THE FACTORIZATION OF $H^p$ ON THE SPACE OF HOMOGENEOUS TYPE

AKIHITO UCHIYAMA
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Let $K$ be a Calderon-Zygmund singular integral operator with smooth kernel. That is, there is an $\Omega(x)$ defined on $\mathbb{R}^n \setminus \{0\}$ which satisfies

\[ \int_{|x|=1} \Omega = 0, \quad \Omega \neq 0, \]

(*) \quad $\Omega(rx) = \Omega(x)$ when $r > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$, \quad $|\Omega(x) - \Omega(y)| \leq |x - y|$ when $|x| = |y| = 1$,

and that

\[ Kf(x) = P.V. \int_{\mathbb{R}^n} \Omega(x-y) |x-y|^{-n} f(y) dy. \]

Let

\[ Kf(x) = P.V. \int_{\mathbb{R}^n} \Omega(y-x) |y-x|^{-n} f(y) dy. \]

R. Coifman, R. Rochberg and G. Weiss showed the weak version of the factorization theorem of $H^1(\mathbb{R}^n)$ and that was refined by Uchiyama in the following form.

**THEOREM A.** If $1 < q < \infty$ and $1/q + 1/r = 1$, then

\[ c_{K,q} \|f\|_{H^1(\mathbb{R}^n)} \leq \inf \left\{ \sum_{i=1}^{\infty} \|g_i\|_{L^q} \|h_i\|_{L^r} : \right. \]

\[ f = \sum_{i=1}^{\infty} (h_i K g_i - g_i K' h_i) \}

\[ \leq c'_{K,q} \|f\|_{H^1(\mathbb{R}^n)}. \]

In this note, we extend Theorem A to $H^p(X)$, where $p \in (1 - \varepsilon_X, 1]$ and $X$ is a space of homogeneous type with certain assumptions.

1. Preliminaries. In the following, $A > 1$ and $\gamma \leq 1$ are positive constants depending only on the space $X$.

Let $X$ be a topological space endowed with a Borel measure $\mu$ and a quasi-distance $d$ such that

(1) \quad $d(x, y) \geq 0$

(2) \quad $d(x, y) > 0$ iff $x \neq y$

(3) \quad $d(x, y) = d(y, x)$

(4) \quad $d(x, z) \leq A(d(x, y) + d(y, z))$
\[ |d(x, z) - d(x, y)|/d(x, y) \leq A(d(z, y)/d(x, y)) \]
\[ \text{if } d(z, y) < d(x, y)/(2A) \]
\[ t/A \leq \mu(B(x, t)) \leq t \]

for all \( x, y, z \) in \( X \) and all \( t \in (0, A\mu(X)) \), where \( B(x, t) = \{ y \in X : d(x, y) < t \} \). We postulate that \( \{B(x, t)\}_{t \in (0, A\mu(X))} \) form a basis of open neighborhoods of the point \( x \).

Let \( \varphi(t) \in C^\infty(0, \infty) \) be a fixed nonnegative function such that \( \varphi(t) = 0 \) on \( (0, 1/2) \), \( \varphi(t) = 1 \) on \( (1, \infty) \) and \( |d\varphi/dt| < 3 \).

Further, we assume that \( X \) is endowed with a function \( k(x, y) \) defined on \( X \times X \) such that

\[ |k(x, y)| \leq 1/d(x, y) \quad \text{for all } x, y \in X \]
\[ \sup \{|k(x, y)| : y \in X \text{ satisfying } A^{-2}t \leq d(x, y) \leq t \} \geq 1/(At) \]
\[ \sup \{|k(y, x)| : y \in X \text{ satisfying } A^{-2}t \leq d(x, y) \leq t \} \geq 1/(At) \]
\[ \text{for all } x \in X \text{ and all } t \in (0, A\mu(X)) \]
\[ |k(x, y) - k(x, z)|, |k(y, x) - k(z, x)| \leq (d(y, z)/d(x, y))^{1/2}/d(x, y) \]
\[ \text{if } d(y, z) < d(x, y)/(2A) \]

and that for any \( f \in L^p(X) \)

\[ Kf(x) = \lim_{t \to 0} \int k(x, y, t)f(y)d\mu(y) \]
\[ K'f(x) = \lim_{t \to 0} \int k(y, x, t)f(y)d\mu(y) \]

exist almost everywhere and

\[ \|Kf\|_2 \leq \|f\|_2, \quad \|K'f\|_2 \leq \|f\|_2, \]

where

\[ k(x, y, t) = k(x, y)\varphi(d(x, y)/t) \]

and \( \|g\|_p \) denotes \( \left( \int_X |g(y)|^p d\mu(y) \right)^{1/p} \).

For \( x \in X \) and \( t \in (0, A\mu(X)) \), let

\[ T(x, t) = \{ \Psi \in C(X) : \}

\[ \sup \Psi \subset B(x, t) \]
\[ |\Psi(y)| \leq 1/t \]
\[ |\Psi(y) - \Psi(z)| \leq (d(y, z)/t)^{1/2}/t \quad \text{for any } y, z \in X \}

For \( f \in L^1(X) \) and \( p > 1/(1 + \gamma) \), let

\[ f^\#(x) = \sup_{t \in (0, A\mu(X))} \sup_{\Psi \in T(x, t)} \left| \int f(y)\Psi(y)d\mu(y) \right| \]
\[ \|f\|_{L^p} = \|f^\#\|_p \].
If $p > 1$, then $\|f\|_{H^p} \approx \|f\|_p$ by the Hardy-Littlewood maximal theorem and we define $H^p(X) = L^p(X)$. If $1/(1 + \gamma) < p \leq 1$, then we define $H^p(X)$ to be the completion of $\{f \in L^1(X) : \|f\|_{H^p} < \infty\}$ by the metric $\|f - g\|_{H^p}$.

A comment on notation: The letter $C$ denotes a positive constant depending only on $A$ and $\gamma$. The various uses of $C$ do not all denote the same constant. All the functions considered are real valued functions.

2. The results. Our results are the following.

**Theorem 1.** If $1/p = 1/q + 1/r < 1 + \gamma$, $0 < 1/q < 1 + \gamma$, $0 < 1/r < 1 + \gamma$, $g \in H^q \cap L^2$ and $h \in H^r \cap L^2$, then

$$\|hKg - gK'h\|_{H^p} \leq c_{q,r} \|g\|_{H^q} \|h\|_{H^r},$$

where $c_{q,r}$ is a positive constant depending on $q$, $r$ and $X$.

**Remark 1.** As a consequence of this theorem, for any $g \in H^q$ and any $h \in H^r$ we can define $hKg - gK'h$ as the limit of $\{h_jKg_j - g_jK'h_j\}_{j=1}^\infty$ in $H^p$, where $\{g_j\}_{j=1}^\infty \subset H^q \cap L^2$ converges to $g$ in $H^q$ and $\{h_j\}_{j=1}^\infty \subset H^r \cap L^2$ converges to $h$ in $H^r$.

**Theorem 2.** If $\mu(X) = \infty$, $1/p = 1/q + 1/r < 1 + \gamma$, $0 < 1/q < 1 + \gamma$, $0 \leq 1/r < 1 + \gamma$ and $f \in H^p$, then there exist $\{g_j\}_{j=1}^\infty \subset H^q$ and $\{h_j\}_{j=1}^\infty \subset H^r$ such that

$$f = \sum_{j=1}^\infty (h_jKg_j - g_jK'h_j),$$

$$(\sum (\|g_j\|_{H^q} \|h_j\|_{H^r})^p)^{1/p} \leq c_{q,r} \|f\|_{H^p}.$$

As a result of these theorems, we get

**Corollary 1.** If $\mu(X) = \infty$, $1/p = 1/q + 1/r < 1 + \gamma$, $0 < 1/q < 1 + \gamma$, $0 < 1/r < 1 + \gamma$ and $f \in H^p$, then

$$c_{q,r} \|f\|_{H^p} \leq \inf \left\{ \left( \sum_{j=1}^\infty (\|g_j\|_{H^q} \|h_j\|_{H^r})^p \right)^{1/p} \right\} ;$$

$$f = \sum_{j=1}^\infty (h_jKg_j - g_jK'h_j) \leq c_{q,r} \|f\|_{H^p}.$$

**Example 1.** Let $X = \mathbb{R}^n$, $d(x, y) = |x - y| \omega_n = (\sum_{j=1}^n (x_j - y_j)^2)^{n/2} \omega_n$, $\mu$ be the Lebesgue measure and let $k(x, y) = \Omega(x - y) |x - y|^{-n}$, where $\omega_n$ is the volume of the unit ball of $\mathbb{R}^n$ and $\Omega$ satisfies (*). Then, by taking $\gamma = 1/n$ and by taking $A$ sufficiently large depending on
Ω, the conditions (1) ~ (10) can be satisfied. In this case, the above definition of $H^1$ coincides with the definition of $H^1(R^n)$ given by Fefferman-Stein [5]. Thus, Corollary 1 is an extension of Theorem A.

**Example 2.** Let $A_t = t^p (0 < t < \infty)$ be a group of linear transformation on $R^n$ with infinitesimal generator $P$ satisfying $(Px, x) \geq (x, x)$, where $(, )$ is the usual inner product in $R^n$. For each $x \in R^n$ let $\rho(x)$ denote the unique $t$ such that $|A_t^{-1}x| = 1$. Let $\Omega(x)$ be such that

$$\int_{|x| = 1} \Omega(x)(Px, x) = 0 \quad (\Omega \neq 0) ,$$

$$\Omega(A_t x) = \Omega(x) \text{ when } t > 0 \text{ and } x \in R^n \setminus \{0\}$$

$$|\Omega(x) - \Omega(y)| \leq |x - y| \text{ when } |x| = |y| = 1 .$$

Let $X = R^n$, $d(x, y) = \rho(x - y)\omega_n$, $\mu$ be the Lebesgue measure and let $k(x, y) = \Omega(x - y)/d(x, y)$, where $\nu = \text{tr } P$. Then, by taking $\gamma = 1/\nu$ and by taking $A$ sufficiently large depending on $P$ and $\Omega$, the conditions (1) ~ (10) can be satisfied. [See Riviere [12].]

If we remove the condition $\mu(X) = \infty$, we can show the following a little weaker result.

**Theorem 2'.** If $\mu(X) < \infty$, $X$ is connected, $1 \leq 1/p = 1/q + 1/r < 1 + \gamma, 1 < q, 1 < r, f \in H^p$ and $\int fd\mu = 0$, then there exist $\{g_j\}_{j=1}^\infty \subset L^q$ and $\{h_j\}_{j=1}^\infty \subset L^r$ such that

$$f = \sum_{j=1}^\infty (h_j K g_j - g_j K'h_j)$$

$$(\sum (||g_j||_q ||h_j||_r)^p)^{1/p} \leq c_{q,r} ||f||_{H^p} .$$

**Corollary 1'.** If $\mu(X) < \infty$, $X$ is connected, $1 \leq 1/p = 1/q + 1/r < 1 + \gamma, 1 < q < \infty, 1 < r < \infty, f \in H^p$ and $\int fd\mu = 0$, then

$$c_{q,r} ||f||_{H^p} \leq \inf \left\{ \left( \sum_{j=1}^\infty (||g_j||_q ||h_j||_r)^p \right)^{1/p} : f = \sum_{j=1}^\infty (h_j K g_j - g_j K'h_j) \right\} \leq c'_{q,r} ||f||_{H^p} .$$

**Remark 2.** When $\mu(X) < \infty$, for $f \in L^1(X)$ we can easily show

$$\left| \int fd\mu \right| \leq C \inf_{x \in X} f^*(x) .$$

Thus, for any $f \in H^p$ we can define $\int fd\mu$ by $\lim_{n \to \infty} \int f_n d\mu$, where $\{f_n\} \subset$
$L^1 \cap H^p$ and $\lim f_n = f$ in $H^p$. And it follows easily that

$$\left| \int f \, d\mu \right| \leq c_p \|f\|_{H^p}.$$  

3. The basic lemmas.

**Definition 1.** If $1/(1 + \gamma) < p \leq 1$, we say that a function $a(y)$ is a $p$-atom if there exists a ball $B(x, t)$ such that

$$(20) \quad \text{supp } a \subset B(x, t), \|a\|_\infty \leq t^{-1/p}, \int_a a \, d\mu = 0.$$  

We can show easily that $\|a\|_{H^p} \leq c_p$.

**Definition 2.** For $f \in L^1 + L^2$, $q > 0$ and $\alpha > 0$, let

$$M_q f(x) = \sup_{t > 0} \left( \int_{B(x, t)} |f|^q \, d\mu / t \right)^{1/q},$$

$$K^* f(x) = \sup_{t > 0} \left| \int_k(x, y, t) f(y) \, d\mu(y) \right|,$$

$$K^{\alpha} f(x) = \sup_{t > 0} \left| \int_k(y, x, t) f(y) \, d\mu(y) \right|,$$

$$f^{[*\alpha]}(x) = \sup_{t > 0} \sup_{\phi \in T_\alpha(x, t)} \left| \int f \phi \, d\mu \right|$$

where

$$(21) \quad T_\alpha(x, t) = \{ \phi \in C(X): |\phi(z)| \leq t^{-1}(t + d(x, z))^{-1-\gamma} \}.$$

**Lemma 1.** If $p > q$, then

$$\|M_q f\|_p \leq c_{p, q} \|f\|_p.$$  

This is an immediate consequence of the Hardy-Littlewood maximal theorem. We omit the proof.

**Lemma 2.** If $d(y, z) \leq d(x, y)/(2A)$, then

$$d(x, y)/(2A) \leq d(x, z) \leq 2Ad(x, y).$$  

This follows easily from (4). We omit the proof.

**Lemma 3.** If $t > 0$ and if $d(y, z) \leq d(x, y)/(2A)$, then

$$|\varphi(d(x, y)/t) - \varphi(d(x, z)/t)| = 0 \quad \text{if } d(x, y) \in (t/(4A), 2At),$$

$$\leq C(d(y, z)/d(x, y)^\gamma$$
otherwise.

Proof. Set \( w = \varphi(d(x, y)/t) - \varphi(d(x, z)/t) \). If \( d(x, y) \leq t/(4A) \), then, by Lemma 2, \( d(x, z) \leq t/2 \). Thus, \( w = 0 - 0 = 0 \). If \( d(x, y) \geq 2At \), then, by Lemma 2, \( d(x, z) \geq t \). Thus, \( w = 1 - 1 = 0 \). If \( t/(4A) < d(x, y) < 2At \), then, by (5),

\[
|w| \leq C|d(x, y) - d(x, z)|/t \leq C(d(y, z)/d(x, y))
\]

**Lemma 4.** If \( t > 0 \) and if \( d(y, z) \leq d(x, y)/(2A) \), then

\[
|k(x, y, t) - k(x, z, t)| \leq Cd(y, z)^{d(x, y)}^{-1/2}
\]

\[
|k(y, x, t) - k(z, x, t)| \leq Cd(y, z)^{d(x, y)}^{-1/2}
\]

Proof. We show only the first inequality. Note that

\[
|k(x, y, t) - k(x, z, t)| \leq |k(x, y) - k(x, z)| \varphi(d(x, y)/t)
\]

\[
+ |k(x, z)| \varphi(d(x, y)/t) - \varphi(d(x, z)/t)|.
\]

By (9), the first term of (22) is dominated by \( d(y, z)^{d(x, y)}^{-1/2} \). By Lemma 2, Lemma 3 and (7), the second term of (22) is also dominated by \( Cd(y, z)^{d(x, y)}^{-1/2} \).

**Lemma 5.** Let \( 1/(1 + \gamma) < p \leq 1 \) and \( u \in H^p \). Then, there exist a sequence of real numbers \( \{\lambda_j\}_{j=1}^{\infty} \) and a sequence of \( p \)-atoms \( \{a_j\}_{j=1}^{\infty} \) such that

\[
u(23) \quad u(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) \quad \text{in } H^p \quad \text{when } \nu(X) = \infty,
\]

\[
u(24) \quad u(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) + \int u \, d\nu(X) \quad \text{in } H^p \quad \text{when } \nu(X) < \infty,
\]

\[
\left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \leq c_p \|u\|_{H^p}.
\]

This is the atomic decomposition of \( H^n(X) \) which was shown by Macias-Segovia [10].

**Lemma 6.** Let \( 1/(1 + \gamma) < p \leq 1 \), \( u \in L^1 \), \( \text{supp } u \subset B(x_0, t) \) and \( t \in (0, A\mu(X)) \). Then, there exists a sequence of real numbers \( \{\lambda_j\}_{j=1}^{\infty} \) and a sequence of \( p \)-atoms \( \{a_j\}_{j=1}^{\infty} \) such that

\[
u(25) \quad u(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) + \lambda_0 a_0(x)
\]

\[
\left( \sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} \leq c_p \left( \int_{\mathbb{R}^n \cap [x_0, 2A]} u^p \, d\mu \right)^{1/p},
\]

where
\[ \lambda_0 = \int \mu \frac{1}{r} \mu(B(x_0, t)) \text{, } a_0(x) = t^{-1/p} \chi_{B(x_0, t)}(x) \]

and \( \chi_E \) denotes the characteristic function of a measurable set \( E \).

Note that \( \int \mu \mu \frac{1}{t} \leq C \inf_{x \in B(x_0, 2At)} \mu(x) \). Then, applying Lemma 5 to \( u - \lambda_0 a_0 \), we get Lemma 6.

**Lemma 7.** Let \( 1/(1 + \gamma) < p \leq \infty \). Then,

\[
\| f^{[a]} \|_p \leq c_{p, \alpha} \| f \|_{H^p} .
\]

**Proof.** It can be shown easily that

\[ f^{[a]}(x) \leq c_a |f(x)| . \]

Thus, if \( p > 1 \), (25) follows from the Hardy-Littlewood maximal theorem.

Let \( 1/(1 + \gamma) < p \leq 1 \). Note that if \( \mu(X) < \infty \), then it is trivial that \( \| \chi_x^{\alpha} \|_p \leq c_p \| \chi_x^{\alpha} \|_{\infty} \leq c_{p, \alpha} \). Thus, by Lemma 5, it suffices to show (25) for a \( p \)-atom \( a(y) \) satisfying (20). If \( y \in B(x, t/\alpha)^{s}, s > 0 \) and \( \Psi \in T_a(y, s) \),

\[
\int |a(z)\Psi(z)| d\mu(z) \\
= \left| \int a(z)(\Psi(z) - \Psi(x))d\mu(z) \right| \\
\leq \int_{B(x, t)} t^{-1/p} d(z, x)^{r} d(x, y)^{-1-\gamma} d\mu \text{ by (21)} \\
\leq t^{1-1/p+\gamma} d(x, y)^{-1-\gamma} .
\]

Thus,

\[
a^{[\alpha]}(y) \leq t^{1-1/p+\gamma} d(x, y)^{-1-\gamma} .
\]

If \( y \in B(x, t/\alpha) \), then

\[
a^{[\alpha]}(y) \leq c_a t^{-1/p} .
\]

Hence, by (26) and (27),

\[
\| a^{[\alpha]} \|_p \leq c_{p, \alpha} .
\]

**Lemma 8.** Let \( 1/(1 + \gamma) < p < \infty \). Then,

\[
\| K^* f \|_p \leq c_p \| f \|_{H^p} \\
(28)
\]

and

\[
\| K^{**} f \|_p \leq c_p \| f \|_{H^p} .
\]
Proof. If \( p > 1 \), then these follow from the well known argument about the maximal singular integral.

Let \( 1/(1 + \gamma) < p \leq 1 \). We show only (28). Note that if \( \mu(X) < \infty \), then it follows easily that

\[
||K^*\mathcal{X}_\gamma||_p \leq c_p \|K^*\mathcal{X}_\gamma\|_2 \leq c_p \|\mathcal{X}_\gamma\|_2 \leq c_p.
\]

Thus, by Lemma 5, it suffices to show (28) for a \( p \)-atom \( a(y) \) satisfying (20). If \( d(x, y) > 2At \) and \( s > 0 \), then

\[
\left| \int k(y, z, s)a(z)d\mu(z) \right| = \left| \int (k(y, z, s) - k(y, x, s))a(z)d\mu(z) \right| \leq C \int_{B(x,2At)} d(x, z)^{-1}(x, y)^{-1+\gamma}d\mu \text{ by Lemma 4}
\]

\[
\leq Ct^{1+\gamma}d(x, y)^{-1+\gamma}.
\]

Thus,

\[ K^*a(y) \leq Ct^{1+\gamma}d(x, y)^{-1+\gamma}. \]

On the other hand, since (28) has been known for \( p = 2 \), we get

\[ \int_{B(x,2At)} |K^*a|^p d\mu \leq Ct^{1-p/2}(\int |K^*a|^2 d\mu)^{p/2} \leq Ct^{1-p/2}||a||_p^p \leq C. \]

Thus, by (30) and (31), we get

\[ \int |K^*a|^p d\mu \leq c_p. \]

**Lemma 9.** Let \( \zeta(x, y) \) be a function defined on \( X \times X \) such that

\[
|\zeta(x, y)| \leq d(x, y)^{\gamma-1}
\]

\[
|\zeta(x, y) - \zeta(x, z)| \leq d(y, z)^{\gamma}/d(x, y)
\]

if \( d(y, z) < d(x, y)/(2A) \). Let \( u \in L^2 \), \( \text{supp } u \subset B(x_0, t), t \in (0, A\mu(X)) \)

\[ v(x) = \int \zeta(x, y)u(y)d\mu(y) \]

and \( 1 + \gamma > 1/s_1 > \gamma \). Then,

\[ \left( \int_{B(x_0,t)} |v|^2 d\mu \right)^{1/s_2} \leq c_{s_1} \left( \int_{B(x_0,2At)} (u^*)^2 d\mu \right)^{1/s_1} \]

where \( 1/s_2 = 1/s_1 - \gamma \).

**Proof.** If \( s_1 > 1 \), this can be shown in the same way as [13]
120. Let \(1/(1 + \gamma) < s_1 \leq 1\). Applying Lemma 6 to \(u(x)\) and \(p = s_1\), we get \(\{\lambda_j \}_{j=0}^{\infty}\) and \(\{a_j(x)\}_{j=0}^{\infty}\) such that (24). For \(j = 1, 2, 3 \ldots\), let

\[
B(x_j, t_j) \supset \text{supp } a_j, \quad t_j^{-1/s_1} \geq ||a_j||_{\infty}.
\]

For \(j = 0, 1, 2, \ldots\), let

\[
v_j(x) = \int \zeta(x, y) a_j(y) d\mu(y).
\]

Then,

\[
|v_0(x)| \leq C t_j^{-1/s_1},
\]

\[
|v_j(x)| \leq C \min(t_j^{-1/s_1}, t_j^{1+\gamma-1/s_1}/d(x, x_j)) \quad \text{for } j \geq 1.
\]

Thus, by (24) and \(s_1 \leq 1 < s_2\),

\[
\left( \int_{B(x_0, t)} |v|^s d\mu \right)^{1/s} \leq \sum_{j=0}^{\infty} |\lambda_j| \left( \int_{B(x_0, t)} |v_j|^s d\mu \right)^{1/s} \leq c_{s_1} \sum_{j=0}^{\infty} |\lambda_j| \leq c_{s_1} (\sum_{j=0}^{\infty} |\lambda_j|^{s_1})^{1/s_1} \leq c_{s_1} \left( \int_{B(x_0, 2A\mu)} (u^*)^s d\mu \right)^{1/s_1}.
\]

4. Proof of Theorem 1. We may assume \(q \leq r\). Then \(r > 1\). Let \(x \in X\) be fixed. Let \(t \in (0, \Lambda \mu(X))\) and \(\Psi \in T(x, t)\). Then

\[
\int \Psi(y) (h(y)Kg(y) - g(y)K'h(y)) d\mu(y)
\]

\[
= \int (\Psi(y)Kg(y) - K(\Psi g)(y)) h(y) d\mu(y).
\]

Set

\[
\eta(y, z) = k(y, z)(\Psi(y) - \Psi(z))g(z).
\]

Note that

\[
\Psi(y)Kg(y) - K(\Psi g)(y) = \int \eta(y, z) d\mu(z).
\]

Let

\[
d(x, y) > 16A^t.
\]

Then \(\Psi(y) = 0\). Set

\[
\int \eta(y, z) d\mu(z) = -k(y, x) \int \Psi(z)g(z) d\mu(z)
\]

\[
+ \int (k(y, x) - k(y, z)) \Psi(z)g(z) d\mu(z)
\]
If $z \in \text{supp } \Psi$, then, by (41),
\[ d(x, z) < d(y, x)/(2A) \]
Hence, by (9) and (12),
\[ |\zeta_2(y, z)| \leq d(x, y)^{-1-\tau t^{-1}}. \]
If $z_1, z_2 \in B(x, 2At)$, then, by (41) and Lemma 2,
\[ d(x, z_1) < d(y, x)/(2A) \text{ and } d(z_1, z_2) < d(y, z_1)/(2A). \]
Hence, by (9), (12) and (13),
\[ |\zeta_2(y, z_1) - \zeta_2(y, z_2)| \leq |k(y, x) - k(y, z_1)| |\Psi(z_1) - \Psi(z_2)| + |k(y, z_1) - k(y, z_2)| |\Psi(z_2)| \leq C(d(z_1, z_2)/t)^{t^{-1}}d(x, y)^{-1-\tau}. \]
Thus, by (43), (44) and $\text{supp } \zeta_2(y, \cdot) \subset B(x, t)$,
\[ Ct^{-\tau}d(x, y)^{1+\tau}\zeta_2(y, \cdot) \in T(x, t). \]
So,
\[ |\eta_2(y)| \leq Cd(x, y)^{-1-\tau t^\tau}g^s(x). \]
Let
\[ d(x, y) \leq 16A't. \]
Set
\begin{align*}
\eta_2(y, z)d\mu(z) &= \Psi(y) \int k(x, z)\varphi(d(x, z)/(\beta t))g(z)d\mu(z) \\
&\quad + \Psi(y) \int (k(y, x) - k(x, z))\varphi(d(x, z)/(\beta t))g(z)d\mu(z) \\
&\quad + \int k(y, z)(\Psi(y) - \Psi(z))\varphi'(d(x, z)/(\beta t))g(z)d\mu(z) + \Psi(y) \int \zeta_2(y, z)g(z)d\mu(z) \chi_{B(x, 16A't)}(y) \\
&= \Psi(y) \int k(x, z, \beta t)g(z)d\mu(z) + \Psi(y) \int \zeta_2(y, z)g(z)d\mu(z) \\
&\quad + \int \zeta_2(y, z)\varphi'(d(x, z)/(\beta t))g(z)d\mu(z) + \Psi(y) \int \zeta_2(y, z)\varphi'(d(x, z)/(\beta t))g(z)d\mu(z) \chi(y) \\
&= \eta_3(y) + \eta_4(y) + \eta_5(y),
\end{align*}
where $\beta = 128A^\epsilon$ and $\varphi' = 1 - \varphi$.
Since $\beta$ is sufficiently large, if $\varphi(d(x, z)/(\beta t)) \neq 0$, then
\[ d(x, y) < d(x, z)/(2A). \]
Hence, by (9),
\[ |\zeta_t(y, z)| \leq Ct(t + d(x, z))^{-1-r}. \]

Let
\[ d(z_1, z_2) < d(x, z_1)/(2A)^{\gamma}. \]

Set
\[ d(z, z_1) - |k(y, z) - k(x, z)| \leq \varphi(d(x, z_1)/(\beta t)) \]
\[ + |k(y, z_1) - k(y, z)| \varphi(d(x, z_1)/(\beta t)) \]
\[ + (|k(y, z_2)| + |k(x, z_2)|) \varphi(d(x, z_1)/(\beta t)) - \varphi(d(x, z_2)/(\beta t)) | \]
\[ = \zeta_{41} + \zeta_{42} + \zeta_{43}. \]

By (49) and (9),
\[ \zeta_{41} \leq Cd(x, z_1)^{\gamma}d(x, z_1)^{-1-r}. \]

Since \( \beta \) is sufficiently large, if \( \varphi(d(x, z_1)/(\beta t)) \neq 0 \), then, by (46) and Lemma 2,
\[ d(x, z_1)/(2A) \leq d(y, z_1). \]

Hence, by (49) and (9),
\[ \zeta_{42} \leq d(z_1, z_2)^{\gamma}d(y, z_1)^{-1-r}\varphi(d(x, z_1)/(\beta t)) \]
\[ \leq Cd(x, z_2)^{\gamma}d(x, z_1)^{-1-r}. \]

By Lemma 2,
\[ d(x, z_2) \geq d(x, z_1)/(2A). \]

If \( \zeta_{43} > 0 \), then, by Lemma 3,
\[ d(x, z_2) > \beta t/(4A). \]

So
\[ d(x, y) \leq 16A^{t} \leq 64A^{\delta}d(x, z_2)/\beta = d(x, z_2)/(2A). \]

Thus, by Lemma 2 and (53),
\[ d(y, z_2) \geq d(x, z_2)/(2A) \geq d(x, z_1)/(2A)^{\gamma}. \]

Hence, by (7), Lemma 3, (53) and (54),
\[ \zeta_{43} \leq (d(y, z_2)^{-1} + d(x, z_2)^{-1})C(d(z_1, z_2)/d(x, z_1))^{\gamma} \]
\[ \leq Cd(x, z_1)^{\gamma}d(x, z_1)^{-1-r}. \]

So, by (48), (51), (52) and (55),
\[ C\zeta_t(y, \cdot) \in T_{(2,4)^{-2}}(x, t). \]

Thus,
\begin{equation}
|\gamma_1(y) - \delta(y)| \leq C |\Psi(y)| g^{*[1/(2,4)^{-2}]}(x).
\end{equation}

By (7) and (13),
\begin{equation}
|\zeta_0(y, z)| \leq t^{-1-\gamma}d(y, z)^{-1-\gamma}.
\end{equation}

If \( d(z_1, z_2) < d(y, z_1)/(2A) \), then by (7), (9) and (13),
\begin{equation}
|\zeta_0(y, z_1) - \zeta_0(y, z_2)|
\leq |k(y, z_1)(\Psi(z_1) - \Psi(z_2))| + |k(y, z_1) - k(y, z_2)| |\Psi(z_2) - \Psi(y)|
\leq d(y, z_1)^{-1-\gamma}d(z_1, z_2)^{-\gamma} + d(y, z_1)^{-1-\gamma}d(z_1, z_2)^{-\gamma}t^{-1-\gamma}d(z_1, z_2)^{-\gamma}
\leq C d(y, z_1)^{-1-\gamma}d(z_1, z_2)^{-\gamma}.
\end{equation}

So, by (57) and (58), \( C^{1+\gamma} \zeta_0(y, z) \) satisfies the hypothesis of Lemma 9. Note that if \( z \in B(x, 2A\beta t) \),
\begin{equation}
(\phi'(d(x, \cdot))/(\beta t))g(\cdot)^*(z) \leq C g^*(z).
\end{equation}

Thus, by Lemma 9, we get
\begin{equation}
\left( \int_{B(x, 16A^4 t)} |\gamma_5|^{s_2} d\mu \right)^{1/s_2}
\leq C c_{s_1} t^{-1-\gamma} \left( \int_{B(x, 2A\beta t)} (g^*)^{s_1} d\mu \right)^{1/s_1},
\end{equation}

where \( \gamma < 1/s_1 < 1 + \gamma \) and \( 1/s_2 = 1/s_1 - \gamma \).

By (42), (45), (47) and (56),
\begin{equation}
\int \gamma(y, z) d\mu(z) = -\int g \Phi(y, x) \Phi(d(y, x)/t) + \gamma_0(y) + \gamma_0(y) + \gamma_0(y)
\end{equation}
where
\begin{equation}
|\gamma_0(y)| \leq C g^{*[1/(2,4)^{-2}]}(x) t^{\gamma}(t + d(x, y))^{-1-\gamma}.
\end{equation}

Thus,
\begin{equation}
|\gamma_0| \leq \left| \int \int \gamma(y, z) d\mu(z) h(y) d\mu(y) \right|
\leq C \left\{ g^*(x) K^* h(x) + h^*(x) K^* g(x) \\
+ \left\{ \gamma_0(y) h(y) d\mu(y) + g^{*[1/(2,4)^{-2}]}(x) M(h(x)) \right\} \right\}.
\end{equation}

Since \( 1/p = 1/q + 1/r \) and \( 1/p < 1 + \gamma \), we can take \( s_i \) such that
\begin{equation}
1 + \gamma > 1/s_1 > \max(1/q, \gamma), 1/s_2 = 1 - s_2 > 1/r.
\end{equation}
Then, by (59),

$$\int \eta_\xi(y) h(y) d\mu(y)$$

(62)

$$\leq \left( \int_{B(x, \lambda^2 t)} |\eta_\xi(y)|^2 d\mu(y) \right)^{1/2} \left( \int_{B(x, \lambda^2 t)} |h|^2 d\mu \right)^{1/2}$$

$$\leq c_\xi M_{\eta_\xi} g^*(x) M_{\eta_\xi} h(x) .$$

By (40), (60) and (62), we get

$$(hKg - gK'h)^s(x) \leq C\{g^s(x)K^s h(x) + h^s(x)K^s g(x)$$

$$+ M_{\eta_\xi} g^s(x) M_{\eta_\xi} h(x) + g^s[[2, 2]^{-1}] (x) M_{\eta_\xi} h(x) \} .$$

All the terms on the right hand side belong to $L^r$ by Lemma 1, Lemma 7, Lemma 8 and (61).

5. Proof of Theorem 2. By Lemma 5, we may assume that $f$ is a p-atom such that

$$\text{supp } f \subset B(x_0, t), \| f \|_\infty < t^{-1/p} \quad \text{and} \quad \int f d\mu = 0 .$$

Let $q \leq r$. Then $r > 1$.

Let $N$ be a large number depending only on $X$ and $p$. Then, by (8), there exists $y_0$ such that

$$A^{-2}Nt \leq d(x_0, y_0) \leq Nt, |k(y_0, x_0)| > 1/(ANt) .$$

By (9),

$$\inf \{d(x, x_0) < t, d(y, y_0) < t \} > 1/(2ANt) .$$

Let

(70)

$$h(x) = \chi_{B(y_0, t)}(x) N .$$

Then, $|K'h(x)| > C$ on $B(x_0, t)$. Let

$$g(x) = -f(x)/K'h(x_0) .$$

Then, $g \in H^q$, $h \in H^r$ and

$$\| g \|_{H^q} \| h \|_{H^r} \leq C(t^{-1/p+1/q})N t^{1/r} = CN .$$

Set

$$w(x) = f(x) - (h(x)Kg(x) - g(x)K'h(x))$$

$$= f(x)(K'h(x_0) - K'h(x))/K'h(x_0) - h(x)Kg(x)$$

$$= w_1(x) + w_2(x) .$$

Since $\text{supp } w_1 \subset B(x_0, t)$ and $\| w_1 \|_\infty \leq t^{-1/p} N^{-1}$, we see that
\[
\int_{B(x_0,4A^2 Nt)} w_{W_1}(x) d\mu(x) \\
\leq \int_{B(x_0,4A^2 Nt)} t^{-1} N^{-r_p} (1 + d(x_0, x)/t)^{-p} d\mu(x) \\
\leq c_p N^{-r_p + 1 - p} \log N.
\]

A similar estimate holds for \( w_2 \). Thus,
\[
\int_{B(x_0,4A^2 Nt)} w_{W_1}(x) d\mu(x) \leq \int_{B(x_0,4A^2 Nt)} w_{W_1} + w_{W_2} d\mu \\
\leq c_p N^{-r_p + 1 - p} \log N \to 0 \quad \text{as} \quad N \to \infty.
\]

Since \( \text{supp} \ w \subset B(x_0, 2A N t) \) and \( \int w d\mu = 0 \), by taking \( N \) sufficiently large and applying Lemma 6 to \( w(x) \), we get
\[
w(x) = \sum_{j=1}^{\infty} \lambda_j f_j(x),
\]
where \( \{f_j\}_{j=1}^{\infty} \) are \( p \)-atoms and
\[
\sum_{j=1}^{\infty} |\lambda_j|^p < 1/2.
\]

Hence,
\[
f = (hKg - gK'h) + \sum_{j=1}^{\infty} \lambda_j f_j.
\]

Applying the same argument to each \( f_j \) and repeating this process, we get the desired result.

6. Proof of Theorem 2'. Since \( \mu(X) < \infty \) and \( X \) is connected, we can easily see that for any \( \varepsilon > 0 \) and any \( p \)-atom \( a(x) \), there exist \( \{a_j(x)\}_{j=1}^{\varepsilon \text{p},} \) such that
\[
a(x) = \sum_{j=1}^{\varepsilon \text{p},} a_j(x)
\]
and that each \( a_j \) is a \( p \)-atom supported on the ball with radius \( < \varepsilon \).

Thus, for the proof of Theorem 2', we may assume that \( f \) is a \( p \)-atom such that the radius of its support is less than \( N^{-1} \mu(X) \), where \( N \) is a sufficiently large number, depending only on \( X \) and \( p \), to be determined later.

Following the proof of Theorem 2, we define \( h(x) \) by (70) and \( g(x) \) by
\[
g(x) = -f(x)/K'h(x).
\]

Then,
\[ w(x) = f(x) - (h(x)Kg(x) - g(x)K'h(x)) = -h(x)Kg(x). \]

Note that if \( y \in B(y_0, t) \), then
\[
|Kg(y)| \leq \left| \int k(y, z)f(z)d\mu(z)/K'h(x_0) \right| \\
+ \left| \int k(y, z)f(z)(1/K'h(x_0) - 1/K'h(z))d\mu(z) \right| \\
\leq C \int |k(y, z) - k(y, x_0)| |f(z)|d\mu(z) \\
+ \int |k(y, z)| |f(z)|N^{-r}d\mu(z) \\
\leq C \int (Nt)^{-1}N^{-r} |f(z)|d\mu(z) \leq CN^{-1-r}t^{-1/p}.
\]

Thus,
\[
\|g\|_q \|h\|_r \leq C \|f\|_q \|h\|_r \leq Ct^{-1/p+1/q}Nt^{1/r} = CN,
\]
\[
\int wd\mu = 0,
\]
\[
supp w \subset B(y_0, t)
\]
\[
\|w\|_\infty \leq \|h\|_\infty \sup_{y \in B(y_0, t)} |Kg(y)| \leq NCN^{-1-r}t^{-1/p}.
\]

If \( N \) is sufficiently large, then \( 2w \) is a \( p \)-atom and the radius of its support is less than \( N^{-1/\mu(X)} \). Iterating this process, we get desired result.

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