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We devise an upper bound for $B(n)$, the number of nonisomorphic groups whose orders are square-free and no larger than n, and a lower bound for $T(n)$, the number of nonisomorphic groups whose orders are no larger than n. It is then noted that $B(n) = o(T(n))$.

An open problem is to find a formula for $N(n)$, the number of nonisomorphic groups of order n. Balash [1] discovered such a formula in the special case where n is square-free, and Higman [4] and Sims [6] developed an asymptotic formula for the number of groups of order a power of a prime. In this paper we use Balash's result to determine an upper bound for $B(n)$, where

$$B(n) = \sum_{\substack{k \leqq n \\ k \text{ square-free}}} N(k) ,$$

and the work of Higman and Sims to bound $T(n)$, given by

$$T(n) = \sum_{k \leqq n} N(k) ,$$

from below.

Higman's result, as refined by Sims, is

LEMMA 1. *Let $A = A(n, p)$ be defined by $\log_p (N(p^n)) = An^3$. Then $A = 2/27 + O(n^{-1/3})$.*

Higman originally offered 2/27 as the function in the lower bound for A with error term $O(n^{-1})$ and 2/15 in the upper bound. Sims reduced the upper bound to $2/27 + O(n^{-1/3})$. The lower bound is all we need, and the constant is not important as long as it is positive.

THEOREM 1. *There exists a positive constant c such that*

$$T(n) \gg n^{c \log^2 n} .$$

Proof. Let $2^m < n \leqq 2^{m+1}$. Then for $n > 1$,

$$T(n) \geqq T(2^m) \gg 2^{km^3} \geqq n^{c \log^2 n} .$$

Murty and Murty [5] show that $T(n) \gg n \log\log\log n$, which is enough to conclude, with a result of Erdös and Szekeres [2], that abelian groups are scarce. They then ask about nilpotent groups. Their lower bound grows more slowly than n^2, whereas the bound in Theorem 1 grows faster than any polynomial in n.

An upper bound for $T(n)$ of

$$n^{an^{2/3}\log n},$$

with a explicitly given, was provided by Gallagher [3]. Every group counted in the proof of Theorem 1 is a p-group and hence nilpotent. For $\mathcal{N}(n)$ the number of nonisomorphic nilpotent groups of order no greater than n, then,

$$n^{c\log^2 n} \ll \mathcal{N}(n) \leqq n^{an^{2/3}\log n},$$

and if the lower bound were the correct order of magnitude of $\mathcal{N}(n)$ then we could say that almost all groups are nilpotent.

If n is square-free, the number of groups of order n is determined by the unitary congruences among the prime divisors of n. Such a congruence exists if, for p and q prime factors of n, $p \equiv 1 (\mathrm{mod}\, q)$. If none exist, then $(n, \phi(n)) = 1$, where $\phi(n)$ is the totient function of Euler, and in that case it was shown by Szele [7] that there is exactly one group of order n. Roughly, the more congruences there are the more nonisomorphic groups of order n there can be. Balash's formula below gives the number of groups of a square-free order n in terms of the unitary congruences among n's prime factors.

LEMMA 2. *For k square-free and $m|k$, let $I(k/m, p)$ be the number of prime divisors q of k/m for which $q \equiv 1 (\mathrm{mod}\, p)$. Then*

$$N(k) = \sum_{m|k} \prod_{p|m} \frac{p^{I(k/m,p)} - 1}{p - 1}.$$

Thus, for instance, $N(6) =$ (summand for 1) + (summand for 2) + (summand for 3) + (summand for 6) $= 1 + 1 + 0 + 0 = 2$.

This is used in

THEOREM 2. $B(n) \ll n^2 \log\log n$.

Proof. For k square-free, Lemma 2 gives

$$N(k) \leqq \sum_{m|k} \prod_{p|m} p^{I(k/m,p)}.$$

But

$$\begin{aligned}\prod_{p|m} p^{I(k/m,p)} &= \prod_{q|k/m} \prod_{p|(q-1,m)} p \\ &= \prod_{q|k/m} (q-1, m) \leqq \prod_{q|k/m} q = k/m.\end{aligned}$$

So

$$N(k) \leqq \sum_{m \mid k} k/m = \sigma(k) \ll k \log \log k$$

and

$$B(n) = \sum_{\substack{k \leqq n \\ k \text{ square-free}}} N(k) \ll n^2 \log \log n \ .$$

Now we have

THEOREM 3. $B(n) = o(T(n))$.

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