SEQUENTIAL TESTING OF SEVERAL SIMPLE HYPOTHESES FOR A DIFFUSION PROCESS AND THE CORRESPONDING FREE BOUNDARY PROBLEM

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An \( n \)-dimensional stochastic process \( \xi(t) \) is observed. It is known that \( \xi(t) \) has the statistics of an \( n \)-dimensional Brownian motion with any one of possibly \( n+1 \) drifts \( \lambda_0, \ldots, \lambda_n \) (\( \lambda_i \) are given \( n \)-vectors). We observe the process at a running cost, per unit time, given by \( c_i \) when the drift is \( \lambda_i \) and after some (stopping) time \( \tau \) make a decision which hypothesis to accept; the hypothesis \( H_i \) means accepting the drift \( \lambda_i \); the drift changes in time in accordance with a Markov process with \( n+1 \) states and a given transition probability matrix. The problem of finding the optimal stopping time and optimal final decision leads to a variational inequality for a degenerate elliptic operator. In this paper we study this variational inequality and the corresponding free boundary. We also consider, by purely probabilistic methods, the case where \( \xi(t) \) is \( k \)-dimensional, \( k \neq n \). The outline of the main results is given at the end of \( \S \) 2.

1. The sequential testing problem. Let \( q_{i,j}(0 \leq i, j \leq n) \) be real numbers such that \( q_{i,j} \geq 0 \) if \( i \neq j \), \( q_{i,i} \leq 0 \), and \( \sum_{j=0}^n q_{i,j} = 0 \) for \( 0 \leq i \leq n \). In a probability space \( (\Omega, \mathcal{F}, P) \) we are given a Markov process \( \theta(t) = \theta(t, w) \) taking values \( 0, 1, \ldots, n \) and having the infinitesimal matrix \( (q_{i,j}) \). We are also given an \( n \)-dimensional Brownian motion \( w(t) \) (with \( w(0) = 0 \)) independent of the process \( \theta(t) \). Let \( \lambda_0, \lambda_1, \ldots, \lambda_n \) be \( n \)-dimensional vectors which span \( R^n \), that is

\[
(1.1) \quad \lambda_1 - \lambda_0, \lambda_2 - \lambda_0, \ldots, \lambda_n - \lambda_0 \quad \text{are linearly independent}.
\]

Consider the process \( \xi(t) \) in \( R^n \) given by

\[
(1.2) \quad d\xi(t) = dw(t) + \sum_{j=0}^n I_{\{\theta(t) = j\}} \lambda_j dt
\]

that is, on the set \( \theta(t, \omega) = j \) \( d\xi(t, \omega) = dw(t, \omega) + \lambda_j dt \). We set \( \mathcal{F}_s = \sigma(\xi(s), 0 \leq s \leq t) \).

When \( \theta(t, \omega) = j \) we say that the hypothesis \( H_j \) is satisfied (at time \( t \)). We shall be concerned with the problem of deciding which hypothesis to accept at a minimal cost. We follow Bayes' formulation in setting up the problem:
The observed process is $\xi(t)$. We are given an a priori probability $\pi$ for $\theta(0)$, that is, we are given

$$\pi = (\pi_0, \pi_1, \ldots, \pi_n), \quad \pi_i \geq 0, \quad \sum_{i=0}^{n} \pi_i = 1$$

and make the initial assumption that $\theta(0) = j$ with probability $\pi_j$. This determines a probability $P^\pi$ on the space of paths $(\theta(t), w(t))$ with $w(0) = 0$, and

$$P^\pi(\theta(0) = j) = \pi_j, \quad 0 \leq j \leq n.$$ 

We shall denote the expectation with respect to $P^\pi$ by $E^\pi$. The running cost (per unit time) of the observation of $\xi(t)$ is a given positive number $c_j$ if $\theta(t) = j$. We observe the process $\xi(t)$ for an amount of time $\tau$, where $\tau$ is a stopping time with respect to $\mathcal{F}_t$; the incurred cost is then

$$E^\pi \left[ \int_0^\tau f(\theta(t))dt \right], \quad \text{where} \quad f(j) = c_j \quad (0 \leq j \leq n).$$

At the time $t = \tau$ we make a terminal decision $d(\omega)$ as to which hypothesis to accept; $d(\omega) = j$ means accepting the hypothesis $H_j$. The variable $d(\omega)$ is taken to be $\mathcal{F}_\tau$ measurable. Set

$$W(\theta, d) = a_i \quad \text{if} \quad d = i, \theta \neq i \quad (a_i > 0),$$

i.e., $a_i$ is the cost for erroneously accepting the hypothesis $H_i$. The cost of the terminal decision is

$$E^\pi[ W(\theta(\tau, \omega), d(\omega)) ]$$

and the total cost for the decision $\delta = (\tau, d)$ is

$$J^\pi(\delta) = E^\pi \left[ \int_0^\tau f(\theta(t))dt + W(\theta(\tau, \omega), d(\omega)) \right].$$

More generally, introducing a discount factor $\alpha, \alpha \geq 0$, the total cost becomes

$$J^\pi(\delta) = E^\pi \left[ \int_0^\tau e^{-\alpha t} f(\theta(t))dt + e^{-\alpha \tau} W(\theta(\tau, \omega), d(\omega)) \right].$$

The problem is to study the least cost function

$$V(\pi) = \inf_{\delta} J^\pi(\delta)$$

and to find an optimal decision $\hat{\delta} = (\hat{\tau}, \hat{d})$, that is,

$$J^\pi(\hat{\delta}) = V(\pi).$$
This problem is called a \textit{sequential testing problem of }n + 1 \textit{simple hypotheses }\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_n. \textit{The case where}

\begin{equation}
\theta \text{ does not depend on } t, \text{ that is, } q_{i,j} = 0 \text{ for } 0 \leq i, j \leq n; \ c_i = c > 0 \ \text{ for } \leq i \leq n
\end{equation}

\text{will be called the \textit{special case}; more refined results will be proved for this case.}

The sequential testing problem in the special case with \(n = 2\) has been studied in detail (see Shiryayev [15] and the references given there). In the case of discrete times the problem (in the special case) was studied by Wald [16], Chow and Robins [7], Shiryayev [14], Kiefer and Sacks [11] and others.

Analogously to the case \(n = 2\) we introduce the a posteriori probability process

\[\pi(t) = (\pi_0(t), \pi_1(t), \cdots, \pi_n(t))\]

Introducing the simplex in \(\mathbb{R}^{n+1}\)

\[
\Pi_{n+1} = \left\{ \pi = (\pi_0, \pi_1, \cdots, \pi_n); \pi_i \geq 0, \ \sum_{i=1}^{n} \pi_i = 1 \right\}
\]

it is clear that \(\pi(t) \in \Pi_{n+1}\) for all \(t > 0\). The process \(\pi(t)\) was studied by Shiryayev (see [13]) and by Anderson and Friedman [2]. It is shown in these references that \(\pi(t)\) is a Markov process with generator

\[
\mathbf{M}_u(\pi) = \frac{1}{2} \sum_{i,j=0}^{n} \pi_i \pi_j \left( \lambda_i - \sum_{k=0}^{n} \lambda_k \pi_k \right).
\]

(1.10)

\[-\sum_{i,j=0}^{n} \pi_i \pi_j \frac{\partial^2 u(\pi)}{\partial \pi_i \partial \pi_j} + \sum_{i,j=0}^{n} q_{i,j} \pi_i \frac{\partial u(\pi)}{\partial \pi_j}\]

and (in [2]) explicit formulas are given for \(\pi_j(t)\) in terms of \(\xi(t)\). In particular, when (1.9) holds,

\begin{equation}
\pi_j(t) = \pi_j \left\{ \sum_{k=0}^{n} \pi_k z_{j,k}(t) \right\}^{-1},
\end{equation}

(1.11)

\[z_{j,k}(t) = \exp \left\{ (\lambda_k - \lambda_j) \cdot \xi(t) - \frac{1}{2} (||\lambda_k||^2 - ||\lambda_j||^2)t \right\}.\]

As in [2] [13; p. 167] we can express \(J_\pi(\delta)\) in terms of the process \(\pi(t)\):

\begin{equation}
J_\pi(\delta) = E^\pi \left\{ \int_0^\tau e^{-\alpha t} h(\pi(t))dt + e^{-\alpha \tau} \sum_{i=0}^{n} (1 - \pi_i(\tau))a_i I_{d(\omega) = i} \right\},
\end{equation}

(1.12)
where \( h(\pi) = \sum_{i=0}^{n} c_i \pi_i \).

Set

\[
J_\pi(\tau) = \inf_{\delta} J_\pi(\delta) \quad \text{where} \quad (\tau, d) = \delta .
\]

For a given \( \tau \) the optimal \( d = d(\omega) \) is such that it minimizes \( \sum (1 - \pi_i(\tau(\omega)))a_i \). Consequently,

\[
J_\pi(\tau) = E^x \left[ \int_0^\tau e^{-\alpha t}h(\pi(t))dt + e^{-\alpha \tau}g(\pi(\tau)) \right]
\]

where

\[
g(\pi) = \min_{0 \leq t \leq n} \{a_i(1 - \pi_i)\} .
\]

The problem associated with (1.7), (1.8) thus reduces to the problem associated with

\[
V(\pi) = \inf_{\tau} J_\pi(\tau)
\]

(where \( V(\pi) \) is the same as in (1.17)) and

\[
J_\pi(\tilde{\tau}) = V(\pi) ,
\]

where \( \tau, \tilde{\tau} \) are stopping times with respect to \( \mathcal{F}_t \).

In the sequel we shall study the hypothesis testing problem in its formulation (1.15), (1.16). For simplicity we shall also always assume that \( \alpha > 0 \); the results in case \( \alpha = 0 \) are still valid, but require some changes in the proofs; we consider this case briefly in § 10.

2. The variational inequality. Let \( \hat{II}_{n+1} = \text{int} II_{n+1} \).

As in [2], the function \( V(\pi) \) in \( \hat{II}_{n+1} \) can be characterized as the bounded solution \( u \) of a certain system of differential in equalities:

\[
Mu - \alpha u + h \geq 0 \quad \text{a.e. in } \hat{II}_{n+1} ,
\]

\[
u(\pi) \leq h(\pi) \quad \text{in } II_{n+1} ,
\]

\[
(Mu - \alpha u + h)(u - g) = 0 \quad \text{a.e. in } II_{n+1} .
\]

Such a system is called a \textit{variational inequality} (for a general study of variational inequalities see, for instance, [3] [9]).

We recall [2] that, because of (1.1), \( M \) is a nondegenerate elliptic operator (in \( n \) independent variables) in \( \hat{II}_{n+1} \). It degenerates however on the boundary \( \partial II_{n+1} \).

\textbf{Lemma 2.1.} (a) \textit{If } \pi = \pi(0) \text{ belongs to } \hat{II}_{n+1} \text{ then } \pi(t) \in \hat{II}_{n+1} \text{ for}
all \( t > 0 \), and (b) if (1.9) holds and if \( \pi_i = \pi_i(0) = 0 \) for some \( i \), then \( \pi_i(t) = 0 \) for all \( t > 0 \).

The assertion (a) follows from the formula for \( \pi_j(t) \) given in [2]. The assertion (b) follows from (1.11).

From (a) it follows that no boundary Dirichlet conditions are needed to be given on \( \partial \Pi_{n+1} \) in order to solve the variational inequality (2.1). The solution of (2.1) can be constructed as follows (cf. [2]):

For any \( \delta > 0, \varepsilon > 0 \), let

\[
H_{n+1}^\delta = \{ \pi \in H_{n+1}, \pi_i > \delta \text{ for } 0 \leq i \leq n \}
\]

and let \( \alpha_i(t) \) be a \( C^\infty \) function in \( t \) satisfying:

\[
\begin{align*}
\beta_i'(t) &\geq 0, \quad \beta_i''(t) \geq 0; \\
\beta_i(t) &\to 0 \text{ if } t < 0, \quad \varepsilon \downarrow 0, \\
\beta_i(t) &\to \infty \text{ if } t > 0, \quad \varepsilon \downarrow 0.
\end{align*}
\]

Consider the elliptic problem

\[
-Mu + \alpha u + \beta_i(u - g) = h \quad \text{in } \Pi_{n+1}^\delta,
\]

\[
u = \varphi \quad \text{on } \partial \Pi_{n+1}^\delta
\]

where \( \varphi \) is any smooth function such that

\[
0 \leq \varphi \leq g.
\]

This problem has a unique solution \( u = u_{\delta, \varepsilon} \). If \( g(\pi) \) were a function in \( W^{2,p} \), for any \( 2 \leq p < \infty \), then one can show, by standard techniques for variational inequalities, that

\[
u_{\delta, \varepsilon} \to u_{\delta} \text{ uniformly as } \varepsilon \to 0,
\]

where \( u_{\delta} \) is the unique solution of the variational inequality

\[
-Mu + \alpha u \leq h \quad \text{a.e. in } \Pi_{n+1}^\delta,
\]

\[
u \leq g \quad \text{in } \Pi_{n+1}^\delta,
\]

\[
(-Mu + \alpha u - h)(u - g) = 0 \quad \text{a.e. in } \Pi_{n+1}^\delta,
\]

\[
u = \varphi \quad \text{on } \partial \Pi_{n+1}^\delta,
\]

\[
u \in W^{2,p}_{\text{loc}}(\Pi_{n+1}^\delta), \quad u \in C(\Pi_{n+1}^\delta).
\]

In the present case \( g \) is not even continuously differentiable. Since however it is the minimum of linear functions in the \( \pi_i \), it is convex. Thus, in terms of, say, \( \pi_i, \ldots, \pi_n \), \n
\[
\left( \frac{\partial^2 g}{\partial \pi_i \partial \pi_j} \right)
\]

is negative semidefinite matrix,

where \( \partial^2 g/\partial \pi_i \partial \pi_j \) is taken in the sense of distributions. By [4] it
follows that
\[ |u_{\delta, \epsilon}|_{W^{2, \infty}(G)} \leq C \quad \text{if} \quad G \subset \tilde{\Omega}_{n+1} \]
where \( C \) is a constant independent of \( \delta, \epsilon \), and then (2.5) is still valid. It follows that

(2.7) \[ |u_\delta|_{W^{2, \infty}(G)} \leq C. \]

We now take \( \delta \to 0 \) and deduce (as in [2]) that

(2.8) \[ u_\delta \to u \quad \text{uniformly in compact subsets of} \quad \tilde{\Omega}_{n+1} \]

where \( u \) is a solution of the variational inequality (2.1); further (by a probabilistic argument),

(2.9) \[ u \text{ has a continuous extension into } \Omega_{n+1}, \]

and, by (2.7),

(2.10) \[ u \in W^{2, \infty}_{\text{loc}}(\tilde{\Omega}_{n+1}). \]

The uniqueness of the solution \( u \) subject to the smoothness conditions (2.9), (2.10) follows (as in [2]) by using Ito's formula.

We recall that \( u \) can also be obtained as follows:

(2.11) \[ u = \lim_{\epsilon \to 0} \lim_{\delta \to 0} u_{\delta, \epsilon}. \]

Let

\[ S = \{ \pi \in \Omega_{n+1}; \ u(\pi) = g(\pi) \}, \quad C = \{ \pi \in \Omega_{n+1}; \ u(\pi) < g(\pi) \}. \]

As in [2], \( V(\pi) \) defined by (1.15) coincides in \( \tilde{\Omega}_{n+1} \) with the solution \( u \) of (2.1), and an optimal stopping time \( \tilde{\tau} \) (as in (1.16)) is given by

(2.12) \[ \tilde{\tau} = \text{hitting time of } S \text{ by the process } \pi(t). \]

Thus the optimal strategy is to continue while \( \pi(t) \) is in \( C \) and to stop when \( \pi(t) \) hits \( S \). For this reason the set \( S \) is called the stopping set and the set \( C \) is called the continuation set.

In the terminology of variational inequalities, \( S \) is called the coincidence set, \( C \) is called the noncoincidence set, and \( g \) is called the obstacle. The set

\[ \Gamma = \tilde{\Omega}_{n+1} \cap \partial C \quad (\partial C = \text{boundary of } C) \]

is called the free boundary.

The purpose of this paper is to study the sets \( C, S \) or, equivalently, the free boundary \( \Gamma \).

We shall denote by \( e_i \) the vertex
(δ_{i0}, δ_{i1}, \cdots, δ_{in})

of \Pi_{n+1}(0 \leq i \leq n).

In § 3 we prove that each vertex \( e_i \) has a \( \Pi_{n+1} \)-neighborhood \( \tilde{S}_i \) such that \( \tilde{S}_i \subset S \). In § 4 we prove some auxiliary results needed for the following section.

In § 5 we study the set

\[(2.13) \quad S_i = S \cap \{ \pi \in \Pi_{n+1}; u(\pi) = a_i(1 - \pi_i) \}\]

under the assumption that

\[q_{i,k} = 0 \text{ for } 0 \leq k \leq n.\]

Introducing the coordinates

\[(2.14) \quad y_j = \frac{\pi_j}{\pi_0} \quad (1 \leq j \leq n)\]

we prove that \( \Gamma_i = \Pi_{n+1} \cap \partial S_i \) can be represented in the form

\[(2.15) \quad y_i = \psi_i(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)\]

where \( \psi_i \) is analytic.

In § 6 we specialize to the case (1.9) and prove that each \( S_i \) is a convex set and \( u(\pi) \) is a concave function.

In §§ 7, 8 we study the asymptotic behavior of the solution when (1.9) holds and \( c \to 0 \). It is shown that \( \partial S_i \) lies within a \( \delta_c \)-neighborhood of \( e_i \) and outside a \( \delta_c \)-neighborhood of \( e_i \). Further,

\[(2.16) \quad E^{e_\tau} \sim \left( \sum_{i=0}^{\infty} \gamma_i \pi_i \right) \log \frac{1}{c}, \quad \frac{1}{2} \gamma_i = \left( \min_{k \neq i} |\lambda_k - \lambda_i| \right)^{-1}\]

where \( \tau \) is the optimal stopping time, and

\[(2.17) \quad \frac{1}{c} u(cy) \longrightarrow \tilde{u}(y) \quad (u(y) = u(\pi))\]

where \( \tilde{u}(y) \) is the solution of a certain variational inequality; the free boundary for \( \tilde{u} \) is also studied.

In § 9 we consider the behavior of the solution as \( c \to \infty \). The case \( \alpha = 0 \) is considered in § 10. Finally, in § 11, we extend some of the results of the previous sections to the case where \( w(t) \) is \( k \)-dimensional, for any \( k \); here the methods are purely probabilistic.

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3. \( S \) contains a neighborhood of the vertices. We always denote by \( u(\pi) \) the solution of (2.1) (which satisfies (2.9), (2.10); re-
call that \( u(\pi) = V(\pi) \) if \( \pi \in \Pi_{n+1} \).

The operator \( Mu \) can be written in terms of the tangential operators of \( \Pi_{n+1} \) (considered as a submanifold in \( \mathbb{R}^{n+1} \)). Observe that on \( \Pi_{n+1} \)

\[
\sum_{i=0}^{n} \pi_i = 1
\]

and, consequently, the operator

\[
\sum_{i=0}^{n} \alpha_i \frac{\partial}{\partial \pi_i}
\]

is tangential if and only if \( \sum_{i=0}^{n} \alpha_i = 0 \). We introduce the tangential operators

\[
D_{0m} = \frac{\partial}{\partial \pi_0} - \frac{\partial}{\partial \pi_m} \quad (1 \leq m \leq n)
\]

and the normal operator

\[
\bar{D} = \sum_{i=0}^{n} \frac{\partial}{\partial \pi_i}.
\]

Substituting

\[
(3.1) \quad \frac{\partial}{\partial \pi_i} = - \sum_{m=1}^{n} \left( \delta_{im} - \frac{1}{n+1} \right) D_{0m} + \frac{1}{n+1} \bar{D}
\]

into \( Mu \) we discover that the coefficients of \( \bar{D}^2 \), \( D_{0m} \bar{D} \) vanish (as indeed they should) and that \( Mu \) takes the form

\[
(3.2) \quad Mu = \frac{1}{2} \sum_{i,j=1}^{n} \pi_i \pi_j \left( \lambda_i - \sum_{k=0}^{n} \lambda_k \pi_k \right) \left( \lambda_j - \sum_{l=0}^{n} \lambda_l \pi_l \right) D_{0i} D_{0j} u
- \sum_{j=1}^{n} \sum_{i=0}^{n} q_{i,j} \pi_i D_{0j} u.
\]

Another useful coordinate system is given by (2.14), i.e.,

\[
y_i = \frac{\pi_i}{\pi_0} \quad (1 \leq i \leq n).
\]

(The role of \( \pi_0 \) is incidental; one can similarly work with the coordinates \( y_i = \pi_i/\pi_j \), \( 0 \leq i \leq n, \ 0 \neq j \), for any fixed \( j \).) It maps \( \Pi_{n+1} \) onto

\[
R^*_n = \{y = (y_1, \cdots, y_n); \ y_j \geq 0 \text{ for } 1 \leq j \leq n\}.
\]

It is easy to compute that
(3.4) \[ \frac{\partial u}{\partial y_i} = \pi_0 \left( D_{0i} - \sum_{k=1}^{n} \pi_k D_{0k} \right) \]

and that in the \( y \)-coordinates \( Mu \) becomes (cf. [2])

\[
Lu = \frac{1}{2} \sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial^2 u}{\partial y_i \partial y_j} + \frac{1}{Y} \sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial u}{\partial y_j} + \sum_{j=1}^{n} \sum_{i=0}^{n} (q_{i,j} - q_{i,0}) y_i \frac{\partial u}{\partial y_j}
\]

where \( y_0 = 1 \) and

(3.6) \[ Y = 1 + y_1 + \cdots + y_n \]
(3.7) \[ \mu_{ij} = (\lambda_i - \lambda_0) \cdot (\lambda_j - \lambda_0) \]

We shall need the following comparison lemma:

**Lemma 3.1.** Suppose that \( \tilde{u} \) is a function satisfying the variational inequality (2.1) in a region \( \bar{\Omega} \subset \Omega_{n+1} \) with \( g \) replaced by \( \bar{g} \). If

\[
\bar{g} \geq g \text{ on } \bar{\Omega}, \quad \tilde{u} \geq u \text{ on } \partial \bar{\Omega} \cap \bar{\Omega}_{n+1},
\]

and \( \tilde{u} \) is uniformly continuous in \( \bar{\Omega} \), then \( \tilde{u} \geq u \) in \( \bar{\Omega} \). Similarly, if \( \bar{g} \leq g \) on \( \bar{\Omega} \), \( \tilde{u} \leq u \) on \( \partial \bar{\Omega} \cap \bar{\Omega}_{n+1} \), then \( \tilde{u} \leq u \) in \( \bar{\Omega} \).

Notice that we do not assume that \( \tilde{u} \geq u \) (or \( \tilde{u} \leq u \)) on \( \partial \bar{\Omega} \cap \partial \Omega_{n+1} \).

**Proof.** The function \( \tilde{u} \) can be obtained as the limit of solutions \( \tilde{u}_\delta \) of variational inequalities in \( \Omega_{n+1} \cap \bar{\Omega} \) (cf. (2.6)); the proof is the same as for \( u \). By a standard comparison theorem for variational inequalities, \( \tilde{u}_\delta \geq u \). Taking \( \delta \to 0 \), the assertion follows.

**Theorem 3.2.** Assume that \( c_j > a_i q_{j,i} \) for \( 0 \leq j \leq n \) and some \( i \). Then there exists a \( \Omega_{n+1} \)-neighborhood \( \tilde{S}_i \) of \( e_i \) such that \( \tilde{S}_i \subset S \).

**Proof.** It suffices to prove the assertion for \( i = 0 \). The proof is by comparison of \( v = u - g \) with a function \( z \) which vanishes in an \( R_n^+ \)-neighborhood of \( y = 0 \). Notice that near \( y = 0 \)

(3.8) \[ g = a_0 (1 - \pi_o) = \frac{a_0}{Y} (y_1 + \cdots + y_n) \]

Since \( M(1 - \pi_o) = -\sum_{i=0}^{n} q_{i0} \pi_i \), \( v \) satisfies:
\[ -Lv + \alpha v \leq \mu_1, \]
\[ v \leq 0, \]
\[ (-Lv + \alpha v - \mu_1)v = 0 \]
a.e., if \( y \in R_n^+, |y| \leq R^*, \) where \( R^* \) is sufficiently small and
\[
\mu_1 = \sum_{i=0}^{n} (c_i - a_0 q_{i0})\pi_i - \alpha g > c^*, \quad c^* > 0.
\]
We have to show that
\[ v(y) = 0 \quad if \quad y \in R_n^+, |y| < R \]
for a sufficiently small \( R. \)
Let (cf. [10])
\[
z(y) = \begin{cases} \frac{N}{1 - \theta} \left( \frac{\log \frac{1}{r}}{\log \frac{1}{R}} \right)^\theta - \theta \frac{\log \frac{1}{r}}{\log \frac{1}{R}} & \text{if } R < r < R_0, \\ 0 & \text{if } r < R \end{cases}
\]
where \( 0 < \theta < 1, N > 0, r = |y|. \) We compute that \( (\partial z / \partial r) < 0 \) if \( R < r < R_0, \) so that \( z < 0. \) Also
\[
z = \frac{\partial z}{\partial r} = 0 \quad if \quad r = R.
\]
If we show that
\[ \gamma \equiv -Lz + \alpha z \quad \text{satisfies} \quad \gamma \leq \mu_1 \quad (R < r < R_0) \]
and if also
\[ z(R_0) \leq -K \quad \text{where} \quad K = \sup v \]
then, by Lemma 3.1, \( z \leq v \) if \( 0 < r < R_0. \) This implies that \( 0 \leq v \)
if \( 0 < r < R, \) and (3.11) follows.
To establish (3.12), (3.13) we compute
\[
\left| L\left( \log \frac{1}{r} \right) \right| \leq C
\]
\[
\left| L\left( \log \frac{1}{r} \right)^\theta \right| \leq \frac{C}{\left( \log \frac{1}{r} \right)^{1-\theta}}.
\]
It follows that
\[ \gamma = -Lz + \alpha z \leq \frac{CN}{\log \frac{1}{R}} + \alpha z \leq \frac{CN}{\log \frac{1}{R}}. \]

Thus it suffices to satisfy (using (3.10))

\[ (3.14) \quad \frac{CN}{\log \frac{1}{R_0}} = c_* \]

and

\[ (3.15) \quad \frac{N}{1 - \theta} (M^\theta - \theta M) - N \leq -K \left( M = \frac{\log \frac{1}{R_0}}{\log \frac{1}{R}} < 1 \right). \]

Choosing \( M \) sufficiently small so that
\[ \frac{M^\theta - \theta M}{1 - \theta} \leq \frac{1}{2} \]

and taking \( N > 2K \), (3.15) follows. Defining \( R_0 \) by (3.14), the proof is complete.

4. Auxiliary results.

**Definition.** A point \( \pi \in \hat{H}_{n+1} \) is said to belong to the ridge \( R \) of the obstacle \( g \) if \( g \) is not \( W^{2,\infty} \) in any neighborhood of \( \pi \).

Thus, \( \pi = (\pi_0, \cdots, \pi_n) \in R \) if and only if
\[ a_i(1 - \pi_i) = a_j(1 - \pi_j) \quad \text{for some } i \neq j. \]

The above definition is analogous to the definition used in elastic-plastic torsion problems [6] where \( g \) is the distance function from the boundary of the domain.

**Theorem 4.1.** The ridge is contained in \( C \).

**Proof.** Suppose \( \vec{\pi} = (\vec{\pi}_0, \cdots, \vec{\pi}_n) \in R \) and, say,
\[ a_i(1 - \vec{\pi}_i) = a_j(1 - \vec{\pi}_j). \]

If \( \vec{\pi} \in S \) then
\[ \nabla (u - a_i(1 - \pi_i)) = 0, \]
\[ \nabla (u - a_j(1 - \pi_j)) = 0 \]

at \( \vec{\pi} \), since \( u - a_i(1 - \pi_i) \leq 0 \) in \( \Pi_{n+1} \) and \( u(\vec{\pi}) - a_i(1 - \vec{\pi}_i) = 0, i = \)
1, 2. Thus
\[ F(a_1\pi_1 - a_2\pi_2) = 0 \text{ at } \pi. \]

But
\[
\left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \right)(a_1\pi_1 - a_2\pi_2) = \left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \right) \frac{a_1y_1 - a_2y_2}{Y} = \frac{a_1 + a_2}{Y} > 0;
\]
a contradiction.

**Lemma 4.2.** Assume that \( q_{i,k} = 0 \) for \( 0 \leq k \leq n \) and some \( i \geq 1 \). Then
\[
(4.1) \quad \frac{\partial}{\partial y_i} \left[ Y(u - a_i(1 - \pi_i)) \right] \geq 0.
\]

**Proof.** It suffices to prove (4.1) for \( i = 1 \). In § 2 we may replace \( \Pi_{n+1}^i \) by any other sequence of domains which increase to \( \Pi_{n+1}^i \) and the boundary values \( \phi \) on \( \partial\Pi_{n+1}^i \) by any continuous function \( \phi \) satisfying \( 0 \leq \phi \leq g \). We shall choose \( \Pi_{n+1}^i \) so that in the \( y \)-coordinates it becomes
\[
(4.2) \quad G_\delta = \{ y; \delta < y_i < \frac{1}{\delta} \text{ for } 1 \leq i \leq n \}.
\]

Let
\[
v = u_\delta - a_i(1 - \pi_i)
\]
\[
z = Yv
\]
and choose \( \phi \) as follows:

\[
u_\delta = 0 \text{ on } y_i = \delta;
\]
\( Yu_\delta \), on each face \( y_i = \delta \) or \( y_i = 1/\delta \) \((2 \leq i \leq n)\), is a monotone increasing function of \( y_i \) such that \( Yu_\delta \leq Yg \), and \( Yu_\delta = 0 \) at \( y_i = \delta \), \( Yu_\delta = Yg \) at \( y_i = 1/\delta \); \( Yu_\delta = Yg \) on \( y_i = 1/\delta \).

Then, on \( y_i = \delta \)
\[
\frac{\partial}{\partial y_i}(Yu_\delta) \geq 0 \text{ (since } u_\delta = 0 \text{ on } y_i = \delta, u_\delta \geq 0 \text{ elsewhere).}
\]

Also
\[ a_i Y(1 - \pi_i) = a_i(1 + y_i + \cdots + y_n) \]

so that

\[
(4.4) \quad -\frac{\partial}{\partial y_i}(a_i Y(1 - \pi_i)) = 0 .
\]

Consequently

\[
(4.5) \quad \frac{\partial z}{\partial y_i} \geq 0 \quad \text{on} \quad y_i = \delta .
\]

On \( y_i = \delta \) or \( y_i = 1/\delta \) \((2 \leq i \leq n)\) we have, by (4.3),

\[
\frac{\partial}{\partial y_i}(Y u_i) \geq 0 .
\]

Using (4.4) we again get

\[
(4.6) \quad \frac{\partial z}{\partial y_i} \geq 0 \quad \text{on} \quad y_i = \delta \quad \text{or} \quad y_i = \frac{1}{\delta} \quad (2 \leq i \leq n) .
\]

On \( y_i = 1/\delta, \quad z = 0 \) by (4.3). Since \( z \leq 0 \) elsewhere, we obtain

\[
(4.7) \quad \frac{\partial z}{\partial y_i} \geq 0 \quad \text{on} \quad y_i = \frac{1}{\delta} .
\]

Denote by \( C_0 \) the set where \( u_0 < g \). Then, in \( C_0 \),

\[
Mv - \alpha v = -\sum_{i=0}^{n} (c_i - a_i q_{i,1}) \pi_i + \alpha a_i (1 - \pi_i) .
\]

Recalling that \( L v = M v \) where \( L \) is defined by (3.5), and substituting

\[
\frac{\partial^2 v}{\partial y_i \partial y_j} = \frac{1}{Y} \frac{\partial z}{\partial y_i} - \frac{1}{Y^2} z ,
\]

\[
\frac{\partial^2 v}{\partial y_i \partial y_j} = \frac{1}{Y} \frac{\partial^2 z}{\partial y_i \partial y_j} - \frac{1}{Y^2} \frac{\partial z}{\partial y_i} - \frac{1}{Y^2} \frac{\partial z}{\partial y_j} + \frac{2}{Y^2} z .
\]

we find that

\[
(4.8) \quad L v - \alpha z = -\sum_{i=0}^{n} (c_i - a_i q_{i,1}) y_i + \alpha a_i (Y - y_i)
\]

where

\[
(4.9) \quad L v z = \sum_{i,j=1}^{n} \mu_{i,j} y_i y_j \frac{\partial^2 z}{\partial y_i \partial y_j} + \sum_{j=1}^{n} \sum_{i=0}^{n} (q_{i,j} - q_{i,0} y_j) y_i \frac{\partial z}{\partial y_j} + \sum_{i=0}^{n} q_{i,0} y_i z .
\]
Differentiating (4.8) with respect to $y_1$, we obtain the following equation for $w = \partial z / \partial y_1$:

\begin{equation}
L_0 w + \sum_{i=1}^{n} \mu_i y_i \frac{\partial w}{\partial y_i} + \sum_{i=0}^{n} \sum_{j=1}^{n} (q_{i,j} - q_{i,0} y_j) y_i \frac{\partial w}{\partial y_j} - \alpha w = -(c_i - a_i g_{i,0}) = -c_i.
\end{equation}

From the maximum principle it then follows that $w > 0$ in $C_\delta$ provided $w \geq 0$ on $\partial C_\delta$. In view of (4.5)-(4.7), $w \geq 0$ on $\partial C_\delta \cap G_\delta$. We next show that

\begin{equation}
w(y) \geq 0 \quad \text{if} \quad y \in \partial C_\delta \cap (\text{int } G_\delta).
\end{equation}

Indeed, since $y \in \partial C_\delta \cap (\text{int } G_\delta)$,

\begin{equation}
u_\delta = a_i (1 - \pi_i), \quad F u_\delta = F(a_i (1 - \pi_i)) \quad \text{at } y,
\end{equation}

for some $i$ for which $g = a_i (1 - \pi_i)$ at $y$. Writing

\[
w = \frac{\partial z}{\partial y_1} = \frac{\partial}{\partial y_1} [Y u_\delta - a_i Y (1 - \pi_i)] + \frac{\partial}{\partial y_1} [a_i Y (1 - \pi_i)]
- \frac{\partial}{\partial y_1} [a_i Y (1 - \pi_i)],
\]

we note that the first term on the right hand side vanishes by (4.12), the third one vanishes by (4.4), and the middle one is equal to

\[
a_i \frac{\partial}{\partial y_1} (Y - y_i) = a_i \quad \text{if} \quad i \neq 1,
= 0 \quad \text{if} \quad i = 1,
\]

we conclude that $w(y) \geq 0$.

It follows that

\[
\frac{\partial}{\partial y_1} [Y (u_\delta - a_i (1 - \pi_i))] \geq 0 \quad \text{in } C_\delta; \text{ hence also in } G_\delta.
\]

Taking $\delta \to 0$, the assertion of the lemma follows.

**Remark.** Recalling (3.4), we can rewrite the assertion of Lemma 4.2 as follows:

\begin{equation}
\left( D_{0i} - \sum_{k=1}^{n} \pi_k D_{0k} \right) [Y (u - a_i (1 - \pi_i))] \geq 0.
\end{equation}

If we replace the role of $e_0$ by another vertex, say $e_n$, the corresponding differential inequality
(where $D_{nj} = \partial/\partial \pi_n - \partial/\partial \pi_j$) coincides with (4.13); thus we do not get any new inequality.

5. The free boundary is analytic. We continue to use the $y$ coordinates (3.3).

Denote by $G_i$ ($0 \leq i \leq n$) the open components of $\Pi_{n+1}\setminus R$ with $\partial G_i \supset e_i$ and set

$$(5.1) \quad S_i = S \cap G_i, \quad C_i = C \cap G_i,$$

the definition of $S_i$ is the same as in (2.13). Denote by $\tilde{G}_i$, $\tilde{S}_i$, $\tilde{C}_i$ the images of $G_i$, $S_i$, $C_i$ respectively in the $y$-coordinates. Denote by $\tilde{R}$ the image of the ridge $R$ in the $y$-coordinates. It is easy to check that if $\tilde{y} = (\tilde{y}_0, \ldots, \tilde{y}_n) \in \tilde{G}_i$ then there is a line segment

$$\gamma = \{y; \ y_j = \tilde{y}_j \text{ if } j \neq i, \ \tilde{y}_i - \beta < y_i \leq \tilde{y}_i\} \quad (\beta > 0)$$

which belongs to $\tilde{G}_i$ and its left end point lies on $\tilde{R}$.

Suppose now that

$$(5.2) \quad q_{i,k} = 0 \text{ for } 0 \leq k \leq n \text{ and some } i \geq 1.$$  

By Lemma 4.2 we then deduce that if

$$(5.3) \quad u(\tilde{y}) - g(\tilde{y}) < 0, \quad \tilde{y} \in \tilde{G}_i$$

then

$$(5.4) \quad u(y^*) - g(y^*) < 0 \text{ for any } y^* \in \gamma.$$  

Thus the open set $C_i$ is connected to $R$. Since $R$ belongs to $C$, by Theorem 4.1 it follows that $C_i$ is connected.

The previous argument involving (5.2), (5.3) shows also that there exists a function $\psi_i(y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ such that

$$(5.5) \quad \tilde{C}_i = \{y \in \tilde{G}_i; \ y_i < \psi_i(y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)\}.$$  

We can thus state:

**Theorem 5.1.** If (5.2) holds then $C_i$ is connected and it is a subgraph in the sense of (5.5).

We next prove that $\psi_i$ is analytic:

**Theorem 5.2.** If (5.2) holds then $\psi_i$ is analytic. More precisely,
\[ \bar{\mathcal{U}}_t = \psi_1(\bar{\mathcal{U}}_1, \ldots, \bar{\mathcal{U}}_{t-1}, \bar{\mathcal{U}}_{t+1}, \ldots, \bar{\mathcal{U}}_n), \]
\[ (\bar{\mathcal{U}}_1, \ldots, \bar{\mathcal{U}}_n) \in \operatorname{int} R^+_n \]

then \( \psi_t \) is analytic at \((\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_{i+1}, \ldots, \bar{y}_n)\). Thus the free boundary in the interior of \( G_t \) is analytic.

The proof of Theorem 5.2 given below is based on a method of Alt [1].

**Proof.** It suffices to prove the theorem for \( i = n \). Let \( y' = (y_1, \ldots, y_{n-1}) \) and consider the function
\[ \zeta = a_n(1 - \pi_n) - u \]
in
\[ D_{\rho_0} = \{ \beta < y_n < \psi_n(y'), |y' - y'_0| < \rho_0 \}. \]

Here \( y'_0 \) is a fixed point with positive coordinates, \( \beta < \psi_n(y'_0) \), and \( \psi_n(y'_0) - \beta, \rho_0 \) are sufficiently small so that \( D_{\rho_0} \) is contained in \( \bar{C}_n \). We have \( \zeta > 0 \) in \( D_{\rho_0} \), \( \zeta = 0 \) on \( y_n = \psi_n(y') \). By Lemma 4.2,
\[ \frac{\partial \zeta}{\partial y_n} < 0 \text{ in } D_{\rho_0}. \]

Consider the function
\[ \eta = \sum_{k=1}^{n-1} \alpha_k \frac{\partial \zeta}{\partial y_k} - A \frac{\partial \zeta}{\partial y_n} \]
in \( D_{\rho_0} \), where \( \sum \alpha_k \leq 1 \) and \( A \) is a sufficiently large positive constant to be determined later on. We have (cf. (4.8))
\[ L_0 \zeta - \alpha \zeta = \sum_{t=0}^{n} (c_t - a_n q_{t,n}) y_t - \alpha a_n (Y - y_n) \equiv \tilde{k}. \]

Differentiating with respect to \( y_k \), first when \( k = n \) and then when \( 1 \leq k \leq n - 1 \), we get
\[ L_0 \left( \frac{\partial \zeta}{\partial y_n} \right) + \sum_{t=1}^{n} \mu_{tn} y_t \frac{\partial}{\partial y_t} \left( \frac{\partial \zeta}{\partial y_n} \right) - \alpha \frac{\partial \zeta}{\partial y_n} = \frac{\partial \tilde{k}}{\partial y_n} = c_n, \]
\[ L_0 \left( \frac{\partial \zeta}{\partial y_k} \right) + \sum_{t=1}^{n} \mu_{tn} y_t \frac{\partial}{\partial y_t} \left( \frac{\partial \zeta}{\partial y_k} \right) - \alpha \frac{\partial \zeta}{\partial y_n} = \frac{\partial \tilde{k}}{\partial y_k} + M_k \]
where
\[ M_k = \sum_{t=1}^{n} (\mu_{tn} - \mu_{tk}) y_t \frac{\partial^2 \zeta}{\partial y_t \partial y_k} + \sum_{t=1}^{n} \lambda_{kt} \frac{\partial \zeta}{\partial y_t} + \bar{\zeta} k \]
where \( \lambda_{tk}, \bar{\zeta} k \) are linear functions, and
\[
\frac{\partial \tilde{k}}{\partial y_k} = c_k - a_n q_{k,n} - \alpha a_n .
\]

Since \( \zeta \in W_{10}^{1,\infty}, M_k \) is bounded and consequently,

(5.8) \[
\tilde{L} \eta = L_0 \eta + \sum_{i=1}^{n} \mu_i y_i \frac{\partial \eta}{\partial y_i} - \alpha \eta = -Ac_n + B
\]

where \( B \) is bounded independently of the \( \alpha \), and \( A \).

We choose \( A \) sufficiently large so that

(5.9) \[
\tilde{L} \eta \leq -1 \quad \text{in} \quad D_{\rho_0} .
\]

Now let \( \rho \) be any number < \( \rho_0 \) (for instance \( \rho = \rho_0/2 \)) and define

\[
D_{\rho} = \{ \beta < y_n < \psi_n(y'), |y' - y'| < \rho \} .
\]

Denote by \( \partial D_{\rho} \) the boundary of \( D_{\rho} \) and set

\[
\Gamma_{\rho,\sigma} = \{ y \in \partial D_{\rho}; y_n < \psi_n(y') - \sigma \} ,
\]

\[
\tilde{\Gamma}_{\rho,\sigma} = \{ y \in \partial D_{\rho}; \psi_n(y') - \sigma < y_n < \psi_n(y') \} .
\]

Define \( \partial D_{\rho_0}, \Gamma_{\rho_0,\sigma}, \tilde{\Gamma}_{\rho_0,\sigma} \) similarly with respect to \( D_{\rho_0} \).

For any sufficiently small \( \sigma > 0 \) we have, by (5.6),

(5.10) \[
\eta > 0 \quad \text{in} \quad \Gamma_{\rho,\sigma} \cup \tilde{\Gamma}_{\rho,\sigma}
\]

provided \( A = A(\sigma) \) is sufficiently large. We claim that if \( \sigma \) is sufficiently small depending on \( \rho, \rho_0 \) then

(5.11) \[
\eta \geq 0 \quad \text{on} \quad \Gamma_{\rho,\sigma} .
\]

Indeed, suppose (5.11) is not true. Then there exists a point \( y^* \in \tilde{\Gamma}_{\rho,\sigma} \) such that \( \eta(y^*) < 0 \).

Consider the function

\[
\tilde{\eta} = \eta + \gamma |y - y^*|^2 \quad (\gamma > 0) .
\]

If \( \gamma \) is sufficiently small then \( \tilde{L} \tilde{\eta} < 0 \). Therefore, \( \tilde{\eta} \) cannot take negative minimum in \( D_{\rho_0} \). But since \( \tilde{\eta}(y^*) < 0, \tilde{\eta} > 0 \) on \( \Gamma_{\rho_0,\sigma} \) (by (5.10)) and on \( y_n = \psi_n(y') \), there must exist a point \( \hat{y} \in \tilde{\Gamma}_{\rho_0,\sigma} \) such that \( \tilde{\eta}(\hat{y}) < 0 \). Thus

\[
\sum_{k=1}^{n} \alpha_k \frac{\partial \zeta}{\partial y_k} - A \frac{\partial \zeta}{\partial y_n} + \gamma |\hat{y} - y^*|^2 < 0 .
\]

Recalling that \( -A(\partial \zeta/\partial y_n) > 0, \) and that

\[
\frac{\partial \zeta}{\partial y_k} = 0(\sigma) \quad \text{on} \quad \tilde{\Gamma}_{\rho_0,\sigma} .
\]
(since $\zeta \in W^{2,\infty}_{loc}$ and $\nabla \zeta = 0$ on $y_n = \psi_n(y')$), we deduce that

$$\tag{5.12} (\rho_0 - \rho)^2 \leq C\sigma$$

where $C$ is a constant independent of $\rho_0$, $\rho$, $\sigma$, $A$. Consequently, if $\sigma$ is sufficiently small so that (5.12) is not true then the inequality (5.11) is valid. It follows that $\eta \geq 0$ on $\partial D_\rho$. Applying the maximum principle we conclude that $\eta > 0$ in $D_\rho$, i.e.,

$$\tag{5.13} \sum_{k=1}^{n-1} \alpha_k \frac{\partial \zeta}{\partial y_k} - A \frac{\partial \zeta}{\partial y_n} > 0 \text{ in } D_\rho.$$  

Denote by $K$ the cone

$$\{ y; y_n < -\frac{|y'|}{A} \}.$$  

The inequality (5.13) implies that if

$$\tilde{y} = (\tilde{y}', \tilde{y}_n), \quad \tilde{y}_n = \psi_n(\tilde{y}')$$

then $\zeta > 0$ in the cone $K + \tilde{y}$. Thus, if $\tilde{y}_n = \psi_n(\tilde{y}')$ then

$$\hat{y} \in K + \tilde{y},$$

i.e.,

$$\hat{y}_n > \tilde{y}_n - \frac{|\tilde{y}' - \tilde{y}'|}{A},$$

or equivalently,

$$\psi_n(\hat{y}') > \psi_n(\tilde{y}') - \frac{|\tilde{y}' - \tilde{y}'|}{A}.$$  

Interchanging $\hat{y}$ with $\tilde{y}$ we deduce that

$$|\psi_n(\hat{y}') - \psi_n(\tilde{y}')| \leq \frac{|\tilde{y}' - \tilde{y}'|}{A},$$

that is, $\psi_n$ is Lipschitz continuous.

By a general result of Caffarelli [5] it then follows that $\psi_n$ is a $C^1$ function and then (by Kinderlehrer and Nirenberg [12]) also analytic.

Remark. It is clear that Theorems 5.1, 5.2 extend to the case where (5.2) holds with $i = 0$. Instead of using the coordinate transformation (3.3), we take $y_j = \pi_j/\pi_{j_0}$ for $0 \leq j \leq n$, $j \neq j_0$ for any $j_0$, $j_0 \neq 0$.

6. The special case (1.9). In this section we obtain additional
results in the special case when (1.9) holds. For any $0 \leq j \leq n$, let
\[ \pi'_j = (\pi_0, \pi_1, \ldots, \pi_{j-1}, \pi_{j+1}, \ldots, \pi_n) \]
and denote by $\tilde{u}_j(\pi'_j)$ the solution of (2.1) corresponding to the problem with $n$ hypotheses $H_i$, $0 \leq i \leq n$, $i \neq j$.

**Theorem 6.1.** Suppose $q_{i,j} = 0$ for $0 \leq i, j \leq n$. If $\pi_j > 0$, $\pi_j \downarrow 0$ then
\[ u(\pi) \rightarrow \tilde{u}_j(\pi'_j). \]

**Remark.** Recall that boundary values for $u$ were not prescribed (on $\partial \Pi_{n+1}$); in fact, in $\Pi_{n+1}$,
\[ u(\pi) = \inf_{\tau} \mathbb{E}^\pi \left[ \int_0^\tau e^{-at}h(\pi(t))dt + e^{-a\tau}g(\pi(\tau)) \right] = V(\pi) \]
and, as shown in [2], the middle term is uniformly continuous in $\hat{H}_{n+1}$. This implies that $u$ has a continuous extension into $\partial \Pi_{n+1}$, which is denoted again by $u$. What we have to prove is that this extension, when restricted to $\pi_j = 0$, coincides with $\tilde{u}_j(\pi'_j)$.

**Proof.** If suffices to consider the case $j = n$. Let $\pi' = \pi'_n$ and $\tilde{u}(\pi') = \tilde{u}_n(\pi'_n)$. We denote by $\tau_\lambda$ the exit time of $\pi(t)$ from $\Pi_{n+1}$. We shall compare the cost functions
\[ J_\pi(\tau) = \mathbb{E}^\pi \left[ \int_0^{\tau \land \tau_\lambda} ce^{-at}dt + e^{-a\tau}g(\pi(\tau))I_{\tau \land \tau_\lambda} \right] \]
\[ J_{\pi'}(\tau) = \mathbb{E}^{\pi'} \left[ \int_0^{\tau} ce^{-at}dt + e^{-a\tau}g_1(\pi') \right] \]
where $\bar{\pi}(t) = (\bar{\pi}_0(t), \ldots, \bar{\pi}_{n-1}(t), \bar{\pi}_n(t))$ is the process $\pi(t)$ with $\bar{\pi}(0) = (\pi', 0)$ and
\[ g_1(\pi) = \min_{0 \leq i \leq n-1} \{a_i(1 - \pi_i)\}. \]
Recall [2] that
\[ u_\lambda(\pi) = \inf J_\pi(\tau) \]
\[ \tilde{u}(\pi') = \inf_{\tau} J_{\pi'}(\tau) \]
where $\tau$ varies over all $\mathcal{F}_t$ stopping times.

By Lemma 2.1, $\bar{\pi}_n(t) \equiv 0$.

In what follows we shall use a model of the Markov process associated with $\pi(t)$ in which the probability is fixed, say $P$, and the initial condition $\pi(0) = \pi$ varies; for each $\pi$, $\pi(t)$ is the solution
of the stochastic differential system associated with the generator $M$, and the initial condition $\pi(0) = \pi$. Working with this model, we can replace $E^{\tau,\pi}$, $E^{\tau'}$ by $E$, and we shall compare $J_{\varepsilon}(\tau)$, $J_{\pi}(\tau)$ with the same $\tau$. We have (see, for instance, [8]), for any $T > 0$,

$$(6.3) \quad E\left[ \sup_{0 \leq t \leq T} |\pi(t) - \bar{\pi}(t)|^2 \right] \leq C_T \pi_n^2, \quad C_T \text{ constant.}$$

By Lemma 2.1, for any $\eta > 0$,

$$(6.4) \quad P[\tau_\varepsilon < T] < \eta \quad \text{if} \quad \pi(0) = (\pi_0, \ldots, \pi_n), \quad \pi_n > 0$$

provided $\delta$ is sufficiently small (depending on $\eta, \pi_n$).

Next, by Lemma 2.1 and (6.3),

$$(6.5) \quad E|g(\pi(t)) - g(\bar{\pi}(t))| \leq C_T \pi_n \quad \text{if} \quad 0 < t \leq T.$$
lations,
\[ \lambda J_{\pi_1}(\delta) + (1 - \lambda) J_{\pi_2}(\delta) = \sum_{i=0}^{n} (\lambda \pi_i + (1 - \lambda) \pi_i') E'[\cdots] \]
(6.7)
\[ = \sum_{i=0}^{n} \pi_i E'[\cdots] = E\tilde{\tau}'[\cdots] = J_{\tilde{\tau}}(\delta) ; \]

here \( \pi_i, \pi_i', \tilde{\tau} \) are the \( i \)th coordinates of \( \pi^1, \pi^2, \tilde{\tau} \) respectively and the expression \([\cdots]\) is the same as in (6.6).

Suppose now that \( \pi^1 \) and \( \pi^2 \) belong to \( S_i \). Then
\[ a_i(1 - \pi_i^1) \leq J_{\pi_i^1}(\delta), \quad a_i(1 - \pi_i^2) \leq J_{\pi_i^2}(\delta) . \]

It follows from (6.7) that
\[ a_i(1 - \tilde{\pi}_i) \leq J_{\tilde{\tau}}(\delta) . \]

Thus \( \tilde{\tau} \in S_i \) and, consequently, \( S_i \) is a convex set.

Next, (6.7) gives
\[ \inf_{\delta} J_{\tilde{\tau}}(\delta) \geq \lambda \inf_{\delta} J_{\pi_i^1}(\delta) + (1 - \lambda) \inf_{\delta} J_{\pi_i^2}(\delta) , \]
i.e.,
\[ u(\lambda \pi^1 + (1 - \lambda) \pi^2) = u(\tilde{\tau}) \geq \lambda u(\pi^1) + (1 - \lambda) u(\pi^2) , \]
so that \( u(\tau) \) is concave.

**REMARK 1.** From Theorems 6.2, 5.2, 3.2 we deduce that each \( S_i \)
is a convex domain containing a \( H_{n+1} \)-neighborhood of \( e_i \) and \( \partial S_i \cap \bar{H}_{n+1} \) is an analytic manifold.

**REMARK 2.** For any numbers \( \alpha_{i,k} (0 \leq i \leq n, 1 \leq k \leq l) \) the equations
\[ \sum_{i=1}^{n} \alpha_{i,k} y_i = \alpha_{0,k} \left( 1 \leq k \leq l, y_i = \frac{\pi_i}{\pi_0} \right) \]
hold if and only if
\[ \sum_{i=1}^{n} \alpha_{i,k} \pi_i - \alpha_{0,k} \pi_0 = 0 \quad 1 \leq k \leq l . \]
Since also \( \sum_{i=0}^{n} \pi_i = 1 \), it follows that the mapping (3.3) maps planes onto planes and lines onto lines. It also maps segments onto segments. It follows, in particular, that
(6.8) (3.3) maps convex sets onto convex sets.

Consequently, by Theorem 6.2, the image \( \tilde{S_i}(1 \leq i \leq n) \) in the \( y \)-space of the coincidence set \( S_i \) (in the \( \pi \)-space) is a convex set.
7. Asymptotic estimates for \( c \to 0 \). For any \( \gamma > 0 \) set

\[
N_\gamma = \left\{ \pi \in \Pi_{n+1}; 1 - \gamma \leq \pi_i < 1 \right\}, \quad N_r = \bigcup_{\gamma=0}^n N_\gamma^\epsilon.
\]

**Theorem 7.1.** Assume that (1.9) holds. Then there exist positive constants \( \delta_1, \delta_2 \) independent of \( c, \alpha \), such that for all \( c \) sufficiently small,

\[
N_{\delta_1 \epsilon} \subset S \subset N_{\delta_2 \epsilon}.
\]

**Proof.** Set

\[
r = \sum_{i=1}^n \pi_i.
\]

Using (3.2) and the relation \( \sum_{j=1}^n q_{i,j} = -q_{i,0} \) we find that (for general \( q_{i,j} \))

\[
M(\log r) = -\frac{1}{2r^2} \left| \sum_{i=1}^n \sum_{k=0}^n (\lambda_i - \lambda_k) \pi_i \pi_k \right|^2 - \frac{1}{r} \sum_{i=0}^n q_{i,0} \pi_i.
\]

Since

\[
\sum_{i=1}^n \sum_{j=1}^n (\lambda_i - \lambda_k) \pi_i \pi_k = 0, \quad \text{by symmetry},
\]

and since

\[
\left| \sum_{i=1}^n (\lambda_i - \lambda_0) \pi_i \pi_0 \right|^2 = \pi_0^2 \left| \sum_{i=1}^n (\lambda_i - \lambda_0) \pi_i \right|^2,
\]

we obtain upon recalling (1.1), that

\[
-q_0^* - K_1 \pi_0^2 \leq M(\log r) \leq -K_2 \pi_0^2 + |q_{0,0}| \pi_0/r, \quad (q_0^* = \max_{1 \leq i \leq n} q_{i,0}),
\]

where \( K_1, K_2 \) are positive constants depending only on the \( \lambda_i \).

To prove the second part of (7.2), consider the function, in \( \Pi = \Pi_{n+1} \cap \{ r < 1/(n + 1) \} \),

\[
v(r) = \begin{cases} 
a_\delta r & \text{if } 0 \leq r \leq R, \\ c_\delta \log r + A - r & \text{if } R < r \leq \frac{1}{n + 1} 
\end{cases}
\]

where \( A, R, \delta \) are positive constants. We choose \( A, R, \delta \) as functions of \( \delta \) so that \( v \) becomes \( C^1 \) at \( r = R \), i.e.,

\[
\frac{c_\delta}{R} - 1 = a_\delta, \quad c_\delta \log R + A - R = a_\delta R;
\]

\( \delta \) is a positive constant (independent of \( c \)) to be determined. Thus
\( R = \frac{c\delta}{\alpha_0 + 1} \left( R < \frac{1}{n+1} \text{ if } c \text{ is sufficiently small} \right) , \)

\( A = c\delta + c\delta \log \frac{\alpha_0 + 1}{c\delta} . \)

Notice that \( \bar{I} \subset G_1 \). The condition

\( v \left( \frac{1}{n+1} \right) < 0 \)

is satisfied if

\[ c\delta \log \frac{1}{n+1} + A - \frac{1}{n+1} < 0 , \]

i.e., (in view of (7.7), if \( c \) is sufficiently small. We also easily find that

\( v \leq c\delta \log (\alpha_0 + 1) . \)

Using (7.5), (7.7), (7.9) and the conditions (1.9), we find that, if \( R < r < 1/(n+1) \),

\[ Mv - \alpha v \geq -K_1 \frac{c\delta}{(n+1)^2} - \alpha c\delta \log (\alpha_0 + 1) > -c \]

provided \( \delta \) is sufficiently small (independently of \( c \)). We also have

\[ v < a_0 r = g \text{ if } R < r < \frac{1}{n+1} . \]

Thus, we can apply Lemma 3.1 with \( \bar{u} = v \) and conclude that \( v \leq u \) in \( \bar{I} \). Since \( v < g \) if \( r > R \), the same is true for \( u \). Thus \( S \cap G_0 \) is contained in \( N_{s_0} \). Similarly one can prove that \( S \cap G_i \) is contained in \( N_{s_0} \) for any \( i \geq 1 \).

To prove the first part of (7.2), let

\[ w_0(r) = \begin{cases} a_0 r & \text{if } 0 \leq r \leq R_0 , \\ c\delta \log r + A_0 & \text{if } R_0 < r \leq 1 . \end{cases} \]

This function is \( C^1 \) at \( r = R_0 \) if

\[ R_0 = \frac{c\delta}{a_0} , \]

\[ A_0 = c\delta + c\delta \log \frac{\alpha_0}{c\delta} . \]

Using (7.5) and (1.9) we get, for \( R_0 < r < n/(n+1) \),
if $\delta$ is sufficiently large (independently of $c$).

Similarly, we define functions $w_i$ for each $1 \leq i \leq n$ and take

\begin{equation}
(7.10) \quad w = \min_{0 \leq i \leq n} w_i.
\end{equation}

Note that if $r = n/(n + 1)$ then certainly $w < w_0$. Thus, if $w = w_0$ then $r < n/(n + 1)$ and, consequently,

\[ Mw_0 - \alpha w_0 < -c \quad \text{if further} \quad r > R_0. \]

The corresponding result is true for each $w_i$.

It follows that outside the $(c\delta)$-neighborhoods of the vertices $e_i$,

\[ Mw - \alpha w < -c \]

where $Mw$ is taken in the distribution sense.

We can now apply Lemma 3.1 (whose proof extends, by approximation, to the case where $\tilde{u}$ is only Lipschitz continuous and $M\tilde{u}$ is taken in the distribution sense). It follows that $u \leq w$, and the first part of (7.2) is established.

**Remark 1.** The proof of the second part of (7.2) extends to the case where, for some $i$,

\[ |q_{i,i}| \geq (n + 1)\max_{k \neq i} (q_{k,i} - \alpha); \]

it gives the relation

\[ S \cap G_i \subset N_{\varepsilon} \cdot \]

**Remark 2.** From the proof of the first part of (7.2) we see that the function

\[ W_\varepsilon(r) = \log \frac{r}{\varepsilon} \quad (r = \sum_{i=1}^{n} \pi_i, \varepsilon > 0) \]

satisfies $MW_0 \leq -A$ if $\varepsilon < r < n/(n + 1)$, where $A$ is a positive constant independent of $\varepsilon$. Define $W_i$ in a similar manner with respect to the vertex $e_i$, and set

\[ W = \frac{1}{A} \min_{0 \leq i \leq n} W_i. \]

Then $MW \leq -1$ in

\[ \tilde{N}_i = \Pi_{n+1}\setminus N_i. \]
Also $W = 0$ on $\partial N_\varepsilon \cap \hat{N}_{n+1}$. Denoting by $\tau_\varepsilon$ the hitting time of $N_\varepsilon$ by the process $\pi(t)$ it follows, by comparison, that

$$E^\varepsilon \tau_\varepsilon \leq W(\pi).$$

Thus

$$(7.11) \quad E^\varepsilon \tau_\varepsilon \leq A_1 \log \frac{1}{\varepsilon} \text{ for all } \pi \in \Pi_{n+1}.$$

In the following section we shall obtain a more precise result as $\varepsilon \to 0$.

**Remark 3.** From Theorem 7.1 it follows that

$$(7.12) \quad u(\pi) \leq A_2 c$$

for all $c$ sufficiently small, where $A_2$ is a constant independent of $c$. In the following section we shall obtain a more precise result as $c \to 0$.

8. Asymptotic estimates for $c \to 0$ (continued).

**Theorem 8.1.** Suppose (1.9) holds. Then, for any $\pi \in \Pi_{n+1},$

$$(8.1) \quad E^\varepsilon \tau_\varepsilon = \left( \sum_{i=0}^n \gamma_i \pi_i \right) \log \frac{1}{\varepsilon} + O \left( \left( \log \frac{1}{\varepsilon} \right)^{1/2} \right), \quad \frac{1}{2} \gamma_i = \{\min_{k \neq i} |\lambda_k - \lambda_i|\}^{-1}$$

as $\varepsilon \to 0$.

The analogous result for discrete processes is given, for instance, in Kiefer and Sacks [10].

**Proof.** Set $\tau = \tau_\varepsilon$. Then $\tau$ is the first time $t$ such that

$$\max_{0 \leq j \leq n} \pi_j(t) = 1 - \varepsilon.$$ 

Using (1.11), the last inequality becomes

$$\max_{0 \leq j \leq n} \left\{ 1 + \sum_{k \neq j} \frac{\pi_k}{\pi_j} e^{(\lambda_k - \lambda_j) \cdot \varepsilon(t) - 1/2(\delta_k^2 - \delta_j^2)} \right\}^{-1} = 1 - \varepsilon,$$

or

$$(8.2) \quad \min_{0 \leq j \leq n} \max_{k \neq j} e^{(\lambda_k - \lambda_j) \cdot \varepsilon(t) - 1/2(\delta_k^2 - \delta_j^2)} = C \varepsilon$$

where $C$ is a random variable, $B_1 \leq C \leq B_2$, and $B_1, B_2$ are positive constants independent of $\varepsilon$ (but depending on the initial point $\pi$. Taking the logarithm on both sides of (8.2) we conclude that
\[(8.3) \min_{0 \leq j \leq n} \max_{k \neq j} \left\{ (\lambda_k - \lambda_j) \cdot \xi(\tau) - \frac{1}{2} (|\lambda_k|^2 - |\lambda_j|^2) \right\} E^{\xi \tau} = -\log \frac{1}{\varepsilon} + o(1) .\]

Recalling that

\[(8.4) \quad E^{\xi \tau} = \sum_{i=0}^{n} \pi_i E^{\xi_i \tau} ,\]

we proceed to evaluate \(E^{\xi \tau}\) for a fixed \(l\). With respect to the probability \(P_l\),

\[(8.5) \quad \xi(t) = w(t) + \lambda_l t \text{ a.s.} \]

Thus, the stopping time \(\tau\) is the hitting time of some region \(Q\) by the process \(w(t)\).

We claim that for any hitting time \(\tau\) of a region \(Q\),

\[(8.6) \quad E \xi 2 |w(\tau)|^2 = 2n E \xi 2 + |x|^2 .\]

Indeed, if \(Q^c = R^n \setminus Q\) is a bounded open set then, since both sides of (8.6) are harmonic functions in \(Q^c\) taking the same boundary values \(|x|^2\) on \(\partial Q^c\), they must agree in \(Q^c\). If \(Q^c\) is unbounded then (8.6) follows by approximating \(Q^c\) by bounded open sets.

From (8.6) applied with \(x = 0\) it follows that

\[(8.7) \quad E^{\xi} |w(\tau)| = (2n)^{1/2} (E^{\xi \tau})^{1/2} \leq C_0 \left( \log \frac{1}{\varepsilon} \right)^{1/2} \text{ (C_0 constant)} \]

where (7.11) was used.

Combining (8.3) with (8.5) and using (8.7), we find that

\[(8.8) \quad \min_{0 \leq j \leq n} \max_{k \neq j} \left\{ (\lambda_k - \lambda_j) \cdot \lambda_l - \frac{1}{2} (|\lambda_k|^2 - |\lambda_j|^2) \right\} E^{\xi \tau} = -\log \frac{1}{\varepsilon} + o(1) + 0 \left( \left( \log \frac{1}{\varepsilon} \right)^{1/2} \right) .\]

Next, one easily checks that

\[
\max_{k \neq j} \left\{ (\lambda_k - \lambda_j) \cdot \lambda_l - \frac{1}{2} (|\lambda_k|^2 - |\lambda_j|^2) \right\} = \begin{cases} 
\max_{k \neq l} \left[ -\frac{1}{2} |\lambda_k - \lambda_l|^2 \right] & \text{if } j = l \\
(\lambda_l - \lambda_j) \cdot \lambda_l - \frac{1}{2} (|\lambda_l|^2 - |\lambda_j|^2) & \text{if } j \neq l.
\end{cases}
\]

Using this in (8.8), we obtain

\[
\left[ \frac{1}{2} \min_{k \neq l} |\lambda_k - \lambda_l|^2 \right] E^{\xi \tau} = \log \frac{1}{\varepsilon} + o \left( \left( \log \frac{1}{\varepsilon} \right)^{1/2} \right) ;
\]
recalling (8.4), the assertion (8.1) follows.

We wish to study the behavior of the solution \( u(\pi) \) in a neighborhood of a vertex \( \epsilon \) as \( c \to 0 \). It suffices to take \( i = 0 \). It will be convenient to use the coordinates (3.3). We also set

\[
 u(y) = u(\pi) , \quad |y| = y_1 + \cdots + y_n ,
\]

(8.9)

\[
 L_0 u = \frac{1}{2} \sum_{i,j=1}^n \mu_{ij} y_i y_j \frac{\partial^2 u}{\partial y_i \partial y_j} .
\]

The function \( u(y) \) satisfies the variational inequality, in \( 0 \leq |y| < \delta_0 \), \( \delta_0 = 1/(n + 1) \),

\[
 L_0 u + \frac{1}{Y} \sum_{i,j=1}^n \mu_{ij} y_i y_j \frac{\partial u}{\partial y_j} - \alpha u + c \leq 0 ,
\]

(8.10)

\[
 u \leq \frac{\alpha \vert y \vert}{Y} ,
\]

\[
 \left( L_0 u + \frac{1}{Y} \sum_{i,j=1}^n \mu_{ij} y_i y_j \frac{\partial u}{\partial y_j} - \alpha u + c \right) \left( u - \frac{\alpha \vert y \vert}{Y} \right) = 0 .
\]

Consider the variational inequality in \( R^n_+ \):

\[
 L_0 \bar{u} - \alpha \bar{u} + 1 \geq 0 ,
\]

(8.11)

\[
 \bar{u} \leq \alpha \vert y \vert ,
\]

\[
 (L_0 \bar{u} - \alpha \bar{u} + 1)(\bar{u} - \alpha \vert y \vert) = 0 ,
\]

subject to the growth condition

(8.12)

\[
 \bar{u}(y) = 0(\vert y \vert) \quad \text{as} \quad |y| \to \infty .
\]

**Theorem 8.2.** Let (1.9) hold. Then there exists a unique solution \( \bar{u}(y) \) of (8.11), (8.12); further, \( 0 \leq u(y) \leq C \) for some constant \( C \), and

(8.13)

\[
 \frac{u(cy)}{c} \to \bar{u}(y) \quad \text{as} \quad c \to 0 ,
\]

uniformly in \( y \) in compact subsets of \( \text{int} \, R^n_+ \).

**Proof.** For any \( A > 0 \) let \( \bar{u}_A \) be the solution of the variational inequality, in \( |y| < A \),

\[
 L_0 \bar{u}_A - \alpha \bar{u}_A + 1 \geq 0 ,
\]

(8.14)

\[
 \bar{u}_A \leq \alpha \vert y \vert ,
\]

\[
 (L_0 \bar{u}_A - \alpha \bar{u}_A + 1)(\bar{u}_A - \alpha \vert y \vert) = 0 ,
\]

\[
 \bar{u}_A = 0 \quad \text{on} \quad |y| = A .
\]

It is easily seen that
\[ \bar{u}_A \geq 0, \quad \bar{u}_A(y) \uparrow \text{ if } A \uparrow. \]

It follows that
\[ (8.15) \quad \bar{u}(y) \equiv \lim_{A \uparrow \infty} \bar{u}_A(y) \]
exists and it is a solution of the variational inequality (8.11).

We can represent \( \bar{u}_A(y) \) in the form
\[ (8.16) \quad \bar{u}_A(y) = \inf_{\tau} J_{y,A}(\tau), \]
\[ (8.17) \quad J_{y,A}(\tau) = \mathbb{E}_y \left[ \int_0^{\tau \wedge \tau^d} e^{-\alpha t} dt + a_0 |y(\tau)| e^{-\alpha \tau} I_{\tau < \tau^d} \right] \]
where \( y(t) \) satisfies
\[ (8.8) \quad dy(t) = \sigma(y(t)) dw(t), \quad y(0) = y \quad (y \in \text{int } R^+_n), \]
for some \( n \)-dimensional Brownian motion \( w(t) \), and
\[ \sigma = (\sigma^{ij}), \quad \sigma^{ij} = \nu_{ij} y_i, \quad \sum_{k=1}^n \nu_{ik} \nu_{jk} = \mu_{ij}, \quad \nu_{ij} = \nu_{ji}; \]
\( \tau^d \) is the exit time of \( y(t) \) from the set \( |y| \leq A \).

We claim that for all \( 1 < A < \infty \),
\[ (8.19) \quad \mathbb{E}_y [e^{-\lambda \tau^d}] \leq \frac{C}{A^2} \quad \text{for some constants } \lambda > 1, \quad C > 0. \]
Indeed, by comparison
\[ \mathbb{E}_y [e^{-\lambda \tau^d}] \leq W(y) \]
provided
\[ L_0 W - \alpha W \leq 0 \quad \text{for } |y| < A, \]
\[ W \geq 1 \quad \text{on } |y| = A. \]

Taking
\[ W(y) = \frac{C_0}{A^2} (y_1^2 + \cdots + y_n^2) \]
where \( C_0 \) is a constant independent of \( A \) and \( \lambda > 1, \lambda - 1 \) sufficiently small, the assertion (8.19) follows.

Taking \( \tau = \tau^d \) in (8.17) and using (8.19) and (8.16), (8.15) we conclude that
\[ (8.20) \quad \bar{u}(y) \leq C \quad (C \text{ positive constant}). \]

Using (8.10) we find that the function
\[ w_c(y) = \frac{u(cy)}{c} \]
satisfies the variational inequality, in \(0 < y < \delta_0/c,\)
\[
L_0w_c + \frac{c}{1 + c|y|} \sum_{i,j=1}^n \mu_{ij}y_iy_j \frac{\partial w_c}{\partial y_j} - \alpha w_c + 1 \geq 0 ,
\]
(8.21) \[ w_c \leq \frac{a_0|y|}{1 + c|y|} , \]
\[
\left(L_0w_c + \frac{c}{1 + c|y|} \sum_{i,j=1}^n \mu_{ij}y_iy_j \frac{\partial w_c}{\partial y_j} - \alpha w_c + 1 \right) \left(w_c - \frac{a_0|y|}{1 + c|y|} \right) = 0 .
\]
Hence we can write
(8.22) \[ w_c(y) = \inf_{\tau} J_{y, A}(\tau) \]
where
\[
J_{y, A}(\tau) = E_y \left[ \int_0^{\tau \wedge \tau_A} e^{-\alpha t} dy + \frac{a_0|y_c(\tau)|}{1 + c|y_c(\tau)|} e^{-\alpha \tau_\tau_A} I_{\tau < \tau_A} + w_c(y_c(\tau)) e^{-\alpha \tau_A} I_{\tau \geq \tau_A} \right]
\]
where \(y_c(t)\) is the solution of the stochastic system
(8.24) \[ dy_c(t) = \sigma(y_c(t))d\bar{w}(t) + b_c(t)dt , \quad y_c(0) = y , \]
the matrix \(\sigma(y)\) is defined above,
\[
b_c = (b_{c,i}) , \quad b_{c,i} = \frac{c}{1 + c|y|} \sum_{j=1}^n \mu_{ij}y_iy_j ,
\]
\(\tau_A = \tau^A \wedge \tau^A_c, \tau^A_c\) is the exit time of \(y_c(t)\) from the set \(|y| \leq A\). Notice that \(\tau_A\) is a stopping time with respect to the \(\sigma\)-fields \(\sigma(\bar{w}(s), 0 \leq s \leq t), t \geq 0;\) here \(A\) is any fixed positive number \(\leq \delta_0/c\).

Analogously to (8.22), (8.23) we can write
(8.25) \[ \bar{u}(y) = \inf_{\tau} \bar{J}_{y, A}(\tau) \]
where
\[
\bar{J}_{y, A}(\tau) = E_y \left[ \int_0^{\tau \wedge \tau_A} e^{-\alpha t} dt + a_0|y(\tau)| e^{-\alpha \tau_\tau_A} I_{\tau < \tau_A} + \bar{u}(y(\tau_A)) e^{-\alpha \tau_A} I_{\tau \geq \tau_A} \right].
\]
By standard arguments, for any large \(T > 0\) and small \(\eta > 0,\)
(8.27) \[ E_y[ \sup_{0 \leq t \leq \tau} |y_c(t) - y(t)|^2 ] \leq \eta^2 \quad \text{if} \quad c \leq c_0(\eta, T) .\]
Next, the proof of (8.19) shows also that

$$A^\lambda E_y[e^{-\lambda \varepsilon}] \leq C$$

for some $\lambda > 1$, $C > 0$

provided $cA \leq 1/C^*$ where $C^*$ is a suitably large positive constant (independent of $c$, $A$). Hence

$$A^\lambda E_y[e^{-\lambda \varepsilon}] \leq C$$

provided $cA \leq \frac{1}{C^*}$.

It follows that

$$E_y[e^{-\lambda \varepsilon}] < \eta \quad \text{(if } c \leq \frac{1}{C^*A})$$

provided $A$ is sufficiently large.

Note that

$$|y(\tau)| \leq A, \quad |y_c(\tau)| \leq A \quad \text{if } \tau \leq \tau^4.$$

Now fix $A$ such that (8.29) holds and then fix $T$ sufficiently large (depending on $A$ but not on $c$) such that

$$|y_c(\tau)| - \frac{|y_c(\tau)|}{1 + c|y_c(\tau)|} < \eta \quad \text{if } \tau \leq \tau^4,$$

(8.30)

$$A e^{-\eta} < \eta.$$

Using (8.27), (8.29), (8.30) and (8.20), and recalling also (by (7.12)) that

$$|w_c| \leq A_2,$$

we deduce from (8.23), (8.26) that

$$|J_{y,\alpha}(\tau) - \tilde{J}_{y,\alpha}(\tau)| \leq C\eta$$

provided $c \leq c_*(\eta, A)$; $c_*$ and $C$ are independent of $\tau$ and $C$ is independent of $c$, $A$. Recalling (8.22), (8.25), we get

$$|w_c(y) - \tilde{u}(y)| \leq C\eta,$$

and the assertion (8.13) follows.

It remains to prove that any solution $\tilde{u}(y)$ of (8.11), (8.12) must coincide with $\tilde{u}$. From (8.19) we conclude that

$$E_y[|\tilde{u}(y(\tau^4))| e^{-\lambda \varepsilon \tau^4}] \rightarrow 0 \quad \text{if } A \rightarrow \infty.$$

Using (8.33) we can now repeat the argument which gave (8.32), with $w_c(y)$ replaced by $\tilde{w}(y)$. We thus deduce that
\[ |\tilde{u}(y) - \bar{u}(y)| \leq C\eta \text{ for and } \eta > 0; \]

hence \( \tilde{u} = \bar{u} \).

From Remark 2 at the end of § 6 we have that the component of the coincidence set of \( w_c(y) \) which contains \( y = 0 \) in convex. We also have:

**THEOREM 8.3.** The coincidence set \( \tilde{S} \) of \( \tilde{u}(y) \) is a convex set.

By Caffarelli [5] it then follows that the free boundary \( \partial \tilde{S} \cap \text{int} R^+ \) is analytic.

**Proof.** It is easy to check that if \( y \mapsto \pi \) by (3.3) then \( c y \mapsto \pi^c = (\pi_0^c, \ldots, \pi_n^c) \) where

\[
\begin{align*}
\pi_0^c &= \frac{\pi_0}{\pi_0 + c(1 - \pi_0)}, & \pi_i^c &= \frac{c\pi_i}{\pi_0 + c(1 - \pi_0)} (1 \leq i \leq n).
\end{align*}
\]

Setting \( \tilde{u}(\pi) = \tilde{u}(y), u(\pi) = u(y) \) we then have, by Theorem 8.2,

\[
\frac{u(\pi^c)}{c} \longrightarrow \tilde{u}(\pi) \text{ as } c \longrightarrow 0.
\]

Set \( \bar{\pi} = \pi - e_0, \bar{\pi}^c = \pi^c - e_0 \). Then, as easily checked,

\[
\begin{align*}
\bar{\pi}^c &= \frac{c}{1 + (1 - c)\bar{\pi}_0} \bar{\pi}^c & \text{and}
\bar{\pi} &= \frac{1}{c + (c - 1)\bar{\pi}_0} \bar{\pi}^c
\end{align*}
\]

where \( \bar{\pi} = (\bar{\pi}_0, \ldots, \bar{\pi}_n), \bar{\pi}^c = (\bar{\pi}_0^c, \ldots, \bar{\pi}_n^c) \).

Now, by the concavity of \( u(\pi) \) established in Theorem 6.2, for any two points \( \hat{\pi}, \tilde{\pi} \) and \( 0 < \lambda < 1 \),

\[
\frac{1}{c} \bar{u}(\lambda \hat{\pi}^c + (1 - \lambda)\hat{\pi}) \geq \frac{1}{c} (\lambda \bar{u}(\hat{\pi}^c) + (1 - \lambda)(\bar{u}(\hat{\pi}^c))
\]

where \( \hat{\pi} = \hat{\pi} - e_0, \tilde{\pi} = \tilde{\pi} - e_0 \) and \( \bar{u}(\pi) = u(\pi) \) for any \( \pi \). We can write

\[
\lambda \hat{\pi}^c + (1 - \lambda)\hat{\pi}^c = \hat{\pi}^c, \quad \tilde{\pi} = \hat{\pi} - e_0,
\]

where, by (8.37),
The point $\hat{\pi}$ depends on $c$; as $c \to 0$

\begin{align*}
\hat{\pi} = \frac{\lambda \hat{\pi} \cdot + (1 - \lambda)\hat{\pi} \cdot}{c + (c - 1)(\lambda \hat{\pi} \cdot + (1 - \lambda)\hat{\pi} \cdot)}.
\end{align*}

as seen using (8.36).

Using (8.35) we see that the right hand of (8.38) converges to

\begin{align*}
\lambda \tilde{u}(\hat{\pi}) + (1 - \lambda)\tilde{u}(\hat{\pi}) \quad \text{as} \quad c \to 0.
\end{align*}

As for the left hand side, using (8.39)-(8.41) we find that it converges to $\tilde{u}(\tilde{\pi})$, where $\tilde{u}(\tilde{\pi}) = \tilde{u}(\pi)$ for any $\pi$. Hence

\begin{align*}
\lambda \tilde{u}(\hat{\pi}) + (1 - \lambda)\tilde{u}(\hat{\pi}) \leq \tilde{u}(\tilde{\pi})
\end{align*}

where $\tilde{\pi}$ is the same linear combination of $\pi$, $\hat{\pi}$ as $\tilde{\pi}$ is of $\hat{\pi}$, $\hat{\pi}$ in (8.41). As $\lambda$ varies from 0 to 1, the points $\tilde{\pi}$ fill the entire interval connecting $\hat{\pi}$ to $\hat{\pi}$.

For the obstacle of $\tilde{u}$ we have equality in (8.42) (since it is a linear function). It follows that if $\tilde{\pi}$ and $\hat{\pi}$ are in the coincidence set of $\tilde{u}$, then so is the entire interval connecting them.

Since the coincidence set is convex in the $\pi$-space, it is also convex in the $y$-space.

We denote by $\tilde{C}$ the continuation set for $\tilde{u}$.

**Lemma 8.4.** Suppose (1.9) holds and

\begin{align*}
\mu_j = 0 \quad \text{for} \quad 2 \leq j \leq n.
\end{align*}

Then

\begin{align*}
\frac{\partial}{\partial y_1}(\tilde{u}(y) - a_0|y|) < 0 \quad \text{in} \quad \tilde{C}.
\end{align*}

**Proof.** Denote by $\hat{u}_A$ the solution of (8.14) subject to a different boundary condition, namely,

\begin{align*}
\hat{u}_A = 0(A).
\end{align*}

Representing $\hat{u}_A$ as a cost function and using (8.27), we find that

\begin{align*}
\hat{u}_A(y) - \tilde{u}_A(y) \to 0 \quad \text{if} \quad A \to \infty.
\end{align*}
Hence
\[
\hat{u}_A(y) \to \tilde{u}(y) \quad \text{if} \quad A \to \infty.
\]

Next, suppose we replace the domain \(|y| < A\) by the domain
\begin{equation}
(8.46) \quad \{y \in \mathbb{R}^+_n, 0 < y_i < A \quad \text{for} \quad 1 \leq i \leq n\}
\end{equation}
and denote by \(u^*_A\) the solution of the variational inequality (8.14) in (8.46) subject to boundary condition (8.45). Then again we have
\begin{equation}
(8.47) \quad u^*_A(y) \to \tilde{u}(y) \quad \text{if} \quad A \to \infty.
\end{equation}
(This follows, for instance, by working throughout the proof of Theorem 8.2 with the domains (8.46) instead of the domains \(|y| < A\).)

Denote by \(u_{\delta,A}(y) \quad (0 < \delta < A)\) the solution of the variational inequality (8.14) in the domain
\begin{equation}
(8.48) \quad \{y \in \mathbb{R}^+_n, \delta < y_i < A \quad \text{for} \quad 1 \leq i \leq n\},
\end{equation}
subject to boundary conditions
\begin{equation}
(8.49) \quad u_{\delta,A}(y) = 0(A).
\end{equation}
Then, for each fixed \(A\), we clearly have
\begin{equation}
(8.50) \quad u_{0,A}(y) \to u^*_A(y) \quad \text{if} \quad \delta \to 0.
\end{equation}

Set
\begin{equation}
(8.51) \quad v = u_{\delta,A} - a_0|y|.
\end{equation}

We choose the boundary conditions in (8.49) such that \(v \leq 0\) and
\begin{equation}
(8.52) \quad v_{y_i} \leq 0 \quad \text{on} \quad y_i = \delta, \\
\quad v_{y_i} \leq 0 \quad \text{on} \quad y_i = A \quad (2 \leq i \leq n), \\
\quad v = -C^*A \quad \text{on} \quad y_i = A \quad (C^* \text{ positive constant}).
\end{equation}

Consider the penalized problem corresponding to the variational inequality for \(v\), namely,
\begin{equation}
(8.53) \quad L_0 v_{\varepsilon} - \alpha v_{\varepsilon} - \beta_{v}(v_{\varepsilon}) + 1 - \alpha a_0|y| = 0
\end{equation}
(where \(\beta_{v}\) is as in (2.3) and \(\beta_{v}(0) = 1\)). Using (8.43) and the condition \(v_{\varepsilon} = 0\) on \(y_i = \delta\), we find that
\begin{equation}
(8.54) \quad \mu_{11} \frac{\partial^2 v_{\varepsilon}}{\partial y_i^2} = \beta_{v}(0) + a_0|y| - 1 > 0 \quad \text{if} \quad y_i = \delta,
\end{equation}
and similarly
Differentiating (8.53) with respect to \( y \), and setting \( z = \partial v_i/\partial y_i \), we get

\[
L_0 z - \alpha z - \beta_i'(v) z - \alpha a_0 = 0.
\]

It follows that \( z \) cannot take a positive maximum at an interior point. Furthermore, from (8.54), (8.55) we deduce that \( z \) cannot take a maximum on the parts \( y_i = \delta, y_i = A \) of the boundary. Since, by (8.52) \( z \leq 0 \) in the remaining parts of the boundary, we conclude that

\[
z(y) < 0 \text{ in the domain (8.48)}.
\]

Taking \( \varepsilon \to 0 \) we get

\[
\frac{\partial}{\partial y_i} (u_\varepsilon, A - a_0 |y|) = \frac{\partial v}{\partial y_i} \leq 0.
\]

Taking \( \delta \to 0 \) and using (8.50), and then letting \( A \to \infty \) and using (8.47), the assertion (8.44) follows.

**THEOREM 8.5.** Suppose (1.9) holds. Then the stopping set \( \tilde{S} \) of \( \tilde{u} \) contains an \((\text{int } R_1^r)\)-neighborhood of the origin. If

\[
\mu_{ik} = 0 \text{ for some } i \text{ and all } k \neq i
\]

then the free boundary \( \bar{T} = \partial \tilde{S} \cap (\text{int } R_1^r) \) of \( \tilde{u} \) can be represented in the form

\[
y_i = \varphi_i(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)
\]

where \( \varphi_i \) is analytic.

The proof of the first part is the same as in the case of Theorem 3.2. To prove the second part, say for \( k = 1 \), we use Lemma 8.4 and proceed as in § 5.

**REMARK.** Denote by \( S_\varepsilon \) the connected component of the coincidence set of \( w_\varepsilon(y) = u(cy)/c \) which contains \( y = 0 \). Introduce the free boundaries

\[
\Gamma_\varepsilon = \partial S_\varepsilon \cap (\text{int } R_1^r), \quad \bar{T} = \partial \tilde{S} \cap (\text{int } R_1^r)
\]

where \( \tilde{S} \) is the coincidence set for \( \tilde{u} \). The sets \( S_\varepsilon, \tilde{S} \) are contained in \( |y| < R_0 \) for some \( R_0 > 0 \). Introduce polar coordinates \((|y|, \theta) = \)
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(|y|, θ1, ..., θn−1) in \( R^n_+ \) and a truncated convex cone

\[ K = \{y; 0 < |y| \leq R \theta, \theta \in G_0 \} \]

\( G_0 \) is such that \( \partial K/\{0\} \) is contained in \( \text{int} \ R^n_+ \). Since \( S_c \) and \( S_\hat{c} \) are convex sets we can represent \( \Gamma_c, \hat{\Gamma} \) in the form

\[ \Gamma_c: |y| = \rho_c(\theta); \quad \hat{\Gamma}: |y| = \hat{\rho} (\theta). \]

From Theorem 3.2 we deduce that, for any \( \varepsilon > 0 \),

\[ |w_c(y) - u(y)| < \varepsilon \quad \text{if} \quad y \in K_\delta, \quad c \leq c(\varepsilon, \delta), \]

where \( K_\delta \) is a \( \delta \)-neighborhood of \( K \) intersected with \( \text{int} \ R^n_+; \delta > 0 \).

We claim that

\[ |\rho_c(\theta) - \hat{\rho}(\theta)| < C \varepsilon^{1/2} \quad \text{if} \quad \theta \in G_0; \]

this gives a rate of convergence of the free boundary of \( u(cy)/c \) to that of \( \hat{u}(y) \).

To prove (8.60) note first that

\[ K_\delta \cap \hat{S} \text{ contains } S_{c, \varepsilon} = \{K_\delta \cap S_c \text{ minus a } C \varepsilon^{1/2}\text{-neighborhood of} \ K \cap S_c \}. \]

Indeed, if \( y \in K_\delta \cap \hat{S} \) then (cf. [5])

\[ \sup_B \hat{u} > \varepsilon \]

where \( B \) is a ball with center \( y \) and radius \( C \varepsilon^{1/2} \); hence, by (8.59), \( \sup_B w_c > 0 \), i.e., \( y \in S_{c, \varepsilon} \).

Next \( \rho_c(\theta) \) is uniformly Lipschitz in \( \theta \) for \(|y|, \theta \) in \( K_{\delta/2} \) and small \( c \), since \( K_\delta \cap S_c \) is convex and contains a fixed \( K_\varepsilon \)-neighborhood of \( y = 0 \). Also \( \rho(\theta) \) is Lipschitz in \( \theta \). These facts together with (8.61) and its counterpart with \( \hat{S}, S_c \) interchanged, give the assertion (8.60) with a suitable \( C \).

9. Asymptotic estimates when \( c \to \infty \). Define, for any \( \varepsilon > 0 \),

\[ D_\varepsilon = \{\varepsilon \text{-neighborhood of the ridge}\} \cap \Pi_{n+1}. \]

**Theorem 9.1.** Suppose that (1.9) holds and \( c \geq c_0(0 \leq i \leq n) \). Then there exist positive constants \( B, c^* \) independent of \( \alpha \), such that, if \( c > c^* \),

\[ C \text{ is contained in } D_{B/c}. \]

Thus

\[ u(\pi) = g(\pi) \text{ outside the } B/c \text{-neighborhood of the ridge}. \]
Proof. Suppose \( \pi^0 = (\pi^0_0, \pi^0_1, \cdots, \pi^0_n) \) is in \( G_0 \) and \( \text{dist}(\pi^0, R) = B/c \). Let

\[
w(\pi) = -\delta c|\tilde{\pi} - \tilde{\pi}^0|^2 + a_0(1 - \pi^0)
\]

where \( \tilde{\pi} = (\pi_0, \cdots, \pi_n) \), \( \tilde{\pi}^0 = (\pi^0_0, \cdots, \pi^0_n) \). Clearly \( w \) lies below the obstacle \( g \) in \( G_0 \). Since

\[
\text{dist}(\pi^0, G_i) > \frac{B}{c} \quad (i \geq 1),
\]

in each \( G_i \) \( w \) decreases at a rate

\[
\geq \delta c|\tilde{\pi} - \tilde{\pi}^0| - a_0 \geq B\delta - a_0 > A_0 \quad (A_0 = \max_{0 \leq i \leq n} a_i)
\]

provided

\[
(9.4) \quad B\delta > 2A_0.
\]

This rate of decrease is faster than the linear rate of decrease of the obstacle \( g \) in \( G_i \). Hence \( w \) lies below \( g \).

Next, \( w < 0 \) outside some \((A_i/\sqrt{c})\)-neighborhood \( N \) (in \( \Pi_{n+1} \)) of \( \pi^0 \). We now compare \( w \) with \( u \) in \( N \). By the calculation leading to (7.5) we find that

\[
Mw - \alpha w > -K_i c\delta \pi^0 - \alpha w > -c
\]

if \( \delta \) is sufficiently small independently of \( c \); we use the fact that

\[
\alpha w \leq a a_0(1 - \pi^0) \leq \frac{3}{4} a a_0 c \leq \frac{3}{4} c \text{ in } N.
\]

Since \( u \geq w \) on \( \partial N \), Lemma 3.1 implies that \( u \geq w \) in \( N \).

Since \( w = g \) at \( \pi^0 \) it follows also that \( u(\pi^0) = g(\pi^0) \) provided \( \delta \) is sufficiently small and provided \( (9.4) \) holds. This completes the proof of (9.2) for points in \( C \cap G_0 \); the proof for \( C \cap G_i \) \((i \geq 1)\) is similar.

Denote by \( \tilde{G} \) any compact subset of \( \tilde{H}_{n+1} \) and set

\[
\tilde{D}_c = \{\varepsilon\text{-neighborhood of the ridge}\} \cap \tilde{G}.
\]

**Theorem 9.2.** There exists a positive constant \( A \) (depending on \( \tilde{G} \)) such that, for all \( c \) sufficiently large,

\[
(9.5) \quad C \text{ contains } \tilde{D}_{A/c}.
\]

The proof is similar to the proof of a corresponding result in [6; §4] for the elastic-plastic torsion problem.

**Proof.** Suppose \( \pi^0 \in \tilde{D}_{A/c} \cap G_0 \) and \( \pi^0 \notin C \). Suppose for simplicity
that $\pi^0$ is close to $G_1$ at least as much as it is to any other $G_i, i \geq 1$. Take points $\pi^1 \in G_i, \pi^2 \in G_1$ such that $\pi^0$ is the center of the segment $\overline{\pi^1 \pi^2}, |\pi^1 - \pi^2| = A_0 \sigma, \sigma = \text{dist.}(\pi^0, R); A_0$ is chosen so that

$$-[g(\pi^0) + g(\pi^1) - 2g(\pi^0)] \geq A_0 \sigma;$$

both $A_0, A_1$ are positive constants depending only on $a_0, a_1$. Since $u(\pi^1) \leq g(\pi^1)$, $u(\pi^2) \leq g(\pi^2), u(\pi^0) = g(\pi^0)$, we obtain

$$A_1 \sigma \leq -[u(\pi^0) + u(\pi^1) - 2u(\pi^0)] \leq A_0 \sigma^2 |u|_{W^{2,\infty}(N)},$$

where $N$ is some neighborhood of $\pi^0$. By standard estimates for variational inequalities [4], the right hand side is bounded by $A_0 \sigma^2 e$; here $A_0, A_1$ are positive constants independent of $c$. It follows that $\sigma \geq 1/(A_0 c)$, and the proof is complete.

10. The case $\alpha = 0$. For simplicity we shall assume in this section that (1.9) holds. Since $c > 0$, if $E^\pi \tau$ is sufficiently large then $J_{\pi}(\tau) > V(\pi)$. Thus we may write

$$V(\tau) = \inf_{E^\tau \tau < K_0} J_{\pi}(\tau)$$

where $K_0$ is some sufficiently large positive constant (depending on $c$).

The existence of a bounded solution (and, in fact, uniformly continuous in $\tilde{H}_{n+1}$) for the variational inequality (2.1) with $\alpha = 0$ is proved in the same way as for $\alpha > 0$. Theorem 3.2 remains valid with the same proof when $\alpha = 0$. Defining $\tilde{\tau}$ by (2.12) and recalling (7.11) we conclude that $E^{\pi \tilde{\tau}} < \infty$. But then we can apply Ito’s formula in order to deduce that $u(\pi) = J_{\pi}(\tilde{\tau})$. We also get, by Ito’s formula,

$$u(\pi) \leq J_{\pi}(\tau)$$

for any stopping time $\tau$ with $E^{\pi \tau} < K_0$. Using (10.1) we deduce that

$$u(\pi) = V(\pi) = J_{\pi}(\tilde{\tau}) \quad \text{if} \quad \pi \in \tilde{H}_{n+1}.$$
From Theorem 8.1 we deduce (for $\alpha = 0$) that

$$V(\pi) = \left(\sum_{i=0}^{n} \gamma_i \pi_i \right) c \log \frac{1}{c} + 0 \left(c \left(\log \frac{1}{c}\right)^{1/2}\right)$$
as $c \to 0$.

To generalize Theorem 8.2, consider the variational inequality (8.11) for $\alpha = 0$:

$$L_\alpha u + 1 \geq 0,$$

$$(10.5) \quad \tilde{u} \leq a_0 |y|,$$

$$(L_\alpha \tilde{u} + 1)(\tilde{u} - a_0 |y|) = 0$$
in $R^+_n$. A trivial solution is given by $a_0 |y|$. We exclude this solution by requiring that

$$u(y) = O(|y|^\theta)$$

for some $0 < \theta < 1$.

**Theorem 10.1.** Let (1.9) hold. Then there exists a unique solution $\tilde{u}$ of (10.5), (10.6); further,

$$0 \leq \tilde{u}(y) \leq C \log(|y| + 1)$$

for some positive constant $C$, and

$$\frac{u(cy)}{c} \to \tilde{u}(y)$$

uniformly in $y$ in compact subsets of $\text{int} \, R^+_n$.

**Proof.** Let

$$z(y) = \begin{cases} a_0 |y| & \text{if } |y| \leq \delta, \\ A \log |y| + B & \text{if } |y| > \delta. \end{cases}$$

For suitable positive constants $\delta, A, B$, one finds that $z$ is a supersolution, i.e., $L_\alpha z + 1 \leq 0$. Hence

$$0 \leq \tilde{u}_A(y) \leq z(y)$$

where $\tilde{u}_A$ is the solution of (8.14) with $\alpha = 0$. It follows that

$$0 \leq \tilde{u}(y) \leq C \log(|y| + 1), \quad \tilde{u}(y) = \lim \tilde{u}_A(y),$$

where $C$ is a generic positive constant independent of $c$.

Next

$$u(\pi) \leq w_0(r),$$

where $w_0$ appears in (7.10). Recalling the precise form of $w_0(r)$ we compute that
provided $|y| \leq \delta_0/c$ where $\delta_0$ is any positive constant (independent of $c$).

We are now ready to proceed with the proof of (8.31), (8.32) in the case $\alpha = 0$. From (10.9), (10.10) and the form of the cost functionals corresponding to $\bar{u}$, $w_c$ we see that we may restrict the $\tau$ to satisfy

\begin{equation}
\tau \leq \bar{\tau}^A, \quad E_y[\tau^A] \leq C \log(|y| + 1) \leq C \log(A + 1).
\end{equation}

The last term in (8.23), for $\alpha = 0$, is bounded by

\begin{equation}
I_A = C \log(A + 1) P_y[\tau^A < \tau].
\end{equation}

Now, for any $\beta > 0$,

\begin{equation}
P_y[\bar{\tau}^A < \tau] = P_y[e^{-\beta(\bar{\tau}^A - \tau)} > 1] \leq E_y[e^{-\beta(\bar{\tau}^A - \tau)}]
\leq \{E_y[e^{\beta \tau^A}]\}^{1/p} \{E_y[e^{\beta \tau^A}]\}^{1/q}
\end{equation}

where $1/p + 1/q = 1$, $p > 1$, $q > 1$.

Since the stopping times which minimize the cost functions are exit times, we may take $\tau$ to be an exit time. Using the second inequality in (10.11) it then follows by ([8; p. 43]) that

\begin{equation}
E_y[e^{\beta \tau^A}] \leq C \quad \text{provided} \quad \beta = \frac{1}{C \log(A + 1)}.
\end{equation}

From the proof of (8.19) with $\kappa - 1 = p\beta/C$ we get

\begin{equation}
E_y[e^{-\beta \tau^A}] \leq \frac{C|y|^2}{A^2} \leq \frac{C|y|^2}{A}
\end{equation}

substituting this estimate and (10.14) into (10.13), we get

\begin{equation}
P_y[\bar{\tau}^A < \tau] \leq \frac{C|y|^{2/p}}{A^{2/p}}.
\end{equation}

Consequently, from (10.12), for any $\gamma > 0$,

\begin{equation}
I_A < \gamma \quad \text{if} \quad A \text{ is sufficiently large;}
\end{equation}

$A$ is independent of $c$. From now on $A$ is fixed. Hence, if $c$ is sufficiently large (depending on $A$),

\begin{equation}
E_y \left| \frac{|y_c(\tau)|}{1 + c|y_c(\tau)|} - |y_c(\tau)| \right| < \gamma.
\end{equation}

In order to complete the proof of (8.31), (8.32), it remains to show that
Now, by (10.11), for any $T > 0$,
\[ P_y[\tau > T] \leq \frac{1}{T} E_y \tau \leq \frac{C}{T} \log(\|y\| + 1). \]

Hence, if $y$ varies in a compact subset,
\[ AE_y[\tau > T] \leq \eta \text{ for a suitable } T > 0. \]

Since, on the other hand, (8.27) holds, the estimates (10.16) follow if $c$ is small enough. We have thus completed the proof of (8.31), (8.32).

Suppose finally that $\tilde{u}$ is another solution of (10.5), (10.6). Repeating the preceding proof of (8.31), (8.32) with $w_c(y)$ replaced by $\tilde{u}(y)$ and choosing $p$ in (10.15) such that $1/p > \theta$, we find that $\tilde{u} \equiv \tilde{u}$.

11. The case where $w(t)$ is $k$-dimensional. In this section we extend many of the results of the previous sections to the case where $w(t)$ is $k$-dimensional; the condition (1.1) is dropped. Thus the generator $M$ is generally a degenerate elliptic operator in the entire region $\Pi^{n+1}_{*}$. We assume, however, that (1.9) holds, so that

(11.1) \[ J_x(\tau) = E_y\left[ c \int_0^\tau e^{-at} dt + e^{-at} g(\pi(\tau)) \right]. \]

From (11.1), (1.15) and the strong Markov property we get

(11.2) \[ V(\pi) = \inf_{\tau \leq \tau_N} E_y\left\{ c \int_0^{\tau \wedge \tau_N} e^{-at} dt + e^{-at} g(\pi(\tau)) I_{\tau < \tau_N} + e^{-a\tau_N} V(\pi(\tau_N)) I_{\tau = \tau_N} \right\} \]
for any stopping time $\tau_N$.

**Theorem 11.1.** There exists a $\Pi^{n+1}_{*}$-neighborhood $\tilde{S}_i$ of $e_i$ such that $\tilde{S}_i \subset S$.

**Proof.** Set $W = V - g$ in a neighborhood $N$ of $e_0$ where $g(\pi) = a_\circ(1 - \pi_0)$, and let $\tau_N = \text{exit time from } N$. Thus, for any stopping time $\tau \leq \tau_N$

\[ E^x[e^{-\alpha\tau} g(\pi(\tau)) - g(\pi)] = E^x\left[ \int_0^\tau c e^{-at}(M - \alpha) g(\pi(t)) dt \right] = E^x\left[ \int_0^\tau c e^{-at}(-\alpha g)(\pi(t)) dt \right]. \]
Therefore, by (11.2),
\[ W(\pi) = V(\pi) - g(\pi) = \inf_{\tau \in \tau_N} \mathbb{E}^{\pi} \left\{ \int_0^{\tau_N} e^{-a\tau}(c - \alpha g)(\pi(t))dt + e^{-a\tau_N} W(\pi(\tau_N))I_{\tau=\tau_N} \right\}. \]

(11.3)

Note also that \( W(\pi) \leq 0 \) and that \( c - \alpha g \lesssim C^* > 0 \) if \( N \) is sufficiently small.

The function \( z \) defined following (3.11) satisfies
\[ Mz - \alpha z + \gamma \geq 0 \] (with \( \gamma < c - \alpha g \)),
\[ z \leq 0, \]
\[ (Mz - \alpha z + \gamma)z = 0, \]
and
\[ z \leq w \quad \text{on} \quad |y| = R. \]

Using Ito's formula we obtain
\[ z(\pi) = \inf_{\tau \in \tau_N} \mathbb{E}^{\pi} \left[ \int_0^{\tau_N} e^{-a\tau}r\gamma dt + e^{-a\tau_N} z(\pi(\tau_N))I_{\tau=\tau_N} \right]. \]

Comparing with (11.3) we conclude that
\[ z(\pi) \leq W(\pi). \]

Since \( z(\pi) = 0 \) where \( \pi \) varies in some neighborhood of \( e_0 \), the same follows for \( W \); this completes the proof.

Theorems 6.1, 6.2 remain valid with the same proof.

**Lemma 11.2.** The estimate (7.11) is valid.

**Proof.** Because of the degeneracy of \( M \), we need to choose the functions \( W_1 \) differently than in the proof of (7.11) in \$7. For simplicity we exhibit the construction in case \( n = 2 \). Take
\[ W_0 = \log \frac{\pi_1 + \pi_2}{\varepsilon} + \log \frac{\delta_1 \pi_1 + \delta_2 \pi_2}{\varepsilon} \]
outside an \( \varepsilon \)-neighborhood of \( e_0 \), where \( \delta_1 = 1 \), \( \delta_2 = 3 \). Thus \( W_0 \geq 0 \) and
\[ MW_0 = M \log(\pi_1 + \pi_2) + M \log(\delta_1 \pi_1 + \delta_2 \pi_2) \]
\[ \leq -\frac{A_0}{n^2} [(\pi_1 - \pi_2)^2 + (\delta_1 \pi_1 - \delta_2 \pi_2)^2] \]
\[ \leq -\frac{A}{n^2} (\pi_1 + \pi_2)^2 = -A. \]
where \( r = \pi_1 + \pi_2 \) and \( A_0, A \) are positive constants.

Similarly, we define the \( W_i \) with respect to \( e_i \) and notice that

\[
MW_i \leq 0, \quad W_i \geq 0.
\]

Hence \( W = \sum_{i=0}^{n} W_i \) satisfies \( MW < -A, W > 0 \) outside an \( \varepsilon \)-neighborhood of the vertices. This implies, by Ito's formula,

\[
E^\varepsilon \tau_e \leq \frac{W}{A},
\]

and (7.11) follows.

Using (7.11) we can now derive Theorem 8.1 as before.

Theorem 8.2 asserts that

\[
\frac{u(\epsilon g)}{c} \longrightarrow \tilde{u}(g) \quad \text{as} \quad c \longrightarrow 0
\]

where \( \tilde{u}(g) \) is defined by (8.25), (8.26). The proof can actually be given by comparing the cost functionals and without introducing variational inequalities at all. Notice that the crucial estimate (8.12) remains valid here (with the same proof) and that also the inequality

\[
\frac{u(\epsilon g)}{c} \leq A
\]

which is needed in proving (11.4) is true (in fact, taking \( \tau \rightarrow \infty \) in the cost functional which defines \( u \) we obtain (11.5)).

The proof of Theorem 3.2 extends to \( \tilde{u} \) (cf. the proof of Theorem 11.1), showing that the stopping set \( \tilde{S} \) contains a neighborhood of each vertex. This proves the part

\[
S \supset N_{\epsilon \gamma}
\]

of Theorem 7.1; the other part follows as in the proof of Theorem 7.1, since

\[
M(\log r) \geq -K_1 \pi_0^2.
\]

The convexity of \( \tilde{S} \) (Theorem 8.3) remains unchanged. Finally, the results of \$10\) (the case \( \alpha = 0 \)) extend with minor changes.

We shall now obtain additional information, taking \( n = 2 \) and \( w(t) \) to be 1-dimensional. We also take for simplicity

\[
\lambda_0 = 0, \quad \lambda_1 = -1, \quad \lambda_2 = 1.
\]

Writing \( \pi_i(t) \) in terms of the observed process (see (1.11)) we easily compute that
\[ \xi(t) = \frac{1}{2} \log \frac{\pi_1 \pi_1}{\pi_2 \pi_2} \]
\[ t = \log \frac{\pi_0^2}{\pi_1 \pi_2} \frac{p_1 p_2}{p_0^3} \]

where \( p_j = \pi_j(0) \), \( \pi_j = \pi_j(t) \).

The mapping \( \sigma: (\xi(t), t) \rightarrow \pi(t) \)

is 1-1, mapping the half-plane \( t > 0 \) onto a subset of \( \Pi_3 \) defined by

\[ \pi_0^2 > \pi_1 \pi_2 \frac{p_0^3}{p_1 p_2} . \]

The ridge of \( \Pi_3 \) is not in the stopping set \( S \) and \( \sigma \) maps \( S_t \) onto a set \( \Sigma_t \); see the accompanying figure.

Take a point \( t \) on the \( t \)-axis and mark the point \( A' = (\xi', t) \) on \( \partial \Sigma_0 \) with \( \xi' < 0 \). Denote by \( \rho(t) \) the distance from \( A' \) to \( \partial \Sigma_1 \); it is achieved at \( B' \in \partial \Sigma_1 \). Denote by \( A, B \) the inverse images of \( A', B' \) under \( \sigma \).

**Theorem 11.3.** As \( t \rightarrow \infty \)

\[ \rho(t) \rightarrow \frac{2}{\sqrt{5}} \left| \log \frac{\alpha(1 - \beta)}{\beta(1 - \alpha)} \right| \]
where \( \alpha, \beta \) are positive constants described in the previous figure.

**Proof.** Write

\[
A = (\pi_0, \pi_1, \pi_2), \quad B = (\bar{\pi}_0, \bar{\pi}_1, \bar{\pi}_2).
\]

Then, as \( t \to \infty \),

\[
\begin{align*}
\pi_1 & \to 0, \quad \bar{\pi}_2 \to 0, \quad \pi_0 \to \alpha, \\
\pi_1 & \to 1 - \alpha, \quad \bar{\pi}_0 \to \beta, \quad \bar{\pi}_1 \to 1 - \beta.
\end{align*}
\]

Setting

\[
\pi_2 = \varepsilon, \quad \bar{\pi}_2 = \bar{\varepsilon},
\]

we can then write

\[
\begin{align*}
\pi_0 &= \alpha - \gamma \varepsilon, \quad \pi_1 = 1 - \alpha + \gamma \varepsilon - \varepsilon, \\
\bar{\pi}_0 &= \beta - \delta \bar{\varepsilon}, \quad \bar{\pi}_1 = 1 - \beta + \delta \bar{\varepsilon} - \bar{\varepsilon},
\end{align*}
\]

where

\[
\varepsilon \to 0, \quad \bar{\varepsilon} \to 0, \quad \gamma \varepsilon \to 0, \quad \delta \bar{\varepsilon} \to 0 \text{ as } t \to \infty.
\]

From (11.6) we find that

\[
\rho = \rho(t) = \left\{ \frac{1}{4} \left[ \log \frac{\pi_2}{\pi_1 \pi_2} \right]^2 + \left[ \log \frac{\pi_0}{\pi_0 \pi_2} \left( \frac{\pi_1}{\pi_1 \pi_2} \right) \right]^2 \right\}^{1/2}.
\]

Hence

\[
\rho^2 = \frac{1}{4} \left[ \log \left( \frac{\lambda}{1 - \alpha} \left( 1 + o(1) \right) \right) \right]^2 + \left[ \log \frac{\lambda}{\beta^2 (1 - \alpha)} \left( 1 + o(1) \right) \right]^2.
\]

where \( \lambda = \bar{\varepsilon}/\varepsilon, \ o(1) \to 0 \) as \( t \to \infty \).

From the definition of \( B' \) it follows that

\[
\rho = \bar{\rho}(1 + o(1))
\]

where

\[
\bar{\rho} = \min_{\lambda} \left\{ \frac{1}{4} \left[ \log \lambda \frac{1 - \beta}{1 - \alpha} \right]^2 + \left[ \log \lambda \frac{\beta^2 (1 - \alpha)}{\alpha^2 (1 - \beta)} \right]^2 \right\}^{1/2}.
\]

Set

\[
\rho(\lambda) = \frac{1}{4} \left( \log k \lambda \right)^2 + \left( \log h \lambda \right)^2 \quad (k > 0, \ h > 0).
\]

Then \( \min \rho(\lambda) \) is obtained at

\[
\lambda = h^{-4/3} k^{-15}.
\]
Using this value in our special case of $\tilde{p}$, (11.7) follows. Consider next the point $(\xi, t) = (0, \tau_c)$ on $\partial \Sigma_0$.

**Theorem 11.4.** As $c \to 0$

$$
(11.8) \quad \tau_c \sim \log \frac{1}{\gamma^2 c^2}
$$

where $\gamma$ is a positive constant.

**Proof.** $(0, \tau_c)$ corresponds to $(\pi_0, \pi_i, \pi_2)$ where, by Theorem 8.2 and 8.3,

$$
\pi_0 \sim \frac{1}{1 + \gamma_1 c + \gamma_2 c}
$$

$$
\pi_i \sim \frac{\gamma_i c}{1 + \gamma_1 c + \gamma_2 c} \quad (i = 1, 2)
$$

and $\gamma_1 = \gamma_2 = \gamma$ since $\pi_1 = \pi_2$. Since

$$
\tau_c = \log \frac{\pi_0^2}{\pi_1 \pi_2} = \log \frac{\pi_0^2}{\pi_1^2},
$$

the assertion follows.

**Remark.** Theorems 11.3, 11.4 have the advantage of providing direct information on the observed process $(\xi(t), t)$. In case $n > 2$ we get a similar picture with $n + 1$ regions $\Sigma_i$ in the half-plane $t > 0$ (of the $(\xi, t)$ variable).

**References**


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