Pacific Journal of Mathematics

BOUNDS FOR THE PERRON ROOT OF A NONNEGATIVE IRREDUCIBLE PARTITIONED MATRIX

EMERIC DEUTSCH

Vol. 92, No. 1 January 1981

PACIFIC JOURNAL OF MATHEMATICS
Vol. 92, No. 1, 1981

BOUNDS FOR THE PERRON ROOT OF A NONNEGATIVE IRREDUCIBLE PARTITIONED MATRIX

EMERIC DEUTSCH

It is well-known that the Perron root of a nonnegative irreducible matrix lies between the smallest and the largest row sum of A. This result is generalized to the case when the matrix A is partitioned into blocks.

1. Introduction and notations. If $A = (a_{ij})$ is a nonnegative irreducible $n \times n$ matrix, then the Perron root $r(A)$ of A satisfies the classical inequalities of Frobenius [1, p. 37; 9; 10, p. 63; 21, p. 31]

$$\min_i S_i \leqq r(A) \leqq \max_i S_i \, , \tag{1}$$

where S_i denotes the ith row sum of A, i.e., $S_i = \sum_{j=1}^n a_{ij}$ $(i = 1, \cdots, n)$. Moreover, we have strict inequalities in (1) unless all the S_i's are equal.

Other bounds for $r(A)$ have been found by Ledermann [13], Ostrowski [15], Brauer [2], Ostrowski and Schneider [17], Hall and Porsching [11], Brauer and Gentry [3; 4], and Deutsch [8]. (In some of these papers one has assumed that A is a positive matrix.)

The purpose of this paper is to give some simple generalizations of the inequalities (1), by considering certain partitionings of A.

We introduce a few notations. By $\boldsymbol{R}^m$ we denote the vector space of all column m-tuples of real numbers and $(x)_i$ denotes the ith (scalar) component of the vector $x \in \boldsymbol{R}^m$. By $\boldsymbol{R}^{m \times m}$ we denote the algebra of all $m \times m$ real matrices and $(A)_{ij}$ denotes the (scalar) (i, j)-entry of the matrix $A \in \boldsymbol{R}^{m \times m}$. For two vectors x, $y \in \boldsymbol{R}^m$, the inequality $x \leqq y$ $(x < y)$ means $(x_i) \leqq (y)_i$ $((x)_i < (y)_i)$ for all $i = 1, \cdots, m$. If $X_1, \cdots, X_t \in \boldsymbol{R}^{m \times m}$, then $\bigwedge_{s=1}^t X_s$ $(\bigvee_{s=1}^t X_s)$ denotes the greatest lower bound (least upper bound) of the matrices $X_1, \cdots, X_t$ in the natural (i.e., componentwise) partial ordering of $\boldsymbol{R}^{m \times m}$. In other words,

$$\left(\bigwedge_{s=1}^t X_s\right)_{ij} = \min_{s=1,\cdots,t} (X_s)_{ij} \, , \quad \left(\bigvee_{s=1}^t X_s\right)_{ij} = \max_{s=1,\cdots,t} (X_s)_{ij} \, ,$$

for all i, $j = 1, \cdots, m$.

The transpose of a matrix A (vector u) will be denoted by A^{T} (u^{T}) and the Perron root of a nonnegative matrix $A \in \boldsymbol{R}^{m \times m}$ will be denoted by $r(A)$.

2. Let

$$(2) \qquad A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix}$$

be a nonnegative irreducible $n \times n$ matrix, where A_{ij} is an $n_i \times n_j$ submatrix $(i, j = 1, \cdots, k)$. Clearly, $n_1 + \cdots + n_k = n$.

Let p_{ij} denote the smallest row sum of A_{ij}, let q_{ij} denote the largest row sum of A_{ij} $(i, j = 1, \cdots, k)$ and consider the $k \times k$ matrices

$$(3) \qquad P(A) = (p_{ij})_{i,j=1,\cdots,k} , \quad Q(A) = (q_{ij})_{i,j=1,\cdots,k} .$$

PROPOSITION 1. *We have*

$$(4) \qquad r(P(A)) \leqq r(A) \leqq r(Q(A)) .$$

Proof. Let $x \in \boldsymbol{R}^n$ be a Perron eigenvector of A, i.e.,

$$(5) \qquad Ax = \rho x \quad (x > 0) ,$$

where $\rho = r(A)$. We partition x as

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \boldsymbol{R}^n ,$$

where $x_j \in \boldsymbol{R}^{n_j}$ $(j = 1, \cdots, k)$. Now, equation (5) can be written

$$(6) \qquad A_{i1}x_1 + \cdots + A_{ik}x_k = \rho x_i \quad (i = 1, \cdots, k) .$$

We assume that $(x_i)_{M_i}$ is the smallest (scalar) component of x_i, i.e.,

$$(x_i)_{M_i} = \min\{(x_i)_1, (x_i)_2, \cdots, (x_i)_{n_i}\} .$$

Equating the M_ith components of both sides of (6), we obtain

$$\rho(x_i)_{M_i} = (A_{i1}x_1)_{M_i} + \cdots + (A_{ik}x_k)_{M_i} ,$$

or

$$\rho(x_i)_{M_i} = \sum_{s=1}^{n_1} (A_{i1})_{M_i,s}(x_1)_s + \cdots + \sum_{s=1}^{n_k} (A_{ik})_{M_i,s}(x_k)_s ,$$

whence, replacing $(x_j)_s$ by $(x_j)_{M_j}$ and then replacing the row sums of A_{ij} by p_{ij}, we have

$$(7) \qquad \rho(x_i)_{M_i} \geqq p_{i1}(x_1)_{M_1} + \cdots + p_{ik}(x_k)_{M_k} \quad (i = 1, \cdots, k) .$$

Introducing the vector

$$v = ((x_1)_{M_1}, \cdots, (x_k)_{M_k})^{\mathrm{T}} \in \boldsymbol{R}^k ,$$

inequalities (7) can be written as

$$\rho v \geqq P(A)v \quad (v > 0) ,$$

which implies [1, p. 28; 5; 22, p. 33] $r(P(A)) \leqq \rho = r(A)$.

The right-hand inequality of (4) is proved in an entirely similar manner.

REMARK 1. Since q_{ij} is the row-sum norm [20, p.180] of A_{ij}, the right-hand inequality of (4) follows at once also from the theory of matricial norms [6; 7], (see also [16; 18; 19]).

PROPOSITION 2. *Either* $P(A) = Q(A)$, *or*

$$r(P(A) < r(A) < r(Q(A)) .$$

Proof. Assume $P(A) \neq Q(A)$. We construct a nonnegative irreducible matrix $B \in \boldsymbol{R}^{n\times n}$ by decreasing certain entries of A so that $P(B) = Q(B) = P(A)$. Then $r(B) < r(A)$ [1, p. 27; 21, p. 30] and, by Proposition 1, $r(B) = r(P(B))$. Consequently, $r(P(A)) < r(A)$. Similarly, we construct a nonnegative irreducible matrix $C \in \boldsymbol{R}^{n\times n}$ by increasing certain entries of A so that $P(C) = Q(C) = Q(A)$. Then $r(A) < r(C)$ and, by Proposition 1, $r(C) = r(P(C))$. Consequently, $r(A) < r(Q(A))$.

COROLLARY 1. *The following statements are equivalent*:
(a) $P(A) = Q(A)$;
(b) $r(A) = r(P(A))$;
(c) $r(A) = r(Q(A))$;
(d) $r(P(A)) = r(Q(A))$.

REMARK 2. If a nonnegative irreducible matrix $A \in \boldsymbol{R}^{n\times n}$, partitioned as in (1), satisfies the equivalent conditions of Corollary 1, then it follows from condition (a) that, for each fixed pair $i, j \in \{1, \cdots, k\}$, all the row sums of A_{ij} are equal to $p_{ij}(=q_{ij})$. Thus, A is a so-called *block-stochastic* matrix [12]. In this case, every eigenvalue of $P(A) \in \boldsymbol{R}^{k\times k}$ is an eigenvalue of $A \in \boldsymbol{R}^{n\times n}$ (see [12, Theorem 2]).

EXAMPLE 1. We consider the partitioned matrix

$$A = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ \hline 2 & 3 & 5 \end{array}\right) .$$

We have

$$P(A) = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}, \quad Q(A) = \begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix},$$

and $r(P(A)) = 7$, $r(Q(A)) = 8$. Thus, $7 < r(A) < 8$. This result is better than those obtained by several other methods [4; 14, p. 158].

EXAMPLE 2. We consider the partitioned matrix

$$A = \left(\begin{array}{ccc|cc} 3 & 1 & 5 & 1 & 4 \\ 2 & 2 & 5 & 2 & 3 \\ 1 & 5 & 3 & 1 & 4 \\ \hline 1 & 1 & 3 & 4 & 1 \\ 0 & 2 & 3 & 3 & 2 \end{array}\right).$$

We have

$$P(A) = Q(A) = \begin{pmatrix} 9 & 5 \\ 5 & 5 \end{pmatrix}$$

and thus, in this case Proposition 1 yields the exact value of the Perron root of A: $r(A) = r(P(A)) = r(Q(A)) = 7 + \sqrt{29} \approx 12.38$. The matrix A is block-stochastic (see Remark 2).

3. Let

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix} \tag{8}$$

be a nonnegative irreducible $n \times n$ matrix, where each A_{ij} is a square $k \times k$ matrix. Clearly, $n = kN$.

Denote

$$R_i(A) = \sum_{j=1}^{N} A_{ij} \in \boldsymbol{R}^{k \times k} \quad (i = 1, \cdots, N). \tag{9}$$

PROPOSITION 3. *We have*

$$r\left(\bigwedge_{j=1}^{N} R_j(A)\right) \leqq r(A) \leqq r\left(\bigvee_{j=1}^{N} R_j(A)\right). \tag{10}$$

Proof. Let $y \in \boldsymbol{R}^n$ be a Perron eigenvector of $G = A^{\top}$, i.e.,

$$Gy = \rho y \quad (y > 0), \tag{11}$$

where $\rho = r(A)$. We partition y as

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \in \boldsymbol{R}^n ,$$

where $y_j \in \boldsymbol{R}^k$ for all $j=1, \cdots, N$. Denoting $G_{ij}=A_{ji}^{\top}$ $(i, j=1, \cdots, N)$, equation (11) can be written

$$\sum_{j=1}^{N} G_{ij} y_j = \rho y_i \quad (i = 1, \cdots, N) . \tag{12}$$

Summing the equations (12) with respect to i, we obtain

$$\sum_{i=1}^{N} \sum_{j=1}^{N} G_{ij} y_j = \rho w , \tag{13}$$

where $w = \sum_{i=1}^{N} y_i \in \boldsymbol{R}^k$. Interchanging the order of summation in the left-hand side of (13), we have

$$\rho w = \sum_{j=1}^{N} (R_j(A))^{\top} y_j ,$$

from where one has

$$\left[\bigwedge_{j=1}^{N} (R_j(A))^{\top}\right] w \leqq \rho w \leqq \left[\bigvee_{j=1}^{N} (R_j(A))^{\top}\right] w .$$

This, in turn, implies the inequalities (10) [1, p. 28; 5; 22, p. 33].

PROPOSITION 4. *Either* $R_1(A) = \cdots = R_N(A)$, *or*

$$r\left(\bigwedge_{j=1}^{N} R_j(A)\right) < r(A) < r\left(\bigvee_{j=1}^{N} R_j(A)\right) .$$

Proof. Assume that $R_1(A), \cdots, R_N(A)$ are not equal. We construct a nonnegative irreducible $n \times n$ matrix B by decreasing certain entries of A so that $R_1(B)= \cdots =R_N(B) = \bigwedge_{j=1}^{N} R_j(A)$. Then $r(B) < r(A)$ [1, p. 27; 21, p. 30] and, by Proposition 3, $r(B) = r(\bigwedge_{j=1}^{N} R_j(B))$. Consequently, $r(\bigwedge_{j=1}^{N} R_j(A)) < r(A)$. Similarly, we construct a nonnegative irreducible $n \times n$ matrix C by increasing certain entries of A so that $R_1(C)= \cdots =R_N(C) = \bigvee_{j=1}^{N} R_j(A)$. Then $r(A) < r(C)$ and, by Proposition 3, $r(C) = r(\bigvee_{j=1}^{N} R_j(C))$. Consequently, $r(A) < r(\bigvee_{j=1}^{N} R_j(A))$.

COROLLARY 2. *The following statements are equivalent*:
(a) $R_1(A)= \cdots =R_N(A)$;
(b) $r(A) = r(\bigwedge_{j=1}^{N} R_j(A))$;
(c) $r(A) = r(\bigvee_{j=1}^{N} R_j(A))$;

(d) $r(\bigwedge_{j=1}^{N} R_j(A)) = r(\bigvee_{j=1}^{N} R_j(A))$.

EXAMPLE 3. We consider the partitioned matrix

$$A = \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ \hline 1 & 4 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right).$$

We have

$$R_1(A) = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}, \quad R_2(A) = \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix},$$

and

$$\bigwedge_{j=1}^{2} R_j(A) = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, \quad \bigvee_{j=1}^{2} R_j(A) = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix},$$

the last two matrices having Perron roots $2 + \sqrt{5}$ and 5, respectively. Thus, $4.236 < r(A) < 5$. The classical inequalities (1) yield only $3 < r(A) < 8$.

REMARK 3. The results of § 3 can be obtained from those of § 2. Indeed, if A is the $n \times n$ matrix given in (8) and if we arrange the rows and columns of A in the following positions;

$$\begin{array}{l} 1,\ N+1,\ 2N+1,\ \cdots,\ (k-1)N+1\ , \\ 2,\ N+2,\ 2N+2,\ \cdots,\ (k-1)N+2\ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ N,\ \ 2N,\ \ 3N,\ \cdots,\ kN\ , \end{array}$$

then we obtain a matrix $A' = (A'_{ij})_{i,j=1,\cdots,k} \in \boldsymbol{R}^{n \times n}$, where each A'_{ij} is an $N \times N$ submatrix. It can be easily seen that

$$P(A') = \bigwedge_{i=1}^{N} R_i(A)\ , \quad Q(A') = \bigvee_{i=1}^{N} R_i(A)\ .$$

Since $r(A) = r(A')$, Propositions 3, 4 and Corollary 2 follow at once from Propositions 1, 2 and Corollary 1, respectively.

REMARK 4. It should be noted that the bounds given by Proposition 3 (or Proposition 1) are not always better than those given by the classical bounds (1). For example, considering the partitioned matrix

$$A = \left(\begin{array}{cc|cc} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right),$$

we have

$$R_1(A) = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}, \quad R_2(A) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

$$\bigwedge_{j=1}^{2} R_j(A) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \bigvee_{j=1}^{2} R_j(A) = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix},$$

the last two matrices having Perron roots 2 and $\frac{1}{2}(7 + \sqrt{17})$, respectively. Thus, $2 < r(A) < 5.562$. However, all row-sums of A are equal to 4 and thus $r(A) = 4$.

References

1. A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.

2. A. Brauer, *The theorems of Ledermann and Ostrowski on positive matrices*, Duke Math. J., **24** (1957), 265–274.

3. A. Brauer and I. C. Gentry, *Bounds for the greatest characteristic root of an irreducible nonnegative matrix*, Linear Algebra and Appl., **8** (1974), 105–107.

4. ———, *Bounds for the greatest characteristic root of an irreducible nonnegative matrix II*, Linear Algebra and Appl., **13** (1976), 109–114.

5. L. Collatz, *Einschliessungssatz für die charakteristischen Zahlen von Matrizen*, Math. Z., **48** (1942), 221–226.

6. E. Deutsch, *Matricial norms*, Numer. Math., **16** (1970), 73–84.

7. ———, *On matricial norms subordinate to vectorial norms*, Math. Z., **122** (1971), 142–150.

8. ———, *Nested bounds for the Perron root of a nonnegative irreducible matrix*, manuscript (see: Notices of the Amer. Math. Soc., **26** (1979), A 421-A 422).

9. G. F. Frobenius, *Über Matrizen aus positiven Elementen*, Sitzungsber. Königl. Preuss. Akad. Wiss. Berlin, (1908), 471–476.

10. F. R. Gantmacher, *The Theory of Matrices*, Vol. 2, Chelsea, New York, 1959.

11. C. A. Hall and T. A. Porsching, *Bounds for the maximal eigenvalue of a nonnegative irreducible matrix*, Duke Math. J., **36** (1969), 159-164.

12. E. V. Haynsworth, *Applications of a theorem on partitioned matrices*, J. Res. Nat. Bur. Standards, Sect. B, **62B** (1959), 73-78.

13. W. Ledermann, *Bounds for the greatest latent roots of a positive matrix*, J. London Math. Soc., **25** (1950), 265-268.

14. M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, 1964.

15. A. Ostrowski, *Bounds for the greatest latent root of a positive matrix*, J. London Math. Soc., **27** (1952), 253-256.

16. ———, *On some metrical properties of operator matrices and matrices partitioned into blocks*, J. Math. Anal. Appl., **2** (1961), 161–209.

17. A. Ostrowski and H. Schneider, *Bounds for the maximal characteristic root of a nonnegative irreducible matrix*, Duke Math. J., **27** (1960), 547–553.

18. F. Robert, *Normes vectorielles de vecteurs et de matrices,* Rev. Française Traitement Information (Chiffres), **7** (1964), 261–299.
19. ———, *Sur les normes vectorielles régulières sur un espace vectoriel de dimension finie,* C.R. Acad. Sci. Paris, **261** (1965), 5173–5176.
20. G. W. Stewart, *Introduction to Matrix Computations,* Academic Press, New York, 1973.
21. R. S. Varga, *Matrix Iterative Analysis,* Prentice-Hall, Englewood Cliffs, 1962.
22. H. Wielandt, *Topics in the Analytic Theory of Matrices,* Lecture notes prepared by Robert R. Meyer, Department of Mathematics, University of Wisconsin, Madison, 1967.

Received September 24, 1979.

POLYTECHNIC INSTITUTE OF NEW YORK
BROOKLYN, NY 11201

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 92, No. 1 January, 1981