REPRESENTATION OF COMPACT AND WEAKLY COMPACT OPERATORS ON THE SPACE OF BOCHNER INTEGRABLE FUNCTIONS

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If \( X^* \) has the Radon-Nikodym property, then for every compact operator \( T: L_1(\mu, X) \to Y \) there is a bounded function \( g: \Omega \to L(X, Y) \) that is measurable for the uniform operator topology on \( L(X, Y) \) such that

\[
T(f) = \int_{\Omega} fg \, d\mu
\]

for all \( f \) in \( L_1(\mu, X) \). The same result holds for weakly compact operators if \( X^* \) is separable Schur space. These representations yield Radon-Nikodym theorems for operator valued measures and a generalization of a theorem of D. R. Lewis.

The representation of linear operators on the Banach space \( L_1(\mu, X) \) of Bochner integrable functions, has been the object of much study for the past forty years. Dunford and Pettis began this investigation in 1940 [6] with the representation of weakly compact and norm compact operators on \( L_1(\mu) \) by a Bochner integral. Their work was based on an earlier paper of Pettis [9] and was complemented by the work of Phillips [11]. More recently, the theory of liftings has been used by Dinculeanu [5] and others to obtain a representation for the general linear operator on \( L_1(\mu, X) \). In this paper we will use methods in the spirit of Dunford, Pettis, and Phillips to show that if \( X^* \) has the Radon-Nikodym property, then the compact operators on \( L_1(\mu, X) \) are representable by measurable kernels and if \( X^* \) is a separable Schur space (i.e., weakly convergent sequences converge in norm) then the weakly compact operators on \( L_1(\mu, X) \) are representable by measurable kernels. As corollaries, we obtain a Radon-Nikodym theorem for operator-valued measures and a generalization of a theorem of D. R. Lewis [4, p. 88] on weakly measurable functions that are equivalent to norm measurable functions.

Throughout this paper \((\Omega, \Sigma, \mu)\) is a finite measure space and \( X, Y \) and \( Z \) are Banach spaces with duals \( X^*, Y^*, \) and \( Z^* \) respectively. The space of all bounded linear operators from \( X \) to \( Y \) will be denoted by \( L(X, Y) \). The subspaces of \( L(X, Y) \) consisting of all the weakly compact and norm compact operators from \( X \) to \( Y \) will be denoted by \( W(X, Y) \) and \( K(X, Y) \). The space \( L_1(\mu, X) \) is the space of \( \mu \)-Bochner integrable functions on \( \Omega \) with values in \( X \) and
\(L_\infty(\mu, X)\) is the space of \(X\)-valued \(\mu\)-Bochner integrable functions on \(\Omega\) that are essentially bounded. An operator \(T: L_1(\mu, X) \to Y\) is representable by a measurable kernel if there is a bounded measurable \(g: \Omega \to L(X, Y)\) such that

\[
T(f) = \text{Bochner} - \int_\Omega fgd\mu.
\]

From this, it follows that \(\|T\| = \|g\|_\infty\) [5, p. 283]. Recall that a Banach space is weakly compactly generated if it is the closed linear span of one of its weakly compact sets. Finally, note that if \(\pi\) is a partition of \(\Omega\) into a countable number of disjoint elements of \(\Sigma\) and if \(f\) is in \(L_1(\mu, X)\), then the function \(E_\pi: L_1(\mu, X) \to L_1(\mu, X)\) defined by

\[
E_\pi(f) = \sum_{E \in \pi} \int_E f d\mu \chi_E
\]

(here the convention \(0/0 = 0\) is observed) is a linear operator.

Most of the first lemma is well-known so we omit the proof.

**Lemma 1.** For each countable partition \(\pi\), the operator \(E_\pi\) is a contraction on \(L_1(\mu, X)\) and \(L_\infty(\mu, X)\). Moreover, if the partitions are directed by refinement, then

\[
\lim_{\pi} \|E_\pi(f) - f\|_1 = 0 \quad \text{for all } f \text{ in } L_1(\mu, X)
\]

\[
\lim_{\pi} \|E_\pi(f) - f\|_\infty = 0 \quad \text{for all } f \text{ in } L_\infty(\mu, X).
\]

Before stating the main theorem we require a preliminary definition. A function \(g\) in \(L_\infty(\mu, L(X, Y))\) is said to have its essential range in the uniformly (weakly) compact operators if there is a (weakly) compact set \(C\) in \(Y\) such that \(g(\omega)x \in C\) for almost all \(\omega\) in \(\Omega\) and \(x\) in \(X\) with \(\|x\| \leq 1\).

**Theorem 2.** Let \(X^*\) have the Radon-Nikodym property. Then there is an isometric isomorphism between the space of compact operators \(K(L_1(\mu, X), Y)\) and the subspace of \(L_\infty(\mu, K(X, Y))\) consisting of those functions whose essential range is in the uniformly compact operators. In fact, \(T\) in \(K(L_1(\mu, X), Y)\) and \(g\) in \(L_\infty(\mu, K(X, Y))\) are in correspondence if and only if

\[
T(f) = \int_\Omega fgd\mu \quad \text{for all } f \text{ in } L_1(\mu, X).
\]

**Proof.** Let \(T\) be in \(K(L_1(\mu, X), Y)\). Notice that for any par-
tion $\pi$, $f$ in $L_i(\mu, X)$, and $g$ in $L_\infty(\mu, X^*) = (L_i(\mu, X))^*$, we have that

$$\int_\Omega E_\pi(f)g d\mu = \int_\Omega fE_\pi(g) d\mu.$$ 

It follows from this that the adjoint of $TE_\pi$ is $E_\pi T^*$. Now, if the partitions $\pi$ are countable, we have that

$$\lim_{\pi} E_\pi f = f \quad \text{for all } f \in L_\infty(\mu, X^*)$$

by Lemma 1. Since $\|E_\pi\|_\infty \leq 1$, this limit is uniform on compact sets. By Schauder's theorem, $T^*: Y^* \to L_\infty(\mu, X^*)$ is compact and so

$$\lim_{\pi} E_\pi T^* y^* = T y^*$$

uniformly for $\|y^*\| \leq 1$. Therefore,

$$\lim_{\pi} E_\pi T^* = T^*$$

in the operator norm. Since $E_\pi T^* = (TE_\pi)^*$, it follows that

$$\lim_{\pi} TE_\pi = T$$

in operator norm.

Now, for each countable partition $\pi$, define $g_\pi: \Omega \to L(X, Y)$ by

$$g_\pi(\cdot)x = \sum_{A \in \pi} \frac{T(x\chi_A)}{\mu A}\chi_A(\cdot).$$

Then for each partition $\pi$, $\omega$ in $\Omega$, and $x$ in $X$ with $\|x\| \leq 1$, we have that $g_\pi(\omega)x \subseteq T\{f: f \in L_i(\mu, X), \|f\| \leq 1\}$. Since $T$ is compact, it follows that $g_\pi(\omega)$ is in $K(X, Y)$ for each partition $\pi$ and $\omega$ in $\Omega$. Moreover, one easily sees that

$$TE_\pi(f) = \int_\Omega f g_\pi d\mu$$

for all simple functions $f$ in $L_i(\mu, X)$ and thus for all functions $f$ in $L_i(\mu, X)$. Hence if $\pi_1$ and $\pi_2$ are two partitions, then

$$(TE_{\pi_1} - TE_{\pi_2})(f) = \int_\Omega f(g_{\pi_1} - g_{\pi_2}) d\mu.$$ 

Since

$$\lim_{\pi_1, \pi_2} \|TE_{\pi_1} - TE_{\pi_2}\| = 0,$$

an appeal to [5, p. 283] establishes that
Thus the net \((g_x)\) is Cauchy in the norm of \(L_\infty(\mu, K(X, Y))\). It follows that there is a \(g\) in \(L_\infty(\mu, K(X, Y))\) such that
\[
\lim \| g_x - g \|_\infty = 0
\]
and so
\[
\lim \int \limits_\Omega f g_x d\mu = \int \limits_\Omega f g d\mu
\]
for all \(f\) in \(L_1(\mu, X)\). We also have, for almost all \(\omega\), that
\[
g(\omega)x \subseteq T\{f: f \in L_1(\mu, X), \|f\| \leq 1\}
\]
for all \(x\) in \(X\) with \(\|x\| \leq 1\). Hence the essential range of \(g\) consists of uniformly compact operators. Finally, Lemma 1 ensures that
\[
T(f) = \lim \limits_\pi TE_x(f) = \lim \limits_\pi \int \limits_\Omega f g_x d\mu = \int \limits_\Omega f g d\mu.
\]
Conversely, suppose that \(g: \Omega \to K(X, Y)\) is a bounded measurable function such that there is a compact set \(C \subset Y\) with \(g(\omega)x\) in \(C\) for almost all \(\omega\) in \(\Omega\) and all \(x\) in \(X\) with \(\|x\| \leq 1\). Without loss of generality, we may assume \(g(\omega)x\) is in \(C\) for all \(\omega\) in \(\Omega\). Define
\[
T(f) = \int \limits_\Omega f g d\mu
\]
for \(f \in L_1(\mu, X)\). Another appeal to [5, p. 283] shows \(\|T\| = \|g\|_\infty\). Let
\[
f = \sum \limits_{i=1}^{n} x_i \mathcal{X}_{E_i}
\]
be a simple function in \(L_1(\mu, X)\) with \(\|f\| \leq 1\) i.e.,
\[
\sum \limits_{i=1}^{n} \|x_i\| \mu E_i \leq 1.
\]
Then
\[
T(f) = \int \limits_\Omega g f d\mu = \sum \limits_{i=1}^{n} \int \limits_{E_i} g(\omega)x_i d\mu(\omega)
\]
\[
= \sum \limits_{i=1}^{n} \|x_i\| \mu E_i \cdot \frac{1}{\mu E_i} \int \limits_{E_i} g(\omega) \frac{x_i}{\|x_i\|} d\mu
\]
is in \(\overline{\co} C\) by [4, p. 48]. Since \(\overline{\co} C\) is compact by Mazur's theorem, the operator \(T\) is compact. This completes the proof.
That $X^*$ has the Radon-Nikodym property is necessary as well as sufficient for the first part of the above proof. Indeed, if each $T$ in $K(L_1(\mu, X), Y)$ is representable by a Bochner integrable $g$ in $L_\infty(\mu, K(X, Y))$, then taking $Y$ to be the scalars shows that $L_1(\mu, X)^* = L_\infty(\mu, X^*)$ which implies [4, p. 98] that $X^*$ has the RNP. An immediate consequence of Theorem 2 is a Radon-Nikodym theorem for certain operator valued measures.

**Corollary 3.** Let $X^*$ have the RNP and let $G: \Sigma \to K(X, Y)$ be a $\mu$-continuous vector measure of bounded variation. If, for each $E_1$ in $\Sigma$ with $\mu E_1 > 0$, there exists $E_2$ in $\Sigma$ with $E_2 \subseteq E_1$ and $\mu(E_2) > 0$ such that

$$\left\{ \left( \frac{G(E)x}{\mu(E)} : x \in X, E \in \Sigma, E \subseteq E_2, \mu(E) > 0, \|x\| \leq 1 \right\} \right.$$ is relatively norm compact, then there exists a Bochner integrable $g: \Omega \to K(X, Y)$ such that

$$G(E) = \int_E gd\mu$$

for each $E$ in $\Sigma$.

**Proof.** By exhaustion [4, p. 70], the corollary is established if for each $E_1$ in $\Sigma$ with $\mu(E_1) > 0$ we can find $E_2$ in $\Sigma$ with $E_2 \subseteq E_1$ and $\mu E_2 > 0$ and a Bochner integrable $g$ such that

$$G(E) = \int_E gd\mu$$

for all $E$ in $\Sigma$ with $E \subseteq E_2$. So let $E_1 \in \Sigma$ with $\mu(E_1) > 0$ and select the $E_2 \subseteq E_1$ guaranteed by the hypothesis. Define an operator $T$ on the simple functions in $L_1(\mu, X)$ by

$$T(f) = \sum_{i=1}^n G(A_i \cap E_2)x_i \quad \text{if} \quad f = \sum_{i=1}^n x_i\chi_{A_i}, A_i \quad \text{in} \quad \Sigma, A_i \cap A_j = \emptyset$$

if $i \neq j$. Notice that if $\|f\| \leq 1$

$$\sum_{i=1}^n \|x_i\| \mu A_i \leq 1,$$

then

$$\sum_{i=1}^n \|x_i\| \mu(A_i \cap E_2) \leq 1$$

and so
\[ T(f) = \sum_{i=1}^{n} \frac{\|x_i\|}{\mu(A_i \cap E_z)} \cdot \frac{G(A_i \cap E_z) \frac{x_i}{\|x_i\|}}{\mu(A_i \cap E_z)} \]

is in
\[ \overline{\text{co}} \left\{ \frac{G(E)x}{\mu(E)} : x \in X, E \in \Sigma, E \subseteq E_z, \mu(E) > 0, \|x\| \leq 1 \right\}, \]
a set which is compact by Mazur's theorem. Thus \( T \) has a compact linear extension to all of \( L_i(\mu, X) \). Hence, by Theorem 2, there exists a Bochner integrable \( g : \Omega \to K(X, Y) \) such that
\[ T(f) = \int_{\Omega} fg \, d\mu \]
for all \( f \in L_i(\mu, X) \). In particular, if \( E \) is in \( \Sigma \) and \( E \subseteq E_z \), then
\[ G(E)x = T(x_{|E}) = \int_{E} gx \, d\mu . \]
Since \( g \) is Bochner integrable, we have, by [4, p. 47], that
\[ G(E) = \int_{E} gd\mu \]
as required.

Our next result is a generalization of a theorem of D. R. Lewis [4, p. 88] dealing with the equivalence of weakly measurable and measurable functions. The proof uses the following result of Amir and Lindenstrauss [1, p. 43]: If \( X \) is a weakly compactly generated space and \( X_0 \subseteq X \) and \( Y_0 \subseteq X^* \) are separable subspaces, then there is a bounded projection \( P : X \to X \) with separable range such that \( X_0 \subseteq P(X) \) and \( Y_0 \subseteq P^*(X^*) \).

**Proposition 4.** Let \( X^* \) and \( Y \) be weakly compactly generated Banach spaces. If \( f : \Omega \to K(X, Y) \) is a bounded function such that for each \( y^* \) in \( Y^* \) the function \( y^* f(\cdot) : \Omega \to X^* \) is measurable, then there is a bounded measurable function \( g : \Omega \to K(X, Y) \) such that for each \( y^* \) in \( Y^* \), \( y^* f(\cdot) = y^* g(\cdot) \mu \text{-a.e.} \), (the exceptional set may depend on \( y^* \)).

**Proof.** We claim that the set \( A = \{ y^* f(\cdot) : y^* \in Y^*, \|y^*\| \leq 1 \} \) is compact in \( L_i(\mu, X^*) \). If not, then there is a sequence \( y^*_m \) in the unit ball of \( Y^* \) and \( \delta > 0 \) such that
\[ \|y^*_n f(\cdot) - y^*_m f(\cdot)\|_{L_i(\mu, X^*)} > \delta \]
for \( m \neq n \). Choose a bounded projection \( P_i : Y \to Y \) with separable
range such that \( P_n y_n^* = y_n^* \) for all \( n \). Since each \( y_n f(\cdot) : \Omega \to X^* \) is measurable and hence essentially separably valued, there is a bounded projection \( P_2 : X^* \to X^* \) with separable range and sets \( \Omega_n \) in \( \Sigma \) with \( \mu(\Omega \setminus \Omega_n) = 0 \) and \( y_n^* f(\Omega_n) \subseteq P_2(X^*) \) for every \( n \). Now, since each \( f(\omega) \) is a compact operator we have, for all \( x^{**} \) in \( X^{**} \), that \( f(\omega)^{**} x^{**} \) is in the natural image of \( Y \) in \( Y^{**} \) and so we may define \( h : \Omega \to K(X^{**}, Y) \) by \( h(\omega) x^{**} = P_2 f(\omega)^{**} P_2 x^{**} \). We claim that for each \( x^{**} \) in \( X^{**} \), the function \( h(\cdot) x^{**} : \Omega \to Y \) is measurable. To see this, note that since \( h(\omega) \) is separable valued, the functions \( f(\cdot) x^{**} \) are separably valued and each \( h(\cdot) x^{**} : \Omega \to X^* \) is measurable, the functions \( h(\cdot) x^{**} \) are weakly measurable. An appeal to the Pettis measurability theorem [4, p. 42] establishes the measurability of \( h(\cdot) x^{**} \). Now if \( Y_0 \) is the Banach space obtained by taking the closed linear span of \( P_2 Y \) in \( Y \), then \( Y_0 \) is separable and \( h \) can be viewed as taking its values in \( K(X^{**}, Y_0) \). Moreover, if we define \( S : Y \to Y_0 \) by \( S y = P_2 y \), then \( h(\omega) x^{**} = SP_2 f(\omega)^{**} P_2 x^{**} \). Thus, if \( y_0^* \) is in \( Y_0 \), then \( h(\omega)^{**} y_0^* = P_2 f(\omega)^{**} P_1 S^* y_0^* \) is in \( P_2 X^{**} \), since the range of \( f(\omega)^{**} \) is in \( X^* \) and \( P_2 \) extends \( P_2 \). Let \( Z = P_2 X^{**} \) and \( B = \{ T : T \in K(X^{**}, Y_0), \ T^* Y_0 \subseteq Z \} \). We claim that \( B \) is separable. To see this, let \( U \) and \( V \) denote the closed unit balls of \( Z^* \) and \( Y_0 \) endowed with the weak* topologies. Since \( Y_0 \) and \( Z \) are separable, \( U \) and \( V \) are compact metric spaces, and thus, so is \( U \times V \). For each \( T \) in \( B \), define a function \( JT \) on \( U \times V \) by \( JT(u, v) = u T^* v \). Then the map \( T \to JT \) is a linear isometry of \( B \) into \( C(U \times V) \) [8] and so, by [7, p. 437], \( B \) is separable. Since the values of \( h \) in \( K(X^{**}, Y_0) \) lie in \( B \) and \( ||h(\omega_1) - h(\omega_2)||_{K(X^{**}, Y_1)} = ||h(\omega_1) - h(\omega_2)||_{K(X^{**}, Y_2)} \) for all \( \omega_1, \omega_2 \) in \( \Omega \), the values of \( h \) in \( K(X^{**}, Y) \) form a separable set. Now because \( h(\cdot) x^{**} \) is measurable for each \( x^{**} \) in \( X^{**} \), an appeal to [5, p. 102] establishes that \( h \) is measurable. Since \( h \) is bounded, \( h \) is Bochner integrable and so we may choose a sequence \( h_n \) of \( K(X^{**}, Y) \)-valued simple functions such that

\[
\lim_{\omega} \int_{\Omega} ||h - h_n|| d\mu = 0.
\]

Define operators \( S_n \) and \( S \) from \( L_\infty(\mu, X^{**}) \) to \( Y \) by

\[
S_n(g) = \int_{\Omega} g h_n d\mu \quad \text{and} \quad S(g) = \int_{\Omega} g h d\mu
\]

for \( g \) in \( L_\infty(\mu, X^{**}) \). Since each \( h_n \) takes on only a finite number of values, each \( S_n \) is a compact operator. Moreover, we have that

\[
||S - S_n|| \leq \int_{\Omega} ||g|| ||h - h_n|| d\mu \leq ||g|| \int_{\infty} ||h - h_n|| d\mu
\]
for all $g$ in $L_\infty(\mu, X^{**})$. It follows immediately that the operator $S$ is compact. The adjoint of $S$ is the operator $y^* \rightarrow y^* h(\cdot)$ and hence by Schauder’s theorem is also compact. But $y^*_n h(\cdot) = y^*_n f(\cdot)$ a.e. This contradicts

$$\| y^*_n f(\cdot) - y^*_n f(\cdot) \|_{L_1(\mu, X^*)} > \delta$$

for $m \neq n$ and establishes that the set $A$ is compact.

Now choose $y^*_n$ in $Y^*$ such that $y^*_n(\cdot)$ is dense in $A$. If $h$ is constructed as above for this sequence ($y^*_n$), then $h$ is measurable and so, by Egoroff’s theorem, for all $\delta > 0$ there is a set $E$ in $\Sigma$ with $\mu(\Omega \setminus E) < \delta$ such that $h\chi_E$ can be approximated uniformly by simple functions. Fix $\delta > 0$ and choose such a set $E$. It follows that the sequence $y^*_n f(\cdot)\chi_E = y^*_n h(\cdot)\chi_E$ is relatively compact in $L_\infty(\mu, X^*)$. Since this sequence is $L_\infty(\mu, X^*)$-dense in $\{y^* f(\cdot)_{\chi_E : \|y^*\| \leq 1}\}$, this set is relatively compact in $L_\infty(\mu, X^*)$.

Now define $T: Y^* \rightarrow L_\infty(\mu, X^*)$ by $Ty^* = y^* f(\cdot)\chi_E$. Then $T$ is compact and as an operator on $L_1(\mu, X)$, $T^*: L_1(\mu, X) \rightarrow Y^{**}$ is compact. Notice that the dominated convergence theorem ensures that $T$ is $w^*$ to $w^*$ sequentially continuous. Thus, if $y^{**}$ is in $T^*(L_1(\mu, X))$, then $y^{**}$ is a weak* sequentially continuous functional on $Y^*$. But since $Y$ is weakly compactly generated, this means $y^{**}$ is a $w^*$ continuous functional on $Y^*$ [3, p. 148]. Hence, $T^*(L_1(\mu, X))$ is contained in $Y$. Theorem 2 now produces a Bochner integrable $g: E \rightarrow K(X, Y)$ such that

$$T^*(k) = \int_E kgd\mu$$

for all $k$ in $L_1(\mu, X)$. But, if $y^*$ is in $Y^*$, then $T^{**} y^* = y^* g$. It follows that $y^* g = y^* f$ a.e. on $E$. Since $\mu(\Omega \setminus E) < \delta$, this completes the proof.

Theorem 2 does not hold for weakly compact operators. To see this, let $\Omega$ be the unit interval endowed with Lebesgue measure and let $r_n(\cdot)$ be the $n$th Rademacher function i.e., $r_n(\omega) = \text{signum}(\sin 2^n \pi \omega)$. Consider the function $g: [0, 1] \rightarrow L(\varepsilon_2, \varepsilon_2)$ defined by $g(\omega)(\alpha) = (r_n(\omega)\alpha)$ for all $(\alpha) \in \varepsilon_2$. The function $g$ is not essentially separably valued, since if $\omega_1$ and $\omega_2$ are different numbers in $[0, 1]$ there exists a Rademacher function $r_n$ with $|r_n(\omega_1) - r_n(\omega_2)| = 2$ and hence, $\|g(\omega_1) - g(\omega_2)\|_{L(\varepsilon_2, \varepsilon_2)} \geq 2$. Thus, $g$ is not measurable. Define an operator $T: L_1(\mu, \varepsilon_2) \rightarrow \varepsilon_2$ by

$$T(f) = \int_{[0, 1]} fgd\mu$$

and note that $T$ is weakly compact. If $T$ were representable by a kernel, then that kernel would be equal to $g$ a.e. and so $g$ would be
measurable, which is a contradiction. However, we can use Proposition 4 to obtain a representation theorem for weakly compact operators by imposing further conditions on $X^*$.

**Theorem 5.** Let $X^*$ be a separable Schur space. Then there is an isometric isomorphism between the space of weakly compact operators $W(L_1(\mu, X), Y)$ and the subspace of $L_\infty(\mu, W(X, Y))$ consisting of those functions whose essential range is in the uniformly weakly compact operators. In fact, $T$ in $W(L_1(\mu, X, Y))$ and $g$ in $L_\infty(\mu, W(X, Y))$ are in correspondence if, and only if,

$$T(f) = \int_\Omega f g d\mu$$

for all $f$ in $L_1(\mu, X)$.

**Proof.** Let $T$ be in $W(L_1(\mu, X), Y)$. By the Factorization Lemma [2, p. 314], there is a reflexive space $R$ and operators $S: L_1(\mu, X) \rightarrow R$ and $J: R \rightarrow Y$ such that $T = JS$. Suppose $S$ is representable by a measurable kernel $h: \Omega \rightarrow L(X, R)$. Then $T$ is representable by the measurable kernel $g: \Omega \rightarrow L(X, Y)$ given by $g(\omega)x = Jh(\omega)x$ for all $x$ in $X$ and $\omega$ in $\Omega$. Hence, without loss of generality, we may assume that $Y$ is reflexive.

Let $G: \Sigma \rightarrow L(X, Y)$ be the representing measure of $T$ i.e.,

(i) $G(E)x = T(x\chi_E)$ for all $x$ in $X$ and $E$ in $\Sigma$

(ii) $T(f) = \int_\Omega f dG$ for all $f$ in $L_1(\mu, X)$ and

(iii) $||T|| = \sup_{\mu E > 0} ||G(E)||$.

An appeal to [10, p. 345] produces a bounded function $g: \Omega \rightarrow L(X, Y)$ such that

(1) $g(\cdot): \Omega \rightarrow Y$ is Bochner integrable for all $x$ in $X$ and

(2) $G(E)x = \int_E g(\omega)x d\mu(\omega)$ for all $x$ in $X$ and $E$ in $\Sigma$.

It follows quickly from the density of simple functions in $L_1(\mu, X)$ that

$$T(f) = \int_\Omega g f d\mu$$

for all $f$ in $L_1(\mu, X)$. Consider, for each $y^*$ in $Y^*$, the functions $y^*g(\cdot): \Omega \rightarrow X^*$. Since these functions are separably valued and weak* measurable, they are measurable by [4, p. 42]. Now $L(X, Y) = K(X, Y)$, since $X^*$ is a Schur space and $Y$ is reflexive. Consequently, Proposition 4 now produces a bounded measurable $h: \Omega \rightarrow K(X, Y)$
such that, for each \( y^* \) in \( Y^* \), \( y^* g(\cdot) = y^* h(\cdot) \mu \)-a.e. Thus, for all \( y^* \) in \( Y^* \) and \( f \) in \( L_1(\mu, X) \) we have that
\[
\langle y^*, Tf \rangle = \int_\Omega \langle y^*, g(\omega)f(\omega) \rangle d\mu(\omega)
= \int_\Omega \langle y^*, h(\omega)f(\omega) \rangle d\mu
= y^* \left( \int_\Omega hf d\mu \right)
\]
and so
\[
T(f) = \int_\Omega hf d\mu.
\]
It follows easily that
\[
h(\omega)x \subseteq T\{f: f \in L_1(\mu, X), \|f\|_1 \leq 1\}
\]
for almost all \( \omega \) in \( \Omega \) and all \( x \) in \( X \) with \( \|x\| \leq 1 \). Hence, the essential range of \( h \) consists of uniformly weakly compact operators.

The converse is proved in the same way as in Theorem 2 so we omit the proof.

Our final result follows from Theorem 5 in the same way that Corollary 3 follows from Theorem 2 so the proof is omitted.

**Corollary 6.** Let \( X^* \) be a separable Schur space and let \( G: \Sigma \rightarrow K(X; Y) \) be a \( \mu \)-continuous vector measure of bounded variation. If, for each \( E_1 \) in \( \Sigma \) with \( \mu E_1 > 0 \), there exists an \( E_2 \) in \( \Sigma \) with \( E_2 \subseteq E_1 \) and \( \mu(E_2) > 0 \) such that
\[
\left\{ \frac{G(E)x}{\mu E} : x \in X, E \in \Sigma, E \subseteq E_2, \mu E > 0, \|x\| \leq 1 \right\}
\]
is relatively weakly compact, then there exists a Bochner integrable \( g: \Omega \rightarrow K(X, Y) \) such that
\[
G(E) = \int_E gd\mu
\]
for each \( E \) in \( \Sigma \).

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