POWER SERIES RINGS OVER DISCRETE VALUATION RINGS

JIMMY T. ARNOLD
If $V$ is a discrete valuation ring with Krull dimension $m$, it is shown that the power series ring $V[[x_1, \cdots, x_n]]$ has Krull dimension $mn + 1$.

Throughout the paper all rings are assumed to be commutative with identity and the ring $R$ is not considered to be a prime ideal of $R$. In [1] the author defines a ring to have the SFT (strong finite type) property if for each ideal $A$ of $R$ there exists a finitely generated ideal $B$ and a positive integer $k$ such that $B \subseteq A$ and $a^k \in B$ for each $a \in A$. It is shown in [1, Theorem 1] that if $R$ fails to have the SFT-property then the power series ring $R[[Y]]$ has infinite Krull dimension. On the other hand, if $D$ is a Prüfer domain with $\dim D = m$ and if $D$ has the SFT-property then $\dim D[[Y]] = m + 1$ [2, Theorem 3.8].

Recall that a valuation ring $V$ with finite Krull dimension is discrete if and only if $P \neq P^2$ for each nonzero prime ideal $P$ of $V$ [5, pp. 190–192]. A valuation ring $V$ has the SFT-property if and only if it is discrete [2, Proposition 3.1]. Thus, if $V$ is a valuation ring and $\dim V = m$ then either $V$ is discrete and $\dim V[[Y]] = m + 1$ (this specific result was proved by Fields in [4, Theorem 2.7]) or $V$ is nondiscrete and $\dim V[[Y]] = \infty$. For $\dim R = m$ the author asks in [1, p. 303] if either $\dim R[[Y]] = m + 1$ or $\dim R[[Y]] = \infty$. We show that the answer is no for ring $V[[x_1, \cdots, x_{n-1}]]$, where $V$ is a discrete valuation ring with $\dim V \geq 2$. Specifically, we prove the following theorem.

**Theorem.** If $V$ is a discrete valuation ring with Krull dimension $m$ then the power series ring $V[[x_1, \cdots, x_n]]$ has Krull dimension $mn + 1$.

**Proof.** The proof is by induction on $m$ and the case $m = 1$ is well-known since, in this case, $V$ is Noetherian (cf. Lemma 2.6 of [4]). Thus assume that $m \geq 2$, that the theorem holds if $\dim V = m - 1$, and suppose that $(0) = P_1 \supseteq P_2 \supseteq \cdots \supseteq P_m$ is the set of prime ideals of $V$. Throughout the proof $X$ denotes the set $\{x_1, \cdots, x_n\}$ of analytic indeterminates over $V$, $V[[X]]$ denotes the power series ring $V[[x_1, \cdots, x_n]]$, $p \in P_1 \setminus P_i$, $W = V_{P_i}$, $U = V/P_i$, $F = W/P_i W$ and, even though $P_i = P_i W$, we write $\mathcal{P}$ to denote the ideal $P_i W$. We note that $W$ is a rank one discrete valuation ring with maximal ideal $\mathcal{P} = p W$, $F$ is the quotient field of $U$, and for each integer $k \geq 1$
we have $P_{k+1} \subseteq P^k P_1$. If $\xi \in (W[X])_{W \setminus \{0\}}$ then there exists a nonzero element $a$ in $W$ such that $a\xi \in W[X]$. But then $pa\xi \in V[X]$ and $pa \in V$ so $\xi \in (V[X])_{V \setminus \{0\}}$. This shows that $(W[X])_{W \setminus \{0\}} \subseteq (V[X])_{V \setminus \{0\}}$ and the reverse containment is obvious so equality holds. It follows that the correspondence $Q \to Q \cap V[X]$ is a bijection from the set

$$\{Q \in \text{Spec}(W[X]) \mid Q \cap W = \{0\}\}$$

to the set $\{Q' \in \text{Spec}(V[X]) \mid Q' \cap V = \{0\}\}$ which preserves set containment. Thus, if $Q \in \text{Spec}(W[X])$ and $Q \cap W = \{0\}$, then rank $Q = \text{rank } (Q \cap V[X])$ and it follows that rank $Q' \leq n$ for each $Q' \in \text{Spec}(V[X])$ such that $Q' \cap V = \{0\}$.

Let $(0) \subset Q_1 \subset \cdots \subset Q_t = P_m + (X)$ be a maximal chain of prime ideals of $V[X]$ and choose $k$ so that $Q_k \cap V = \{0\}$ while $Q_{k+1} \cap V \neq \{0\}$. Then, as we have already observed, $k = \text{rank } Q_k \leq n$. Since $p \in Q_{k+1}$, we have $(P_i[X])^* \subseteq P_i^* P_i[X] \subseteq p P_i[X] \subseteq Q_{k+1}$ and hence $Q_{k+1} \supseteq P_i[X]$.

We first consider the case in which $Q_{k+1} \neq P_i[X]$. It follows from Theorem 3.14 of [3] that there exist elements $\lambda_1 = x_1, \lambda_2, \cdots, \lambda_n$ in $x_i F[x_i]$ such that the $U[x_i]$-homomorphism $\phi: U[X] \to U[\lambda_1, \cdots, \lambda_n]$ determined by $\phi(x_i) = \lambda_i$, $1 \leq i \leq n$, is an isomorphism. But $\phi$ extends to an $F[x_i]$-epimorphism $\phi: F[X] \to F[x_i]$ and if $\bar{Q}$ is the kernel of $\phi$ then depth $\bar{Q} = 1$, rank $\bar{Q} = n - 1$ [6, Corollary 1, p. 218], and $\bar{Q} \cap U[X] = \{0\}$. Since $F[X] = (W/\mathcal{P})[X] \cong W[X]/\mathcal{P}[X]$ and $U[X] \cong V[X]/P_i[X]$, $\bar{Q}$ determines a prime ideal $Q$ of $W[X]$ such that depth $Q = 1$, rank$(Q/\mathcal{P}[X]) = n - 1$, and $Q \cap V[X] = P_i[X]$. Therefore, rank $Q \geq n$ and, since $\dim W[X] = n + 1$, it follows that rank $Q = n$. If we choose $f_1, \cdots, f_{n-1} \in X W[X]$ such that the corresponding elements $\bar{f}_1, \cdots, \bar{f}_{n-1}$ in $F[X]$ form a regular system of parameters for $(F[X])_{\bar{Q}}$, then $\{f_i, \cdots, f_{n-1}, p\}$ is a regular system of parameters for $(W[X])_Q$ and the ideal $N'_i = (f_i, \cdots, f_{n-1})(W[X])_Q$ is a prime ideal of $(W[X])_Q$ for $1 \leq i \leq n - 1$ (cf. Corollary 1, p. 302 and Theorem 26, p. 303 of [6]). In particular, $N_{n-1} = N'_n \cap W[X]$ is a prime ideal of $W[X]$ such that rank $N_{n-1} = n - 1$, $N_{n-1} \subseteq Q$, and $N_{n-1} \cap W = \{0\}$. We now have $P_i[X] = Q \cap V[X] \supseteq N_{n-1} \cap V[X]$ and rank $(N_{n-1} \cap V[X]) = rank N_{n-1} = n - 1$—that is, rank $P_i[X] \geq n$. Therefore, $k + 1 = rank Q_{k+1} \geq 1 + rank P_i[X] \geq n + 1$. We have already seen that $k \leq n$, so $k = n$ and rank $P_i[X] = 1$. Thus, $P_i[X] / P_i[X] \subseteq Q_{k+1} / P_i[X] \subseteq \cdots \subseteq Q_i / P_i[X]$ is a maximal chain of prime ideals in $V[X] / P_i[X] \cong \mathcal{U}[X]$ of length $t - k$. By assumption $t - k = (m - 1)n + 1$ and since $k = n$ this implies that $t = mn + 1$.

We now consider the case in which $Q_{k+1} = P_i[X]$. It follows from the previous argument that $n \leq rank P_i[X] = rank Q_{k+1} = k + 1$. We will show that equality holds. Let $\mathcal{P}$ be a valuation overring of $V[X]$ with prime ideals $(0) \subset Q'_1 \subset \cdots \subset Q'_{k+1}$ such that $Q'_i \cap V[X] = Q'_i$.
for each \( i \). Since \( Q_k \cap V = (0) \) we may assume that \( Q'_{k+1} = \text{rad}(p'V') \) and, by localizing if necessary, we assume that \( Q'_{k+1} \) is the maximal ideal of \( V' \). We wish to show that \( V' \supseteq W[X] \). If this is not the case then there exists \( h \in W[X] \) such that \( h^{-1} \in Q'_{k+1} \). If \( f = ph \) then \( f \in P_i[X] \), \( h^{-1} = p/f \), and there exists an integer \( s \) such that \( h^s = p^s/f^s = p\zeta \) for some \( \zeta \in V' \). But \( f^s \in (P_i[X])^s \subseteq p^s-1P_i[X] \) so we have \( p\zeta = p^s/p^{s-1}f_1 \) for some \( f_1 \in P_i[X] \). Therefore, \( 1/f_1 = \zeta \in V' \) contrary to the assumption that \( P_i[X] \not\subseteq V' \). It follows that \( W[X] \subseteq V' \) and if \( Q''_i = Q'_i \cap W[X] \) for \( 1 \leq i \leq k + 1 \) then \( 0 \subset Q''_1 \subset \cdots \subset Q''_{k+1} \) is a chain of prime ideals of \( W[X] \) such that \( Q'' \cap V[X] = Q_i \). In particular, \( Q''_{k+1} \cap V[X] = P_i[X] \). Since \( [P + (X)] \cap V[X] = P_i + (X) \) it follows that \( Q''_{k+1} \) is not maximal in \( W[X] \). Thus, \( n + 1 > \text{rank } Q''_{k+1} \geq k + 1 \) — that is, \( k < n \). It follows that \( k = n - 1 \) and this together with the previous argument shows that, in either case, \( \text{rank } P_1[X] = n \). We now have that \( P_i[X]/P_i[X] \subset P_{i+1}/P_i[X] \subset \cdots \subset Q_i/P_i[X] \) is a maximal chain of prime ideals in \( V[X]/P_i[X] \cong U[X] \) of length \( t - (k + 1) = t - n \). By assumption, \( t - n = (m - 1)n + 1 \), so \( t = mn + 1 \).

**Remark.** The proof of the theorem shows that if \( (0) = P_0 \subset P_1 \subset \cdots \subset P_m \) is the set of prime ideals of a discrete valuation ring \( V \) then each of the prime ideals \( P_i[x_i, \ldots, x_n] \) can be included in a maximal chain of prime ideals of \( V[x_1, \ldots, x_n] \) and for \( 0 < i < m \) we have rank \( (P_i[x_i, \ldots, x_n]/P_{i-1}[x_i, \ldots, x_n]) = n \).

**References**


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