

Pacific Journal of Mathematics

**A NECESSARY CONDITION ON THE EXTREME POINTS OF A
CLASS OF HOLOMORPHIC FUNCTIONS. II**

FRANK JOHN FORELLI, JR.

A NECESSARY CONDITION ON THE EXTREME POINTS OF A CLASS OF HOLOMORPHIC FUNCTIONS II

FRANK FORELLI

We correct an oversight in the paper of the same title by pointing out that a theorem holds which is stronger than the theorem of that paper.

1. Let X be a complex manifold. (We agree that X is connected.) It will be convenient to denote by $H(X)$ the class of all holomorphic functions on X . Let $p \in X$, and let:

$$N(X, p) = \{f: f \in H(X), \operatorname{Re} f > 0, f(p) = 1\}$$

$$W(X, p) = \{g: g \in H(X), |g| < 1, g(p) = 0\}$$

$$\bar{W}(X) = \{g: g \in H(X), |g| \leq 1\}.$$

Thus

$$N(X, p) = \{(1 + g)/(1 - g): g \in W(X, p)\}.$$

Let $g \in W(X)$. We will say that g is irreducible [1] if whenever $g = \varphi\psi$ where $\varphi, \psi \in W(X)$, then either φ or ψ is a constant of modulus one. The purpose of this brief note is to correct an oversight in [2] by pointing out that the following theorem (which is stronger than the theorem of [2]) holds.

THEOREM. *Let $g \in W(X, p)$ and let $f = (1 + g)/(1 - g)$. If $N(X, p) \neq \{1\}$, and if f is an extreme point of $N(X, p)$, then g is irreducible.*

Proof. Our proof is based on the following three identities.

$$(1.1) \quad \frac{1 - zw}{(1 - z)(1 - w)} = \frac{1}{2} \frac{1 + z}{1 - z} + \frac{1}{2} \frac{1 + w}{1 - w}.$$

$$(1.2) \quad \frac{1 + zw}{1 - zw} = \frac{1}{2} \left[\frac{(1 - z)(1 - w)}{1 - zw} + s \right] + \frac{1}{2} \left[\frac{(1 + z)(1 + w)}{1 - zw} - s \right].$$

And

$$(1.3) \quad \frac{1 + [w(z + w)/(1 + zw)]}{1 - [w(z + w)/(1 + zw)]} = \frac{1}{2}(1 - z) \frac{1 - w}{1 + w} + \frac{1}{2}(1 + z) \frac{1 + w}{1 - w}.$$

The identity (1.1) proves that

$$\operatorname{Re} \frac{1 - zw}{(1 - z)(1 - w)} > 0$$

if $|z| < 1, |w| < 1$. This in turn proves that

$$\operatorname{Re} \frac{(1 - z)(1 - w)}{1 - zw} > 0$$

if $|z| < 1, |w| < 1$. Thus if $\operatorname{Re} s = 0$, then

$$(1.4) \quad \operatorname{Re} \left[\frac{(1 - z)(1 - w)}{1 - zw} + s \right] \geq 0$$

if $|z| < 1, |w| \leq 1$.

Let $g = \varphi\psi$ where $\varphi \in W(X, p), \psi \in W(X)$. It is to be proved that ψ is a constant of modulus one. If $t \in T$, then by the identity (1.2),

$$\begin{aligned} f &= \frac{1}{2} \left[\frac{(1 - t\varphi)(1 - \bar{t}\psi)}{1 - \varphi\psi} + s \right] + \frac{1}{2} \left[\frac{(1 + t\varphi)(1 + \bar{t}\psi)}{1 - \varphi\psi} - s \right] \\ &= \frac{1}{2}\alpha + \frac{1}{2}\beta. \end{aligned}$$

We have

$$(1.5) \quad \alpha(p) = 1 - \bar{t}\psi(p) + s.$$

Let t in T satisfy $\operatorname{Re} [\bar{t}\psi(p)] = 0$ and let $s = \bar{t}\psi(p)$. Then by (1.4) and (1.5) we have $\alpha, \beta \in N(X, p)$. Thus $\alpha = \beta$. This gives

$$(1.6) \quad \sigma = \bar{t}\psi = \frac{s - t\varphi}{1 + st\varphi} = \frac{s - \tau}{1 + s\tau}.$$

Thus

$$(1.7) \quad f = \frac{1 + \tau\sigma}{1 - \tau\sigma} = \frac{1 + [\tau(s - \tau)/(1 + s\tau)]}{1 - [\tau(s - \tau)/(1 + s\tau)]}.$$

We have $s = i\gamma, -1 \leq \gamma \leq 1$. By (1.7) and the identity (1.3),

$$(1.8) \quad f = \frac{1}{2}(1 - \gamma) \frac{1 - i\tau}{1 + i\tau} + \frac{1}{2}(1 + \gamma) \frac{1 + i\tau}{1 - i\tau}.$$

If $-1 < \gamma < 1$, then by (1.8), $i\tau = -i\tau$, hence $\tau = 0$. Thus $f = 1$ which contradicts the fact that 1 is not an extreme point of $N(X, p)$ if $N(X, p) \neq \{1\}$. Thus $\gamma = \pm 1$, hence by (1.6), $\bar{t}\psi = s$ which proves that g is irreducible.

2. Let $X = D$. If $g \in W(D, 0)$, then by the lemma of Schwarz, $g(z) = z\psi(z)$ where $\psi \in W(D)$. Thus by the foregoing we have a

quite elementary proof of the fact that if f is an extreme point of $N(D, 0)$, then

$$f(z) = (1 + tz)/(1 - tz)$$

where $t \in T$. There is a different elementary proof of this in [3].

3. The identity (1.1) states that if

$$(3.1) \quad f(z, w) = (1 - z)(1 - w)/(1 - zw),$$

then $1/f$ is not an extreme point of $N(D \times D, 0)$. We will prove that f on the other hand is extreme. Thus if

$$(3.2) \quad g = (f - 1)/(f + 1),$$

then the Cayley transform of g is extreme, whereas the Cayley transform of $-g$ is not.

3.1. If A is a convex set, then we will denote by ∂A the class of all extreme points of A . If B is a compact Hausdorff space, then we will denote by $M_+(B)$ the class of all Radon measures on B . Thus if $\mu \in M_+(B)$ and $E \subset B$, then $\mu(E) \geq 0$.

Let $f \in N(D \times D, 0)$. Then $\operatorname{Re} f$ is the Poisson integral μ^\sharp of a measure μ in $M_+(T \times T)$. It will be convenient to denote this measure by f^* . Thus

$$\operatorname{Re} f = (f^*)^\sharp.$$

Let F be a closed subset of the torus $T \times T$. We will denote by N_F the class of those f in $N(D \times D, 0)$ for which $\operatorname{spt}(f^*) \subset F$.

PROPOSITION. $\partial N_F \subset \partial N(D \times D, 0)$.

Proof. Let $f \in \partial N_F$. It is to be proved that $f \in \partial N(D \times D, 0)$. Thus let $f = 1/2g + 1/2h$ where $g, h \in N(D \times D, 0)$. Then $g^* + h^* = 2f^*$, hence $g^* \leq 2f^*$. This proves that $g \in N_F$. Likewise $h \in N_F$. Thus $f = g = h$.

3.2. Henceforth we let

$$F = \{(t, \bar{t}): t \in T\},$$

and we define $\pi: T \rightarrow F$ by $\pi(t) = (t, \bar{t})$. Let $f \in N_F$ and let $\mu = f^*$. Then $\mu = \pi_*\lambda$ where $\lambda \in M_+(T)$. We have

$$\hat{\mu}(j, k) = \int \bar{z}^j \bar{w}^k d\mu(z, w) = \int \bar{t}^j t^k d\lambda(t) = \hat{\lambda}(j - k).$$

Thus $\hat{\lambda}(j - k) = 0$ if $jk < 0$, hence $\hat{\lambda}(n) = 0$ if $n \neq 0, \pm 1$. This proves that

$$(3.3) \quad d\lambda = \left(\frac{\bar{a}}{2} e^{-i\theta} + 1 + \frac{a}{2} e^{i\theta} \right) \frac{d\theta}{2\pi}$$

where $a \in \mathbf{C}$. We have

$$\frac{\bar{a}}{2} e^{-i\theta} + 1 + \frac{a}{2} e^{i\theta} \geq 0,$$

hence $1 - |a| \geq 0$. Thus we see that N_F may be identified with

$$\bar{D} = \{a: a \in \mathbf{C}, |a| \leq 1\}$$

and that ∂N_F may be identified with

$$T = \partial D = \{a: a \in \mathbf{C}, |a| = 1\}.$$

3.3. Let (3.1) hold. We have

$$f(z, w) = 1 + 2 \left(\sum_1^{\infty} z^k w^k - \frac{1}{2} \sum_0^{\infty} z^{k+1} w^k - \frac{1}{2} \sum_0^{\infty} z^k w^{k+1} \right),$$

hence $f^* = \pi_* \lambda$ if in (3.3) we let $a = -1$. This proves that $f \in \partial N_F$, hence by Proposition 3.1, $f \in \partial N(D \times D, 0)$. Furthermore, we see that

$$\partial N_F = \{(1 - az)(1 - \bar{a}w)/(1 - zw): a \in T\}.$$

3.4. A comment on the foregoing. Let (3.1) and (3.2) hold, let $t \in T$, and let

$$h = (1 + tg)/(1 - tg).$$

Let $t \neq -1$, let $s = (\bar{t} - 1)/2$, and let

$$\varphi(z) = t(z + s)/(1 + \bar{s}z).$$

Then

$$f(\varphi(z), w) = ah(z, w) + ib$$

where $a + ib = f(\varphi(0), 0)$. By Proposition 3.3 of [2], this proves that $h \in \partial N(D \times D, 0)$.

4. A concluding comment. Let G be a region in D . It will be convenient to say that $D - G$ is a Painleve null set if every bounded holomorphic function on G has a holomorphic extension to D . By way of a corollary to Theorem 1, we have the following converse of the lemma of Schwarz.

THEOREM. *Let $W(X)$ separate the points of X , and let φ in $W(X, p)$ satisfy*

$$W(X, p) = \varphi W(X).$$

Then the complex manifold X may be identified with the open unit disc D modulo a Painleve null set.

Proof. If $g \in W(X, p)$, then $g = \varphi\psi$ where $\psi \in W(X)$. Thus by Theorem 1,

$$\partial N(X, p) \subset \{(1 + t\varphi)/(1 - t\varphi) : t \in T\}.$$

This proves, by the Krein-Milman theorem, that if $f \in N(X, p)$, then

$$(4.1) \quad f = \int \frac{1 + t\varphi}{1 - t\varphi} d\mu(t)$$

where $\mu \in M_+(T)$. This in turn proves, since $W(X)$ separates the points of X , that φ is univalent. Thus we may identify X with $\varphi(X)$, in which case (4.1) becomes

$$(4.2) \quad f(z) = \int \frac{1 + tz}{1 - tz} d\mu(t)$$

if $z \in X$. The right side of (4.2), however, belongs to $N(D, 0)$. Thus $D - X$ is a Painleve null set, which completes the proof of Theorem 4.

REFERENCES

1. P. Ahern and W. Rudin, *Factorizations of bounded holomorphic functions*, Duke Math. J., **39** (1972), 767-777.
2. F. Forelli, *A necessary condition on the extreme points of a class of holomorphic functions*, Pacific J. Math., **73** (1977), 81-86.
3. F. Holland, *The extreme points of a class of functions with positive real part*, Math. Ann., **202** (1973), 85-87.

Received February 16, 1979 and in revised form July 3, 1979. Partially supported by the National Science Foundation.

THE UNIVERSITY OF WISCONSIN-MADISON
MADISON, WI 53706

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)

University of California
Los Angeles, CA 90024

HUGO ROSSI

University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ANDREW OGG

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF HAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1981 by Pacific Journal of Mathematics

Manufactured and first issued in Japan

Bruce Allem Anderson and Philip A. Leonard , Sequencings and Howell designs	249
Kevin T. Andrews , Representation of compact and weakly compact operators on the space of Bochner integrable functions	257
James Glenn Brookshear , On the structure of hyper-real z -ultrafilters	269
Frank John Forelli, Jr. , A necessary condition on the extreme points of a class of holomorphic functions. II	277
Richard J. Friedlander, Basil Gordon and Peter Tannenbaum , Partitions of groups and complete mappings	283
Emden Robert Gansner , Matrix correspondences of plane partitions	295
David Andrew Gay and William Yslas Vélez , The torsion group of a radical extension	317
André (Piotrowsky) De Korvin and C. E. Roberts , Convergence theorems for some scalar valued integrals when the measure is Nemytskii	329
Takaâi Kusano and Manabu Naito , Oscillation criteria for general linear ordinary differential equations	345
Vo Thanh Liem , Homotopy dimension of some orbit spaces	357
Mark Mahowald , bo -resolutions	365
Jan van Mill and Marcel Lodewijk Johanna van de Vel , Subbases, convex sets, and hyperspaces	385
John F. Morrison , Approximations to real algebraic numbers by algebraic numbers of smaller degree	403
Caroline Series , An application of groupoid cohomology	415
Peter Frederick Stiller , Monodromy and invariants of elliptic surfaces	433
Akihito Uchiyama , The factorization of H^p on the space of homogeneous type	453
Warren James Wong , Maps on simple algebras preserving zero products. II. Lie algebras of linear type	469