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**BOUNDS FOR THE PERRON ROOT OF A NONNEGATIVE
IRREDUCIBLE PARTITIONED MATRIX**

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It is well-known that the Perron root of a nonnegative irreducible matrix lies between the smallest and the largest row sum of A . This result is generalized to the case when the matrix A is partitioned into blocks.

1. **Introduction and notations.** If $A = (a_{ij})$ is a nonnegative irreducible $n \times n$ matrix, then the Perron root $r(A)$ of A satisfies the classical inequalities of Frobenius [1, p. 37; 9; 10, p. 63; 21, p. 31]

$$(1) \quad \min_i S_i \leq r(A) \leq \max_i S_i,$$

where S_i denotes the i th row sum of A , i.e., $S_i = \sum_{j=1}^n a_{ij}$ ($i=1, \dots, n$). Moreover, we have strict inequalities in (1) unless all the S_i 's are equal.

Other bounds for $r(A)$ have been found by Ledermann [13], Ostrowski [15], Brauer [2], Ostrowski and Schneider [17], Hall and Porsching [11], Brauer and Gentry [3; 4], and Deutsch [8]. (In some of these papers one has assumed that A is a positive matrix.)

The purpose of this paper is to give some simple generalizations of the inequalities (1), by considering certain partitionings of A .

We introduce a few notations. By \mathbf{R}^m we denote the vector space of all column m -tuples of real numbers and $(x)_i$ denotes the i th (scalar) component of the vector $x \in \mathbf{R}^m$. By $\mathbf{R}^{m \times m}$ we denote the algebra of all $m \times m$ real matrices and $(A)_{ij}$ denotes the (scalar) (i, j) -entry of the matrix $A \in \mathbf{R}^{m \times m}$. For two vectors $x, y \in \mathbf{R}^m$, the inequality $x \leq y$ ($x < y$) means $(x)_i \leq (y)_i$ ($(x)_i < (y)_i$) for all $i = 1, \dots, m$. If $X_1, \dots, X_t \in \mathbf{R}^{m \times m}$, then $\bigwedge_{s=1}^t X_s$ ($\bigvee_{s=1}^t X_s$) denotes the greatest lower bound (least upper bound) of the matrices X_1, \dots, X_t in the natural (i.e., componentwise) partial ordering of $\mathbf{R}^{m \times m}$. In other words,

$$\left(\bigwedge_{s=1}^t X_s \right)_{ij} = \min_{s=1, \dots, t} (X_s)_{ij}, \quad \left(\bigvee_{s=1}^t X_s \right)_{ij} = \max_{s=1, \dots, t} (X_s)_{ij},$$

for all $i, j = 1, \dots, m$.

The transpose of a matrix A (vector u) will be denoted by A^\top (u^\top) and the Perron root of a nonnegative matrix $A \in \mathbf{R}^{m \times m}$ will be denoted by $r(A)$.

2. Let

$$(2) \quad A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \dots & \dots & \dots & \dots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix}$$

be a nonnegative irreducible $n \times n$ matrix, where A_{ij} is an $n_i \times n_j$ submatrix ($i, j = 1, \dots, k$). Clearly, $n_1 + \dots + n_k = n$.

Let p_{ij} denote the smallest row sum of A_{ij} , let q_{ij} denote the largest row sum of A_{ij} ($i, j = 1, \dots, k$) and consider the $k \times k$ matrices

$$(3) \quad P(A) = (p_{ij})_{i,j=1,\dots,k}, \quad Q(A) = (q_{ij})_{i,j=1,\dots,k}.$$

PROPOSITION 1. *We have*

$$(4) \quad r(P(A)) \leq r(A) \leq r(Q(A)).$$

Proof. Let $x \in \mathbf{R}^n$ be a Perron eigenvector of A , i.e.,

$$(5) \quad Ax = \rho x \quad (x > 0),$$

where $\rho = r(A)$. We partition x as

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \mathbf{R}^n,$$

where $x_j \in \mathbf{R}^{n_j}$ ($j = 1, \dots, k$). Now, equation (5) can be written

$$(6) \quad A_{i1}x_1 + \dots + A_{ik}x_k = \rho x_i \quad (i = 1, \dots, k).$$

We assume that $(x_i)_{M_i}$ is the smallest (scalar) component of x_i , i.e.,

$$(x_i)_{M_i} = \min\{(x_i)_1, (x_i)_2, \dots, (x_i)_{n_i}\}.$$

Equating the M_i th components of both sides of (6), we obtain

$$\rho(x_i)_{M_i} = (A_{i1}x_1)_{M_i} + \dots + (A_{ik}x_k)_{M_i},$$

or

$$\rho(x_i)_{M_i} = \sum_{s=1}^{n_1} (A_{i1})_{M_i,s}(x_1)_s + \dots + \sum_{s=1}^{n_k} (A_{ik})_{M_i,s}(x_k)_s,$$

whence, replacing $(x_j)_s$ by $(x_j)_{M_j}$ and then replacing the row sums of A_{ij} by p_{ij} , we have

$$(7) \quad \rho(x_i)_{M_i} \geq p_{i1}(x_1)_{M_1} + \dots + p_{ik}(x_k)_{M_k} \quad (i = 1, \dots, k).$$

Introducing the vector

$$v = ((x_1)_{M_1}, \dots, (x_k)_{M_k})^T \in \mathbf{R}^k ,$$

inequalities (7) can be written as

$$\rho v \geq P(A)v \quad (v > 0) ,$$

which implies [1, p. 28; 5; 22, p. 33] $r(P(A)) \leq \rho = r(A)$.

The right-hand inequality of (4) is proved in an entirely similar manner.

REMARK 1. Since q_{ij} is the row-sum norm [20, p.180] of A_{ij} , the right-hand inequality of (4) follows at once also from the theory of matricial norms [6; 7], (see also [16; 18; 19]).

PROPOSITION 2. *Either $P(A) = Q(A)$, or*

$$r(P(A)) < r(A) < r(Q(A)) .$$

Proof. Assume $P(A) \neq Q(A)$. We construct a nonnegative irreducible matrix $B \in \mathbf{R}^{n \times n}$ by decreasing certain entries of A so that $P(B) = Q(B) = P(A)$. Then $r(B) < r(A)$ [1, p. 27; 21, p. 30] and, by Proposition 1, $r(B) = r(P(B))$. Consequently, $r(P(A)) < r(A)$. Similarly, we construct a nonnegative irreducible matrix $C \in \mathbf{R}^{n \times n}$ by increasing certain entries of A so that $P(C) = Q(C) = Q(A)$. Then $r(A) < r(C)$ and, by Proposition 1, $r(C) = r(P(C))$. Consequently, $r(A) < r(Q(A))$.

COROLLARY 1. *The following statements are equivalent:*

- (a) $P(A) = Q(A)$;
- (b) $r(A) = r(P(A))$;
- (c) $r(A) = r(Q(A))$;
- (d) $r(P(A)) = r(Q(A))$.

REMARK 2. If a nonnegative irreducible matrix $A \in \mathbf{R}^{n \times n}$, partitioned as in (1), satisfies the equivalent conditions of Corollary 1, then it follows from condition (a) that, for each fixed pair $i, j \in \{1, \dots, k\}$, all the row sums of A_{ij} are equal to $p_{ij}(=q_{ij})$. Thus, A is a so-called *block-stochastic* matrix [12]. In this case, every eigenvalue of $P(A) \in \mathbf{R}^{k \times k}$ is an eigenvalue of $A \in \mathbf{R}^{n \times n}$ (see [12, Theorem 2]).

EXAMPLE 1. We consider the partitioned matrix

$$A = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ \hline 2 & 3 & 5 \end{array} \right) .$$

We have

$$P(A) = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}, \quad Q(A) = \begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix},$$

and $r(P(A)) = 7$, $r(Q(A)) = 8$. Thus, $7 < r(A) < 8$. This result is better than those obtained by several other methods [4; 14, p. 158].

EXAMPLE 2. We consider the partitioned matrix

$$A = \left(\begin{array}{ccc|cc} 3 & 1 & 5 & 1 & 4 \\ 2 & 2 & 5 & 2 & 3 \\ 1 & 5 & 3 & 1 & 4 \\ \hline 1 & 1 & 3 & 4 & 1 \\ 0 & 2 & 3 & 3 & 2 \end{array} \right).$$

We have

$$P(A) = Q(A) = \begin{pmatrix} 9 & 5 \\ 5 & 5 \end{pmatrix}$$

and thus, in this case Proposition 1 yields the exact value of the Perron root of A : $r(A) = r(P(A)) = r(Q(A)) = 7 + \sqrt{29} \approx 12.38$. The matrix A is block-stochastic (see Remark 2).

3. Let

$$(8) \quad A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}$$

be a nonnegative irreducible $n \times n$ matrix, where each A_{ij} is a square $k \times k$ matrix. Clearly, $n = kN$.

Denote

$$(9) \quad R_i(A) = \sum_{j=1}^N A_{ij} \in \mathbf{R}^{k \times k} \quad (i = 1, \dots, N).$$

PROPOSITION 3. We have

$$(10) \quad r\left(\bigwedge_{j=1}^N R_j(A)\right) \leq r(A) \leq r\left(\bigvee_{j=1}^N R_j(A)\right).$$

Proof. Let $y \in \mathbf{R}^n$ be a Perron eigenvector of $G = A^\top$, i.e.,

$$(11) \quad Gy = \rho y \quad (y > 0),$$

where $\rho = r(A)$. We partition y as

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \in \mathbf{R}^n,$$

where $y_j \in \mathbf{R}^k$ for all $j=1, \dots, N$. Denoting $G_{ij} = A_{ji}^\top$ ($i, j=1, \dots, N$), equation (11) can be written

$$(12) \quad \sum_{j=1}^N G_{ij} y_j = \rho y_i \quad (i = 1, \dots, N).$$

Summing the equations (12) with respect to i , we obtain

$$(13) \quad \sum_{i=1}^N \sum_{j=1}^N G_{ij} y_j = \rho w,$$

where $w = \sum_{i=1}^N y_i \in \mathbf{R}^k$. Interchanging the order of summation in the left-hand side of (13), we have

$$\rho w = \sum_{j=1}^N (R_j(A))^\top y_j,$$

from where one has

$$\left[\bigwedge_{j=1}^N (R_j(A))^\top \right] w \leq \rho w \leq \left[\bigvee_{j=1}^N (R_j(A))^\top \right] w.$$

This, in turn, implies the inequalities (10) [1, p. 28; 5; 22, p. 33].

PROPOSITION 4. *Either $R_1(A) = \dots = R_N(A)$, or*

$$r\left(\bigwedge_{j=1}^N R_j(A)\right) < r(A) < r\left(\bigvee_{j=1}^N R_j(A)\right).$$

Proof. Assume that $R_1(A), \dots, R_N(A)$ are not equal. We construct a nonnegative irreducible $n \times n$ matrix B by decreasing certain entries of A so that $R_1(B) = \dots = R_N(B) = \bigwedge_{j=1}^N R_j(A)$. Then $r(B) < r(A)$ [1, p. 27; 21, p. 30] and, by Proposition 3, $r(B) = r(\bigwedge_{j=1}^N R_j(B))$. Consequently, $r(\bigwedge_{j=1}^N R_j(A)) < r(A)$. Similarly, we construct a nonnegative irreducible $n \times n$ matrix C by increasing certain entries of A so that $R_1(C) = \dots = R_N(C) = \bigvee_{j=1}^N R_j(A)$. Then $r(A) < r(C)$ and, by Proposition 3, $r(C) = r(\bigvee_{j=1}^N R_j(C))$. Consequently, $r(A) < r(\bigvee_{j=1}^N R_j(A))$.

COROLLARY 2. *The following statements are equivalent:*

- (a) $R_1(A) = \dots = R_N(A)$;
- (b) $r(A) = r(\bigwedge_{j=1}^N R_j(A))$;
- (c) $r(A) = r(\bigvee_{j=1}^N R_j(A))$;

$$(d) \quad r(\bigwedge_{j=1}^N R_j(A)) = r(\bigvee_{j=1}^N R_j(A)).$$

EXAMPLE 3. We consider the partitioned matrix

$$A = \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ \hline 1 & 4 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array} \right).$$

We have

$$R_1(A) = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}, \quad R_2(A) = \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix},$$

and

$$\bigwedge_{j=1}^2 R_j(A) = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, \quad \bigvee_{j=1}^2 R_j(A) = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix},$$

the last two matrices having Perron roots $2 + \sqrt{5}$ and 5, respectively. Thus, $4.236 < r(A) < 5$. The classical inequalities (1) yield only $3 < r(A) < 8$.

REMARK 3. The results of § 3 can be obtained from those of § 2. Indeed, if A is the $n \times n$ matrix given in (8) and if we arrange the rows and columns of A in the following positions;

$$\begin{aligned} &1, N + 1, 2N + 1, \dots, (k - 1)N + 1, \\ &2, N + 2, 2N + 2, \dots, (k - 1)N + 2, \\ &\dots\dots\dots \\ &N, 2N, 3N, \dots, kN, \end{aligned}$$

then we obtain a matrix $A' = (A'_{ij})_{i,j=1,\dots,k} \in \mathbf{R}^{n \times n}$, where each A'_{ij} is an $N \times N$ submatrix. It can be easily seen that

$$P(A') = \bigwedge_{i=1}^N R_i(A), \quad Q(A') = \bigvee_{i=1}^N R_i(A).$$

Since $r(A) = r(A')$, Propositions 3, 4 and Corollary 2 follow at once from Propositions 1, 2 and Corollary 1, respectively.

REMARK 4. It should be noted that the bounds given by Proposition 3 (or Proposition 1) are not always better than those given by the classical bounds (1). For example, considering the partitioned matrix

$$A = \left(\begin{array}{cc|cc} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right),$$

we have

$$R_1(A) = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}, \quad R_2(A) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

$$\bigwedge_{j=1}^2 R_j(A) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \bigvee_{j=1}^2 R_j(A) = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix},$$

the last two matrices having Perron roots 2 and $\frac{1}{2}(7 + \sqrt{17})$, respectively. Thus, $2 < r(A) < 5.562$. However, all row-sums of A are equal to 4 and thus $r(A) = 4$.

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