HOMOTOPY DIMENSION OF SOME ORBIT SPACES

Vo Thanh Liem
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The homotopy dimension of a compact absolute neighborhood retract space $X$ is defined to be the least dimension among all the finite CW-complexes which have the same homotopy type of $X$. We show that actions of finite groups or actions of tori (with finite orbit types) on a finite-dimensional compact absolute neighborhood retract $X$ do not raise homotopy dimension if the homotopy dimension of $X$ is not two.

1. Introduction and preliminaries. Through this note, all actions are of finite types.

In [7], Oliver gave an affirmative answer to Conner's conjecture: "The orbit space of an action of a compact Lie group on a finite-dimensional AR is an AR". From West [10], it follows that every compact absolute neighborhood retract $X$ (CANR $X$) has the homotopy type of a finite complex. So, we can define the homotopy dimension (h.d.) of a CANR $X$ by

$$\text{h.d.}(X) = \min \{\dim K | K \text{ is a finite complex and } K \cong X\}.$$  

On the other hand, Conner [5] has shown that the orbit space of an action of a compact Lie group on a finite-dimensional CANR is a CANR. It is natural to wonder whether the actions of a compact Lie group on a CANR can raise the homotopy dimension. We will show that the homotopy dimension does not increase when $\text{h.d.}(X) \neq 2$ and when the action comes from either a finite group or a toral group.

Combining a well-known result of Wall (Thm. F, [8]) and the result of West [10] (mentioned above), we can easily obtain the following lemma that will be needed in the sequel.

**Lemma 0.** A CANR has the homotopy type of a $k$-dimensional finite complex if and only if $H_q(\tilde{X}; Z) = 0$ for all $q > k$ and $H^{k+1}(X; \beta) = 0$ for every coefficient bundle $\beta$ of $Z\pi_1(X)$-modules over $X$ if $k \neq 2$. Moreover, if $H_q(\tilde{X}; Z) = 0$ for $q > 2$ and $H^3(X; \beta) = 0$; then $\text{h.d.}(X) \leq 3$.

2. Orbits of action of finite groups. Let $G$ be a cyclic group of order $p$ with a generator $g$. The notation in [1] will be used as follows $1 - g$ and $1 + g + \cdots + g^{p-1}$ will be denoted respectively by $\tau$ and $\sigma$. If one of these is denoted by $\rho$, the other will be denoted
by \( \bar{\rho} \). If \( \beta \) is a sheaf of \( \mathbb{Z}_p \)-modules over \( X/G \), let \( A \) denote the sheaf 
\[ H^q(X; \pi^*\beta | \pi^{-1}y) \mid y \in X/G \] over \( X/G \),
where \( \pi^*\beta \) is the pull back of \( \beta \) associated with the orbit map \( \pi: X \to X/G \). If \( U \) is an open subset of \( X/G \), let \( A_U \) denote the sheaf 
\[ \{ \cup \{ H^q(X; \pi^*\beta | \pi^{-1}y) \mid y \in U \} \cup \{ 0 \} \mid y \in X/G \} \]
and let \( A_F \) ( \( F \) closed in \( X/G \)) denote \( A/A(F) \) (refer to page 41 of [1]).

It will be convenient to establish the following preliminary lemmas before we begin the proof of the main result.

**Lemma 1.** Let \( G = \mathbb{Z}_p \), \( p \) prime, act on a CANR \( X \) with fixed point set \( F \). Assume that \( m = \dim X < \infty \) and that \( \beta_p \) is a bundle of coefficients of \( \mathbb{Z}_p \pi_1(X/G) \)-modules over \( X/G \). If \( \text{h.d.} (X) \leq k \), then 
\[ H^q(X/G; \beta_p) = 0 \] for all \( q \geq k + 1 \).

**Proof.** Think of \( \rho \) and \( \bar{\rho} \) as endomorphisms of the sheaf \( A \) and denote their images respectively by \( \rho A \) and \( \bar{\rho} A \). Since \( \mathbb{Z}_p \) is a field, it follows that the following sequence of sheaves over \( X/G \)
\[ 0 \longrightarrow \bar{\rho} A \longrightarrow A \overset{\rho \oplus \eta}{\longrightarrow} \rho A \oplus A_F \longrightarrow 0 \]
is exact, where \( \bar{\rho} A \to A \) is the inclusion and where \( \eta: A \to A_F \) is the quotient homomorphism (Lemma 4.1 of [1]). This sequence induces an exact cohomology sequence
\[ \cdots \longrightarrow H^n(X/G; \bar{\rho} A) \longrightarrow H^n(X/G; A) \]
\[ \longrightarrow H^n(H/G; \rho A) \oplus H^n(X/G; A_F) \longrightarrow \cdots . \]

Let \( H^*(\rho) \) denote \( H^*(X/G; \rho A) \). Observe \( H^*(X/G; A_F) = H^*(F; \beta_p | F) \); then, from the above cohomology sequence and the fact that \( H^*(X/G; A) \equiv H^*(X; \pi^*\beta_p) \) (see page 35, [1]), there are the following exact sequences:
\[ H^q(X; \pi^*\beta_p) \longrightarrow H^q(\sigma) \oplus H^q(F; \beta_p | F') \longrightarrow H^{q+1}(\tau), \]
\[ H^{q+1}(X; \pi^*\beta_p) \longrightarrow H^{q+1}(\tau) \oplus H^{q+1}(F; \beta_p | F') \longrightarrow H^{q+2}(\sigma), \]
\[ \vdots \]
\[ H^m(X; \pi^*\beta_p) \longrightarrow H^m(\rho) \oplus H^m(F; \beta_p | F') \longrightarrow H^{m+1}(\bar{\rho}). \]

Since \( \text{h.d.} (X) \leq k \), it follows from Lemma 0 that \( H^*(X, \pi^*\beta_p) = 0 \), for all \( n \geq q \geq k + 1 \). On the other hand, \( H^{m+1}(\bar{\rho}) = 0 \) since \( \dim X = m < \infty \). Thus, we can show inductively that
\[ (1) \quad H^q(X/G, F; \beta_p) = H^q(\sigma) = 0, \]
\[ (2) \quad H^q(F; \beta_p | F) = 0. \]
Hence, from the exact sequence of the pair \((X/G, F)\),
\[
\cdots \to H^q(X/G, F; \beta_p) \to H^q(X/G; \beta_p) \to H^q(F; \beta_p|F) \to \cdots,
\]
it follows that \(H^q(X/G; \beta_p) = 0\); and the proof of lemma is complete.

**Lemma 2.** Let \(G = Z_p, p\) prime, act on a CANR \(X\) with fixed point set \(F \neq \emptyset\). Assume that \(\dim X = m < \infty\) and that \(\beta\) is a bundle of coefficients of \(Z\pi_1\)-modules over \(X/G\). Then \(H^q(X/G; \beta) = 0\) for all \(q \geq k + 1\), if h.d. \((X) \leq k\).

**Proof.** Consider the following diagram
\[
\cdots \to H^q(X/G; \beta) \xrightarrow{\pi^*} H^q(X; \beta) \to H^q(X/G; \beta_p) \to \cdots
\]
where \(\mu^*\) is the transfer map [1] and where the horizontal exact sequence is from the exact sequence of bundles of coefficients over \(X/G\):
\[
0 \to \beta \xrightarrow{\pi^*} \beta \to \beta_p \to 0.
\]
So, it follows easily that \(H^q(X/G; \beta) = 0\) if \(q \geq k + 1\), since \(H^q(X; \pi^*\beta) = 0\) by Lemma 0 and \(H^q(X/G; \beta_p) = 0\) by Lemma 1. The proof is now complete.

**Lemma 3.** Let a finite group \(G\) act on \(X\) with fixed point set \(F \neq \emptyset\). If \(X\) has the homotopy type of a simplicial complex \(K^k\), then \(H_q(X/G; Z) = 0\) for all \(q > k\).

**Proof.** Let \(\pi^*(\tilde{X}/G)\) be the pullback of the universal covering space \(p: \tilde{X}/G \to X/G\) associated with the orbit map \(\pi: X \to X/G\). Then, the induced map \(\tilde{p}: \pi^*(\tilde{X}/G) \to X\) is a covering map and the lifting map \(\pi^*\) of \(\pi\) is the orbit map of the induced action of \(G\) on \(\pi^*(\tilde{X}/G)\). Now, since \(X \cong K^k\), it follows that \(H_q(\pi^*(\tilde{X}/G), Z) = 0\) for \(q \geq k + 1\). Then, the Smith theorem in the integral homology theory shows that \(H_q(\tilde{X}/G, Z) = 0\) for \(q \geq k + 1\). (Similar to the proof of Lemma 2 above by use of the transfer map \(\mu_*\) on page 119 of [3].) Hence, the proof is complete.

**Theorem 1.** Suppose that a finite group \(G\) acts on a finite dimensional CANR \(X\). If h.d. \((X) \leq k\) and \(k \neq 2\), then h.d. \((X/G) \leq k\). If \(k = 2\), h.d. \((X/G \leq 3)\).
Proof. Step 1. \( G = \mathbb{Z}_p, \ p \) prime.

Case 1. \( F = \emptyset. \) See Lemma 2 of [6].

Case 2. \( F \neq \emptyset. \) It follows from Lemma 2 and Lemma 3 above that

1. \( H^q(X/G; \beta) = 0, \ q \geq k + 1 \) and for any bundle coefficient \( \beta \) over \( X/G, \)
2. \( H_q(X/G; \mathbb{Z}) = 0, \ q \geq k + 1. \)

So, it follows from Lemma 0 that h.d. \( (X) \leq k. \)

Step 2. \( G \) is cyclic of order \( p^n, \ p \) prime. We prove inductively on \(|G|, \) the order of \( G. \) Let \( H \) be a subgroup of \( G \) of order \( p^{n-1} \) then, h.d. \( (X/H) \leq k \) by induction hypothesis and the proof is complete by Step 1.

Step 3. \( G \) is a finite \( p \)-group. First, by an inductive proof as in Step 2 we may assume that \( G \) is abelian, since \( G \) is solvable. Therefore, we can write \( G = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p^{n_k}. \) Then, again an inductive proof as above will complete the proof for this case.

Step 4. General case. The proof will be similar to that of Theorem III. 5.2 in [1].

Suppose that \(|G| = p_1^{n_1} \cdots p_s^{n_s}, \) and that \( K_j \) is a \( p_j \)-Sylow subgroups of \( G, \) and denote \( \pi_{s,j} \) the canonical map \( X/K_j \rightarrow X/G \) for \( j = 1, 2, \cdots, s \) as in [1]. Define \( \pi': H^*(X/G; \beta) \rightarrow \sum_{j=1}^s H^*(X/K_j; \pi_{s,j}^* \beta) \) by

\[
\pi' = \pi_{s,1}^* + \cdots + \pi_{s,s}^*.
\]

Observe that \( H^n(X/K_j; \pi^n \beta) = 0 \) for \( q \geq k + 1 \) and \( j = 1, 2, \cdots, s \) by Step 3 above. Hence, if we can show that \( \pi' \) is injective, then \( H^n(X/G; \beta) = 0 \) for \( q \geq k + 1. \) Therefore, the theorem will follow by Lemma 0 and Lemma 3 above.

Now, let \( \mu'_j: H^*(X/K_j; \pi_{s,j}^* \beta) \rightarrow H^*(K/G; \beta) \) be the transfer map [1] such that \( \mu'_j \pi_{s,j}^* \) is the multiplication by \(|G|/|K_j|. \) If \( r \in \text{Ker} \pi', \) then we have \(|G|/|K_j| \cdot r = \mu'_j \pi_{s,j}^*(r) = 0 \) for each \( j = 1, 2, \cdots, s, \) since \( \pi_{s,j}^* = 0. \) Therefore, for each \( j = 1, 2, \cdots, s \)

\[
(p_1^{n_1} \cdots p_{j-1}^{n_{j-1}} p_{j+1}^{n_{j+1}} \cdots p_s^{n_s}) \cdot r = 0.
\]

Since the family \( p_1^{n_1} \cdots p_{j-1}^{n_{j-1}} p_{j+1}^{n_{j+1}} \cdots p_s^{n_s}, \ j = 1, 2, \cdots, s, \) is relatively prime, it follows that \( r = 0, \) and the proof is now complete.

3. Orbits of actions of total groups.

Lemma 4. Suppose that the circle group \( S^1 \) acts on a finite-
dimensional CANR $X$. If h.d. $(X) \leq k$, then $H^q(X/S^1; \beta) = 0$ for all $q \geq k + 1$ and for all bundles of coefficients $\beta$ over $X/S^1$.

**Proof.** Assume that $H_1, \ldots, H_s$ are finite isotropy subgroups of the action. Let $G$ be a finite cyclic subgroup of $S^1$ such that $H_1, \ldots, H_s$ are subgroups of $G$. Then h.d. $(X/G) \leq k$ by the theorem above. So, we may assume that the action is semi-free, i.e., it has only two orbit types $\{e\}$ and $S^1$. Let $\beta$ be a bundle of coefficients of $\mathbb{Z}_\pi$-modules over $X/S^1$, where $\pi = \pi_i(X/S^1)$. From Lemma 0, it follows that $H^q(X; \pi^*\beta) = 0$ for all $q \geq k + 1$, where $\pi: X \to X/S^1$ is the orbit map.

**Case 1.** $F = \emptyset$. Since the action is free, \{H^0(\pi^{-1}y; \pi^*\beta); y \in X/S^1\} = \beta and \{H^1(\pi^{-1}y; \pi^*\beta); y \in X/S^1\} = \beta. An observations on Leray spectral sequence (as in Case 2) proves the lemma for this case.

**Case 2.** $F \neq \emptyset$. Since $\pi^{-1}(y) = \{e\}$ or $S^1$, we have

1. $E_{s,q}^{1,0} = H^q(X/S^1; H^s(\pi^{-1}y; \pi^*\beta \mid \pi^{-1}y)) = H^q(X/S^1; \beta),$
2. $E_{s,1}^{1,1} = H^q(X/S^1; H^s(\pi^{-1}y; \pi^*\beta \mid \pi^{-1}y)) = H^q(X/S^1; F; \beta),$ and
3. $E_{s,0}^{1,0} = 0$ if $s \geq 2$.

We now proceed by induction on $q$. Since dim $X < \infty$, we may assume that $H^q(X/S^1; \beta) = 0$ for $q \geq k + 2$, then we will show that $H^{k+1}(X/S^1; \beta) = 0$.

**Step 1.** To show that $H^q(X/S^1, F; \beta) = 0$ for $q \geq k + 1$. By the induction hypothesis, we observe that for each $q \geq k + 2$, the $E_{s,q}^{1,0}$ term, of the Leray spectral sequence for the map $\pi$ (page 140, [2]) is trivial, since $E_{s,0}^{1,0} = H^q(X/S^1; \beta)$ by (1). Observing the Leray spectral sequence $\{E_{s,q}^{1,0}\}$ of $\pi$, we can show that for all $r \geq 2$

1. $E_{r}^{k+1,1} = E_{s}^{k+1,1},$

and
2. $E_{r}^{k+2,0} = 0$;

therefore,

(a) $E_{\infty}^{k+1,1} = H^{k+1}(X/S^1, F; \beta)$ by (2), and

(b) $E_{\infty}^{k+2,0} = 0.$

Now, from the convergence of \{$E_{s,q}^{1,0}$\} to $H^*(X; \pi^*\beta)$ and from the fact that $H^{k+1}(X; \pi^*\beta) = 0$ by Lemma 0, we can show that $H^{k+1}(X/S^1, F; \beta) = 0$.

**Step 2.** To show that $H^q(X, F; \pi^*\beta) = 0$ for $q \geq k + 2$. Consider the Leray spectral sequence (page 140, [2]) of the map of pairs $\pi: (X, F) \to (X/S^1, F)$. First we observe that the sheaf $\xi = \{H^0(\pi^{-1}y, \pi^{-1}(y \cap F); \pi^*\beta \mid \pi^{-1}y) \mid y \in X/S^1\}$ and the sheaf $\eta = \{H^1(\pi^{-1}y,$
$\pi^{-1}(y \cap F); \pi^*\beta | \pi^{-1}y)$ are the same over $X/S'$, since $\pi^{-1}(y) = \{e\}$ or $S'$. Moreover, from the definition of the relative cohomology (Prop. II. 12.2, [2]), it follows that $H^*(X/S'; F; \beta) = H^*(X/S'; \xi)$. Then, from Step 1 it follows that

$$E_{s,*}^{\alpha} = \begin{cases} H^*(X/S'; F; \beta) = 0 & \text{if } q \geq k + 1 \\ 0 & \text{if } s \geq 2 \end{cases}.$$ 

Therefore, $E_{s,*}^{\alpha} = 0$ when $q + s \geq k + 2$. Consequently, for $q \geq k + 2$ $H^q(X, F; \beta) = 0$, since $\{E_{s,*}^{\alpha}\}$ converges to $H^*(X, F; \beta)$.

**Step 3.** To show that $H^q(X/S'; \beta) = 0$ for $q \geq k + 1$. First, from the exact cohomology sequence of the pair $(X, F)$ and from the fact of $H^q(X, F; \pi^*\beta) = 0$ for $q \geq k + 2$, it follows that $H^q(F; \pi^*\beta | F) = 0$ for $q \geq k + 1$. Then, we observe that $H^q(F; \pi^*\beta | F) = H^q(F; \beta | F)$, since $F$ is the fixed point set. So, $H^q(F; \beta | F) = 0$ for $q \geq k + 1$. Therefore, the exactness of the cohomology sequence of the pair $(X/S', F)$ shows that $H^q(X/S'; \beta) = 0$ for $q \geq k + 1$, since $H^q(X/S', F; \beta) = 0$ by Step 1, and the proof of lemma is now complete.

**Theorem 2.** Suppose that $T^m$ acts on a finite-dimensional $\text{CANR}$ $X$. Then

(1) h.d. $(X/T^m) \leq \text{h.d.} (X)$ if h.d. $(X) \neq 2$,

and

(2) h.d. $(X/T^m) \leq 3$ if h.d. $(X) = 2$.

**Proof.** By induction on $m$, without loss of generality we only consider the actions of $S^1$. By Lemmas 0 and 4, we only have to show that $H_q(\widetilde{X/S^1}; Z) = 0$ for all $q \geq k + 1$. Again, by Lemma 4 above, $H^q(\widetilde{X/S^1}; Z) = 0$ for all $q \geq k + 1$; therefore Ext $(H_{q-1}(\widetilde{X/S^1}); Z) = 0$ and Hom $(H_q(\widetilde{X/S^1}; Z); Z) = 0$ for all $q \geq k + 1$ by the universal-coefficient theorem (Thm. 5.5.3 in [8]). Hence, for each $q \geq k + 1$ Ext $(H_q(\widetilde{X/S^1}; Z); Z) = 0$ and Hom $(H_q(\widetilde{X/S^1}; Z); Z) = 0$; and it follows from Theorem V.13.7 in [2] that $H_q(\widetilde{X/S^1}; Z) = 0$. The proof is now complete.

**Corollary.** Let $G$ be a compact Lie group such that $|G/G_0|$ is finite, where $G_0$ is the torus identity component of $G$. Let $G$ act on a finite-dimensional $\text{CANR}$ $X$. Then,

(1) if h.d. $(X) \neq 2$, then h.d. $(X/G) \leq \text{h.d.} (X)$,

(2) if h.d. $(X) = 2$, then h.d. $(X/G) \leq 3$.

We conclude this paper by some remarks.
REMARKS. (1) It is a well-known problem in infinite-dimensional topology to determine whether the orbit space of an action of compact Lie group on the Hilbert cube $\prod_{n}^\infty [0, 1]$ is a CAR. This explains (maybe) the condition $\dim X < \infty$ in the above statements.

(2) The limitation, when $\text{h.d.}(X) = 2$, is from an unsettled problem.

(3) The author does not see how to extend these results for the case of actions of compact Lie groups on a CANR.

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