$$

is constructed by setting $P = \text{coker}(f \oplus (-\alpha)\Delta)$, where $\Delta: A \rightarrow A \oplus A$ is the diagonal map. We say that β is the pushout of f along α . Pullbacks are obtained dually. A sequence E is a diagram $A \xrightarrow{f} B \xrightarrow{g} C$ such that $gf = 0$. E is *left exact* if f is the kernel of g , *right exact* if g is the cokernel of f , and *exact* if it is both left and right exact. If $\alpha: A \rightarrow A'$, we pushout f along α to construct the sequence αE .

$$\begin{array}{ccccc}
 E: & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 & \alpha \downarrow & & \downarrow \varphi & & \parallel \\
 \alpha E: & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C .
 \end{array}$$

The pushout property gives the existence and uniqueness of $g': B' \rightarrow C$ such that $g'f' = 0$ and $g'\varphi = g$. We obtain $E\beta$ in the dual manner for $\beta: C' \rightarrow C$.

Richman and Walker [3, Theorem 4] assert that $(\alpha E)\beta = \alpha(E\beta)$ for an arbitrary sequence E . This is not true, even if E is exact. Consider the category of abelian p -groups with no elements of infinite height. Let B be a direct sum of cyclic groups of order p^n for $n = 1, 2, 3, \dots$, \bar{B} be the torsion subgroup of the corresponding direct product, and $G[p] = \{g \in G: pg = 0\}$. Then we have

$$\begin{array}{ccccc}
 E: & \bar{B}[p] & \xrightarrow{i} & \bar{B} & \xrightarrow{\beta} & \bar{B}/\bar{B}[p] \\
 & \alpha \downarrow & & \beta \downarrow & & \parallel \\
 \alpha E: & \frac{\bar{B}[p]}{B[p]} & \xrightarrow{0} & \frac{\bar{B}}{B[p]} & \xlongequal{\quad} & \frac{\bar{B}}{B[p]} \\
 & \parallel & & \uparrow \beta & & \uparrow \beta \\
 (\alpha E)\beta: & \frac{\bar{B}[p]}{B[p]} & \xrightarrow{0} & \bar{B} & \xlongequal{\quad} & \bar{B}
 \end{array}$$

where i is the injection map and α and β are the coset maps. The fact that 0 is the pushout of i along α is due to Richman and Walker [3, p. 522]. Note that E is exact, but αE is not left exact. On the other hand, we have

$$\begin{array}{ccccc}
 \alpha(E\beta): & \frac{\bar{B}[p]}{B[p]} & \longrightarrow & \frac{\bar{B}[p]}{B[p]} \oplus \bar{B} & \longrightarrow & \bar{B} \\
 & \alpha \uparrow & & \uparrow & & \parallel \\
 E\beta: & \bar{B}[p] & \longrightarrow & \bar{B}[p] \oplus \bar{B} & \longrightarrow & \bar{B} \\
 & \parallel & & \downarrow \nabla(i \oplus 1) & & \downarrow \beta \\
 E: & \bar{B}[p] & \xrightarrow{i} & \bar{B} & \xrightarrow{\beta} & \bar{B}/\bar{B}[p]
 \end{array}$$

where $\nabla: \bar{B} \oplus \bar{B} \rightarrow \bar{B}$ is the codiagonal map. Hence $(\alpha E)\beta \neq \alpha(E\beta)$.

2. Stable exact sequences. The example motivates following definition.

DEFINITION (Richman and Walker [3, p. 524]). An exact sequence

E is said to be *stable* if αE and $E\beta$ are exact for all maps α and β .

LEMMA 1 (*Richman and Walker [3, Theorem 5]*). *If E is right exact, then αE is right exact. If E is left exact, then $E\beta$ is left exact.*

The objective of [3] was to define Ext so that it is a functor. Thus showing that $(\alpha E)\beta = \alpha(E\beta)$ if E is stable is crucial. Now there is always a morphism $\alpha(E\beta) \rightarrow (\alpha E)\beta$. The problem is to get the morphism back. We now construct the morphism $\alpha(E\beta) \rightarrow (\alpha E)\beta$. Consider the diagram

$$\begin{array}{ccccc}
 E\beta: A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C' \\
 & & \downarrow \xi & & \downarrow \beta \\
 E: A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 \alpha \downarrow & & \downarrow \lambda & & \parallel \\
 \alpha E: A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C.
 \end{array}$$

Construct $(\alpha E)\beta$:

$$\begin{array}{ccccc}
 (\alpha E)\beta: A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C' \\
 & & \downarrow \varphi & & \downarrow \beta \\
 \alpha E: A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C.
 \end{array}$$

Then $\beta g_1 = g' \lambda \xi$ implies there exists $\delta: B_1 \rightarrow B_3$ such that $g_3 \delta = g_1$ and $\varphi \delta = \lambda \xi$ (since g_3 is the pullback of g' along β). Thus the diagram

$$\begin{array}{ccccc}
 E\beta: A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C' \\
 \alpha \downarrow & & \downarrow \delta & & \parallel \\
 (\alpha E)\beta: A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C'
 \end{array}$$

commutes since $g_3(\delta f_1 - f_3 \alpha) = g_3 \delta f_1 = g_1 f_1 = 0$ and $\varphi(\delta f_1 - f_3 \alpha) = \varphi \delta f_1 - \varphi f_3 \alpha = \lambda \xi f_1 - f' \alpha = 0$ imply $f_3 \alpha = \delta f_1$ (again using the fact that g_3 is a pullback). Now construct $E\beta g_3$ and factor the morphism $(\alpha, \delta, 1)$ through $\alpha(E\beta)$, using the pushout property, to obtain the commutative diagram

$$\begin{array}{ccccc}
 E\beta g_3: A & \xrightarrow{f_0} & X & \xrightarrow{g_0} & B_3 \\
 \parallel & & \downarrow \varphi_0 & & \downarrow g_3 \\
 E\beta: A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_2} & C' \\
 \alpha \downarrow & & \downarrow \varphi_1 & & \parallel \\
 \alpha(E\beta): A' & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C' \\
 \parallel & & \downarrow \varphi_2 & & \parallel \\
 (\alpha E)\beta: A' & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C'
 \end{array}$$

where $\varphi_2\varphi_1 = \delta$. We now use this diagram to prove

THEOREM 2. *Let $E: A \xrightarrow{f} B \xrightarrow{g} C$ be stable exact. Then αE and $E\beta$ are stable exact. Furthermore, $\alpha(E\beta) = (\alpha E)\beta$ for all $\alpha: A \rightarrow A'$ and $\beta: C' \rightarrow C$.*

Proof. Now $E\beta$ and αE are exact since E is stable. And since the pull back of a pullback, is a pullback, $(E\beta)\gamma = E(\beta\gamma)$ is exact. Dually $\mu(\alpha E) = (\mu\alpha)E$ is exact. Thus to show that αE is stable requires $(\alpha E)\beta$ to be exact. Dually, the stability of $E\beta$ requires the exactness of $\alpha(E\beta)$. However, $g_2 = \text{coker } f_2$ and $f_3 = \ker g_3$ by Lemma 1. Thus, in order to show that αE is stable, it only remains to show $g_3 = \text{coker } f_3$.

First, we show that φ_2 is an epimorphism. From the diagram, we observe that $g_3\varphi_2\varphi_1\varphi_0 = g_3g_0$, that is, $g_3(\varphi_2\varphi_1\varphi_0 - g_0) = 0$. So there is $\gamma: X \rightarrow A'$ such that $f_3\gamma = \varphi_2\varphi_1\varphi_0 - g_0$ since $f_3 = \ker g_3$. Then $f_3\gamma f_0 = \varphi_2\varphi_1\varphi_0 f_0 = f_3\alpha$. So $\gamma f_0 = \alpha$ since f_3 is a monomorphism. Now $0 = \varphi_1\varphi_0 f_0 - f_2\alpha = \varphi_1\varphi_0 f_0 - f_2\gamma f_0 = (\varphi_1\varphi_0 - f_2\gamma)f_0$. So there is $\nu: B_3 \rightarrow B_2$ such that $\nu g_0 = \varphi_1\varphi_0 - f_2\gamma$ since $g_0 = \text{coker } f_0$. Then $\varphi_2\nu g_0 = \varphi_2\varphi_1\varphi_0 - \varphi_2 f_2\gamma = \varphi_2\varphi_1\varphi_0 - f_3\gamma = g_0$. But $\varphi_2\nu g_0 = g_0$ implies $\varphi_2\nu = 1$ since g_0 is a cokernel and hence an epimorphism. Hence, φ_2 is an epimorphism since $\mu\varphi_2 = 0$ implies $\mu = \mu\varphi_2\nu = 0$.

Suppose $\mu f_3 = 0$. Then $\mu\varphi_2 f_2 = 0$. Since $g_2 = \text{coker } f_2$, there is η such that $\eta g_2 = \mu\varphi_2$. And $\eta g_3\varphi_2 = \eta g_2 = \mu\varphi_2$ implies $\eta g_3 = \mu$. And if $\eta' g_3 = \mu$, then $\eta' g_2 = \eta' g_3\varphi_2 = \mu\varphi_2 = \eta g_2$. And $\eta' = \eta$ since g_2 is an epimorphism. Hence $g_3 = \text{coker } f_3$.

Consider the dual diagram

$$\alpha(E\beta) \longrightarrow (\alpha E)\beta \longrightarrow \alpha E \longrightarrow f_3\alpha E .$$

Now the dual of the above argument gives φ_2 is a monomorphism and $f_2 = \ker g_2$. Consequently, $E\beta$ is stable.

Recall $\varphi_2\nu = 1$. Then $\varphi_2\nu\varphi_2 = \varphi_2$ implies $\nu\varphi_2 = 1$ since φ_2 is a monomorphism. Hence $(\alpha E)\beta = \alpha(E\beta)$ in the sense that φ_2 is an

isomorphism.

Ext in a pre-abelian category can now be pursued as in [3]. That is, the results and proofs of Richman and Walker [3, §3ff.] hold as stated. We conclude with the fact for an exact sequence, stability is equivalent to associativity.

THEOREM 3. *An exact sequence E is stable if and only if $(\alpha E)\beta = \alpha(E\beta)$ for all α and β .*

Proof. Only associativity implies stability needs to be proved. (The following argument is Fred Richman's.) Consider the diagrams

$$\begin{array}{ccc}
 E: A & \longrightarrow & B & \longrightarrow & C \\
 \alpha \downarrow & & \downarrow & & \parallel \\
 \alpha E: A' & \longrightarrow & K & \xrightarrow{\varphi} & C \\
 \parallel & & \uparrow & & \uparrow \\
 (\alpha E)0: A' & \longrightarrow & \ker \varphi & \longrightarrow & 0
 \end{array}
 \qquad
 \begin{array}{ccc}
 E: A & \longrightarrow & B & \longrightarrow & C \\
 \parallel & & \uparrow & & \uparrow \\
 E0: A & \xlongequal{\quad} & A & \longrightarrow & 0 \\
 \alpha \downarrow & & \downarrow & & \parallel \\
 \alpha(E0): A' & \xlongequal{\quad} & A' & \longrightarrow & 0.
 \end{array}$$

So $(\alpha E)0 = \alpha(E0)$ implies A' maps isomorphically onto $\ker \varphi$. So αE is exact. Similarly, $E\beta$ is exact.

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