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POWER SERIES RINGS OVER DISCRETE VALUATION RINGS

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If V is a discrete valuation ring with Krull dimension m , it is shown that the power series ring $V[[x_1, \dots, x_n]]$ has Krull dimension $mn + 1$.

Throughout the paper all rings are assumed to be commutative with identity and the ring R is not considered to be a prime ideal of R . In [1] the author defines a ring to have the SFT (strong finite type) property if for each ideal A of R there exists a finitely generated ideal B and a positive integer k such that $B \subseteq A$ and $a^k \in B$ for each $a \in A$. It is shown in [1, Theorem 1] that if R fails to have the SFT-property then the power series ring $R[[Y]]$ has infinite Krull dimension. On the other hand, if D is a Prüfer domain with $\dim D = m$ and if D has the SFT-property then $\dim D[[Y]] = m + 1$ [2, Theorem 3.8]. Recall that a valuation ring V with finite Krull dimension is discrete if and only if $P \neq P^2$ for each nonzero prime ideal P of V [5, pp. 190–192]. A valuation ring V has the SFT-property if and only if it is discrete [2, Proposition 3.1]. Thus, if V is a valuation ring and $\dim V = m$ then either V is discrete and $\dim V[[Y]] = m + 1$ (this specific result was proved by Fields in [4, Theorem 2.7]) or V is nondiscrete and $\dim V[[Y]] = \infty$. For $\dim R = m$ the author asks in [1, p. 303] if either $\dim R[[Y]] = m + 1$ or $\dim R[[Y]] = \infty$. We show that the answer is no for ring $V[[x_1, \dots, x_{n-1}]]$, where V is a discrete valuation ring with $\dim V \geq 2$. Specifically, we prove the following theorem.

THEOREM. *If V is a discrete valuation ring with Krull dimension m then the power series ring $V[[x_1, \dots, x_n]]$ has Krull dimension $mn + 1$.*

Proof. The proof is by induction on m and the case $m = 1$ is well-known since, in this case, V is Noetherian (cf. Lemma 2.6 of [4]). Thus assume that $m \geq 2$, that the theorem holds if $\dim V = m - 1$, let $\dim V = m$, and suppose that $(0) = P_0 \subset P_1 \subset P_2 \subseteq \dots \subseteq P_m$ is the set of prime ideals of V . Throughout the proof X denotes the set $\{x_1, \dots, x_n\}$ of analytic indeterminates over V , $V[[X]]$ denotes the power series ring $V[[x_1, \dots, x_n]]$, $p \in P_1 \setminus P_1^2$, $W = V_{P_1}$, $U = V/P_1$, $F = W/P_1W$ and, even though $P_1 = P_1W$, we write \mathcal{P} to denote the ideal P_1W . We note that W is a rank one discrete valuation ring with maximal ideal $\mathcal{P} = pW$, F is the quotient field of U , and for each integer $k \geq 1$

we have $P_1^{k+1} \subseteq p^k P_1$. If $\xi \in (W[X])_{W \setminus (0)}$ then there exists a nonzero element a in W such that $a\xi \in W[X]$. But then $pa\xi \in V[X]$ and $pa \in V$ so $\xi \in (V[X])_{V \setminus (0)}$. This shows that $(W[X])_{W \setminus (0)} \subseteq (V[X])_{V \setminus (0)}$ and the reverse containment is obvious so equality holds. It follows that the correspondence $Q \rightarrow Q \cap V[X]$ is a bijection from the set

$$\{Q \in \text{Spec}(W[X]) \mid Q \cap W = (0)\}$$

to the set $\{Q' \in \text{Spec}(V[X]) \mid Q' \cap V = (0)\}$ which preserves set containment. Thus, if $Q \in \text{Spec}(W[X])$ and $Q \cap W = (0)$, then $\text{rank } Q = \text{rank}(Q \cap V[X])$ and it follows that $\text{rank } Q' \leq n$ for each $Q' \in \text{Spec}(V[X])$ such that $Q' \cap V = (0)$.

Let $(0) \subset Q_1 \subset \cdots \subset Q_t = P_m + (X)$ be a maximal chain of prime ideals of $V[X]$ and choose k so that $Q_k \cap V = (0)$ while $Q_{k+1} \cap V \neq (0)$. Then, as we have already observed, $k = \text{rank } Q_k \leq n$. Since $p \in Q_{k+1}$ we have $(P_1[X])^2 \subseteq P_1^2[X] \subseteq pP_1[X] \subseteq Q_{k+1}$ and hence $Q_{k+1} \supseteq P_1[X]$.

We first consider the case in which $Q_{k+1} \neq P_1[X]$. It follows from Theorem 3.14 of [3] that there exist elements $\lambda_1 = x_1, \lambda_2, \dots, \lambda_n$ in $x_1 F[x_1]$ such that the $U[x_1]$ -homomorphism $\phi: U[X] \rightarrow U[\lambda_1, \dots, \lambda_n]$ determined by $\phi(x_i) = \lambda_i$, $1 \leq i \leq n$, is an isomorphism. But ϕ extends to an $F[x_1]$ -epimorphism $\bar{\phi}: F[X] \rightarrow F[x_1]$ and if \bar{Q} is the kernel of $\bar{\phi}$ then $\text{depth } \bar{Q} = 1$, $\text{rank } \bar{Q} = n - 1$ [6, Corollary 1, p. 218], and $Q \cap U[X] = (0)$. Since $F[X] = (W/\mathcal{S})[X] \cong W[X]/\mathcal{S}[X]$ and $U[X] \cong V[X]/P_1[X]$, \bar{Q} determines a prime ideal Q of $W[X]$ such that $\text{depth } Q = 1$, $\text{rank}(Q/\mathcal{S}[X]) = n - 1$, and $Q \cap V[X] = P_1[X]$. Therefore, $\text{rank } Q \geq n$ and, since $\dim W[X] = n + 1$, it follows that $\text{rank } Q = n$. If we choose $f_1, \dots, f_{n-1} \in XW[X]$ such that the corresponding elements $\bar{f}_1, \dots, \bar{f}_{n-1}$ in $F[X]$ form a regular system of parameters for $(F[X])_{\bar{Q}}$, then $\{f_1, \dots, f_{n-1}, p\}$ is a regular system of parameters for $(W[X])_Q$ and the ideal $N'_i = (f_1, \dots, f_i)(W[X])_Q$ is a prime ideal of $(W[X])_Q$ for $1 \leq i \leq n - 1$ (cf. Corollary 1, p. 302 and Theorem 26, p. 303 of [6]). In particular, $N_{n-1} = N'_{n-1} \cap W[X]$ is a prime ideal of $W[X]$ such that $\text{rank } N_{n-1} = n - 1$, $N_{n-1} \subset Q$, and $N_{n-1} \cap W = (0)$. We now have $P_1[X] = Q \cap V[X] \supset N_{n-1} \cap V[X]$ and $\text{rank}(N_{n-1} \cap V[X]) = \text{rank } N_{n-1} = n - 1$ —that is, $\text{rank } P_1[X] \geq n$. Therefore, $k + 1 = \text{rank } Q_{k+1} \geq 1 + \text{rank } P_1[X] \geq n + 1$. We have already seen that $k \leq n$, so $k = n$ and $\text{rank}(Q_{k+1}/P_1[X]) = 1$. Thus, $P_1[X]/P_1[X] \subset Q_{k+1}/P_1[X] \subset \cdots \subset Q_t/P_1[X]$ is a maximal chain of prime ideals in $V[X]/P_1[X] \cong U[X]$ of length $t - k$. By assumption $t - k = (m - 1)n + 1$ and since $k = n$ this implies that $t = mn + 1$.

We now consider the case in which $Q_{k+1} = P_1[X]$. It follows from the previous argument that $n \leq \text{rank } P_1[X] = \text{rank } Q_{k+1} = k + 1$. We will show that equality holds. Let \mathcal{V} be a valuation overring of $V[X]$ with prime ideals $(0) \subset Q'_1 \subset \cdots \subset Q'_{k+1}$ such that $Q'_i \cap V[X] = Q_i$

for each i . Since $Q_k \cap V = (0)$ we may assume that $Q'_{k+1} = \text{rad}(p\mathcal{V})$ and, by localizing if necessary, we assume that Q'_{k+1} is the maximal ideal of \mathcal{V} . We wish to show that $\mathcal{V} \cong W[X]$. If this is not the case then there exists $h \in W[X]$ such that $h^{-1} \in Q'_{k+1}$. If $f = ph$ then $f \in P_1[X]$, $h^{-1} = p/f$, and there exists an integer s such that $h^{-s} = p^s/f^s = p\zeta$ for some $\zeta \in \mathcal{V}$. But $f^s \in (P_1[X])^s \subseteq p^{s-1}P_1[X]$ so we have $p\zeta = p^s/p^{s-1}f_1$ for some $f_1 \in P_1[X]$. Therefore, $1/f_1 = \zeta \in \mathcal{V}$ contrary to the assumption that $P_1[X] \subseteq Q'_{k+1}$. It follows that $W[X] \subseteq \mathcal{V}$ and if $Q''_i = Q'_i \cap W[X]$ for $1 \leq i \leq k+1$ then $(0) \subset Q''_1 \subset \cdots \subset Q''_{k+1}$ is a chain of prime ideals of $W[X]$ such that $Q''_i \cap V[X] = Q_i$. In particular, $Q''_{k+1} \cap V[X] = P_1[X]$. Since $[\mathcal{S} + (X)] \cap V[X] = P_1 + (X)$ it follows that Q''_{k+1} is not maximal in $W[X]$. Thus, $n+1 > \text{rank } Q''_{k+1} \geq k+1$ — that is, $k < n$. It follows that $k = n-1$ and this together with the previous argument shows that, in either case, $\text{rank } P_1[X] = n$. We now have that $P_1[X]/P_1[X] \subset Q_{k+2}/P_1[X] \subset \cdots \subset Q_i/P_1[X]$ is a maximal chain of prime ideals in $V[X]/P_1[X] \cong U[X]$ of length $t - (k+1) = t - n$. By assumption, $t - n = (m-1)n + 1$, so $t = mn + 1$.

REMARK. The proof of the theorem shows that if $(0) = P_0 \subset P_1 \subset \cdots \subset P_m$ is the set of prime ideals of a discrete valuation ring V then each of the prime ideals $P_i[x_1, \dots, x_n]$ can be included in a maximal chain of prime ideals of $V[x_1, \dots, x_n]$ and for $0 < i < m$ we have $\text{rank } (P_i[x_1, \dots, x_n]/P_{i-1}[x_1, \dots, x_n]) = n$.

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