EMBEDDINGS OF THE PSEUDO-ARC IN $E^2$

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In this paper, we show that there exists an embedding, $P_\ast$, of the pseudo-arc in the plane such that any two accessible points lie in distinct composants of $P_\ast$. We also show that there are $c=2^{\omega_0}$ distinct embeddings of the pseudo-arc in the plane, including for each positive integer $n$, one with exactly $n$ composants accessible. This answers some questions and a conjecture of Brechner.

For definitions and notation of chain (from $p$ to $q$), link, crooked, etc., see [1] and [7]. The links of our chains will always be the interiors of disks, and if two links of a chain intersect their intersection is the interior of a disk. When a chain $D$ refines a chain $C$, we shall always require that the closure of each link of $D$ be contained in a link of $C$.

First we describe the special embedding $P_\ast$, then prove Brechner’s conjecture that any two distinct accessible points of $P_\ast$ lie in distinct composants. Let $C_0$ be a chain in $E^2$ from point $p$ to point $q$ which runs straight across from left to right horizontally. Let $C_1$ be a chain also running from $p$ to $q$ which is crooked in $C_0$ and descending, as in Figure 1. If we think of $C_1$ as straightened out with $p$ on the left and $q$ on the right, then $C_2$ is a chain from $p$ to $q$ which is crooked in $C_1$ and ascending. We continue in this manner, alternating descending and ascending chains, so that $C_i$ runs from $p$ to $q$, mesh $(C_i) < 1/2^i$, $C_{i+1}$ refines and is crooked in $C_i$, and $C_{i+1}$ is descending (ascending) in $C_i$ if $i$ is even (odd). The pseudo-arc $P_\ast$ is $\bigcap_{i=0}^\infty C_i^\ast$. (If $A$ is a collection of sets, $A^\ast$ is the union of $A$.)

**Theorem.** Any two distinct accessible points of $P_\ast$ are in distinct components.

**Proof.** We can draw horizontal rays to the left from $p$ and to
the right from \( q \). A top accessible point will be a point of \( P_s \) which is accessible by an arc lying in the upper complementary domain of \( P_s \) plus the two rays (except for the endpoint of the arc in \( P_s \)). A bottom accessible point is defined similarly. We will show that any two distinct top accessible points are in distinct composants. A similar argument will show that any two distinct bottom accessible points are in distinct composants.

Let \( a \) and \( b \) be distinct top accessible points. Suppose \( a = \bigcap_{i \in \omega_0} C_i(a_i) \) and \( b = \bigcap_{i \in \omega_0} C_i(b_i) \). Let \( \alpha \) and \( \beta \) be arcs above \( P_s \) with \( \alpha \cap P_s = a \) and \( \beta \cap P_s = b \). We can suppose without loss of generality that for each \( i \in \omega_0 \), \( \alpha \cap C_i^* \) and \( \beta \cap C_i^* \) are connected.

**Claim.** The subcontinuum \( M \) of \( P_s \) irreducible between \( a \) and \( b \) contains both \( p \) and \( q \) (i.e., for each \( i \in \omega_0 \) and sufficiently large \( j \in \omega_0 \) the subchain of \( C_i \) between \( C_i(a_i) \) and \( C_i(b_i) \) has links in each of \( C_i(0) \) and \( C_i(n_i) \), where \( C_i(n_i) \) is the last link of \( C_i \).

**Proof of claim.** For each \( i \in \omega_0 \) there exists \( k \in \omega_0 \) such that, for \( j \geq k \), \( cl \left( \{C_j(n)|C_j(n) \cap \alpha \neq \emptyset \}^* \right) \subseteq C_i(a_i) \) and \( cl \left( \{C_j(m)|C_j(m) \cap \beta \neq \emptyset \}^* \right) \subseteq C_i(b_i) \). Choose \( i \) large enough that there are at least two links of \( C_i \) between \( C_i(a_i) \) and \( C_i(b_i) \), and \( k > i \) so that the above condition holds and it takes at last three links of \( C_k \) to span between nonadjacent links of \( C_i \) or to reach from \( C_k(0) \) to \( C_i(1) \) or to reach from \( C_k(n_k) \) to \( C_i(n_i - 1) \).

Consider \( j > k \) such that \( j \) is even. Then \( \{C_j(n)|C_j(n) \cap \alpha \neq \emptyset \} \) and \( \{C_j(m)|C_j(m) \cap \beta \neq \emptyset \} \) are separated by several links of \( C_j \). Suppose \( \{C_j(n)|C_j(n) \cap \alpha \neq \emptyset \} \) comes first in \( C_j \). Then because \( C_{j+1} \) is descending in \( C_j \) and \( \alpha \) lie above \( P_s \), \( C_{j+1}(a_{j+1}) \) is in the maximal subchain of \( C_{j+1} \) with no links reaching past \( \{C_j(n)|C_j(n) \cap \alpha \neq \emptyset \}^* \). But \( C_{j+1}(b_{j+1}) \) is not in any of the links of \( C_j \) up to this point (or in fact at least three links beyond), so by crookedness of \( C_{j+1} \) in \( C_j \) there is a link \( \gamma \) of \( C_{j+1} \) between \( C_{j+1}(a_{j+1}) \) and \( C_{j+1}(b_{j+1}) \) with \( \gamma \subseteq C_j(1) \subseteq C_i(0) \).

Similarly, if \( j \) is odd, there is a link \( \delta \) of \( C_{j+1} \) between \( C_{j+1}(a_{j+1}) \) and \( C_{j+1}(b_{j+1}) \) with \( \delta \subseteq C_{j+1}(n_{j+1} - 1) \subseteq C_i(n_i) \). Thus the subcontinuum \( M \) of \( P_s \) irreducible between \( a \) and \( b \) contains both \( p \) and \( q \). Hence \( M = P_s \), and \( a \) and \( b \) are in different composants of \( P_s \).

Similarly any two bottom accessible points of \( P_s \) are in different composants. By Theorem 3.1 of [5] if top and bottom accessible points of \( P_s \) are in the same composant \( C \) of \( P_s \) then either \( p \in C \) or \( q \in C \). Thus the top and bottom accessible points must be the same and be either \( p \) or \( q \). So any two distinct accessible points of \( P_s \) are in different composants.
It follows from [8] that, though $P_8$ has $c = 2^{\omega_0}$ distinct accessible composants, there exists some component of $P_8$ which is not accessible.

2. Other embeddings. We will now show how to obtain $c = 2^{\omega_0}$ distinct embeddings of the pseudo-arc in the plane. These will be distinguished by use of prime ends and accessibility. First however we will describe $c = 2^{\omega_0}$ distinct 0-dimensional closed subsets of the unit circle, $S^1$, which will be associated with these embeddings.

Let $X_j = e^{\pi i j}$ for $j = 1, 2, \cdots$. (This is the only place in the paper where $i$ is not an integer or finite ordinal. Here of course $i = \sqrt{-1}$. In all later discussion, we return to letting $i$ be an integer or ordinal.) Let $X_0 = 1$. This is a simple sequence which divides $S^1$ into a countable number of open intervals. For any subset $A$ of $\{1, 2, \cdots\}$, let $C_A$ be the closed set consisting of $\{X_i\}_{i \in \omega_0}$ together with a Cantor set in the open interval between $X_i$ and $X_{i+1}$ for each $i \in A$. We shall describe how to embed a pseudo-arc $P_A$ in the plane such that its space of prime ends is homeomorphic to $S^1$ by a homeomorphism $h$, where for each open interval $I$ of $S^1 - C_A$ all accessible points which correspond to prime ends in $h^{-1}(I)$ are in the same composant of $P_A$, and accessible points which correspond to prime ends in different intervals are in different composants of $P_A$. Thus if $A$ and $B$ are distinct subsets of $\{1, 2, \cdots\}$ then $P_A$ and $P_B$ are inequivalently embedded in the plane.

For each basic Cantor set $C$ in $C_A$, $C = \bigcap_{i \in \omega_0} C(i)$ where each $C(i)$ is a finite collection of closed intervals in $S^1$ and $C(i+1)$ is obtained by removing open intervals from the middle of each component of $C(i)$. Order the set of all endpoints of components of $C(i)$'s such that each endpoint of a component of $C(i)$ comes before each endpoint of a component of $C(i+1)$ which is not also an endpoint of a component of $C(i)$.

Let $\{y_i\}_{i \in \omega_0}$ be a well-ordering of the set of all end points of components of $S^1 - C_A$ such that:

1. $y_0 = X_1$ and $y_1 = X_0$.
2. For each basic Cantor set $C$ in $C_A$ the restriction of the well-ordering of $\{y_i\}_{i \in \omega_0}$ to points in $C$ is the ordering described above.
3. Both $X_i$ and $X_{i+1}$ come before any point of a Cantor set between these two points.

Let $C_0$ be a chain in $E^2$ running straight across horizontally from a point $z_0$ (in link $L_0$) to a point $z_1$ (in link $L_1$). (Consistent with our previous notation, subscripts will not indicate adjacent links but will rather indicate points $z$ contained in these links.) Suppose inductively that chain $C_{2^k}$ has been formed with distinct nonadjacent
links $L_0, L_1, \ldots, L_{t+1}$ specified so that the ordering of the $L_j$'s along the chain corresponds to the ordering of $\{y_j\}_{j \leq t+1}$ in $S^1$ going from $X_1$ to $X_0$ clockwise. Suppose also that points $z_j$ have been specified in each $L_j$ with $\text{st}(z_j, C_{2i}) = L_j$. We will now describe how to form chains $C_{2i+1}$ (refining $C_{2i}$) and $C_{2i+2}$ (refining $C_{2i+1}$).

Think of chain $C_{2i}$ as straightened out horizontally with $z_0$ on the left and $z_1$ on the right. Let $\{W_n\}_{n \leq i+2}$ be the ordering of $\{y_j\}_{j \leq t+2}$ induced by the order of the points in $S^1$ from $X_1$ to $X_0$ clockwise. Let $\mu$ be a bijection such that $W_n = y_{\mu(n)}$ for each $n \leq i+2$. In chain $C_{2i+1}$ is a chain (see Figure 2) from $z_0$ to $z_1$ which starts by running straight from $L_0$ to $L_1$, then consists of segments $D_n$, for $1 < n \leq i+2$, such that (for $\mu(n) \neq i+2$):

1. $D_n$ runs straight from $L_1$ to $L_{\mu(n)}$ above all previous parts of $C_{2i+1}$, straight back to $L_1$ above all previous parts of $C_{2i+1}$, straight to $L_0$ below all previous parts of $C_{2i+1}$, then straight back to $L_1$ below all previous parts of $C_{2i+1}$.

2. The bend $D_n$ in $L_{\mu(n)}$ contains $z_{\mu(n)}$, where $W_n = Y_{\mu(n)}$.

3. $D_n$ intersects only $D_{n-1}$ and $D_{n+1}$, each of which it intersects in an end link.

If $y_{i+2}$ is a point of a basic Cantor set $C$ of $C_A$ and is either the leftmost point of $C$ or the left end one of the intervals removed in forming $C$ (by the $C(i)$'s), then $D_{i+2}$ (where $\mu(i+2) = i + 2$) satisfies conditions (1) and (3) with $L_{i+2}$ being the link of $C_{2i}$ immediately after $L_{\mu(i+2)}$. Choose $z_{i+2}$ in the bend of $D_{i+2}$ in $L_{\mu(i+2)}$ (and not in either adjacent link of $C_{2i+1}$). Otherwise do the same with $L_{i+2}$ chosen to be the link of $C_{2i}$ immediately before $L_{\mu(i+2)}$. The chain $C_{2i+1}$ is the union of the $D_n$'s and the initial straight segment from $L_0$ to $L_1$.

To get chain $C_{2i+2}$ think of straightening $C_{2i+1}$ out horizontally with $z_0$ on the left, and consider the set $\Gamma$ of links of $C_{2i+1}$ which are either end links of the $D_n$'s, links where the bends of the $D_n$'s occur, or end links of $C_{2i+1}$. In each subchain of $C_{2i+1}$ connecting consecutive elements in $\Gamma$, place a crooked descending chain going

![Figure 2](image-url)

One possible configuration of the nerve of $C_0$ in $C_2$ is shown.
between the two ends (and if a \( z_n \) is in such a subchain place it in the appropriate end link of the crooked chain). This can be done so that the underlying point sets of crooked chains in adjacent subchains intersect exactly in an end link. Chain \( C_{2i+2} \) will be the union of these small crooked chains.

Note that, while \( C_{2i+1} \) is not crooked in \( C_{2i} \), nor is \( C_{2i+2} \) in \( C_{2i+1} \), chain \( C_{2i+2} \) is crooked in \( C_{2i} \). If we do this so that the mesh of the chains gets arbitrarily small, then the intersection is a pseudoarc \( P_A \) [2]. By construction, each \( z_t \) is accessible, and different \( z_t \)'s lie in different composants of \( P_A \).

Let \( h \) be a homeomorphism between the space of prime ends of \( P_A \) and \( S^1 \) such that \( h(z_i) = y_i \) for each \( i \in \omega_0 \), where \( z_i \) is the prime end associated with the accessible point \( z_t \). Suppose \( p \) and \( q \) are accessible points of \( P_A \) with associated prime ends \( \bar{p} \) and \( \bar{q} \), where \( h(\bar{p}) \) and \( h(\bar{q}) \) lie in the same component \( I \) of \( S^1 - C_A \). Let \( a \) (respectively \( b \)) be the accessible point whose associated prime end \( \bar{a} \) (resp. \( \bar{b} \)) is mapped by \( h \) to the largest (smallest) endpoint of \( I \) in the counterclockwise ordering \( (0, 2\pi] \) of \( S^1 \).

Claim. Each of \( p \) and \( q \) is in the same composant of \( P_A \) as the point \( a \).

Proof. Let \( \alpha \) and \( \beta \) be disjoint rays to infinity which intersect \( P_A \) only at their endpoints \( a \) and \( b \) respectively. Let \( \pi \) be a ray, disjoint from \( \alpha \) and \( \beta \), which intersects \( P_A \) only in its endpoint \( p \). We may assume that, for each \( i \in \omega_0 \), \( \pi \cap C_i^* \) is connected (as are also \( \alpha \cap C_i^* \) and \( \beta \cap C_i^* \)). If \( a = y_{m_1} \) and \( b = y_{m_2} \) choose \( N \) bigger than both \( m_1 \) and \( m_2 \) and such that the sets \( \{ l \in C_{2N} | \alpha \cap l \neq \emptyset \}^* \), \( \{ l \in C_{2} | \pi \cap l \neq \emptyset \}^* \), and \( \{ l \in C_{2} | \beta \cap l \neq \emptyset \}^* \) are disjoint. For each \( n > 2N \), let \( M_n \) be the minimum subchain of \( C_n \) containing both \( \{ l \in C_{n} | \alpha \cap l \neq \emptyset \} \) and \( \{ l \in C_{n} | \pi \cap l \neq \emptyset \} \). Then for each \( n > 2N \), \( \text{cl}(M_n^*) \subseteq \text{cl}(M_n^*) \), by our construction of the \( C_i^* \)'s (and since there are no other \( y_j \)'s between \( a \) and \( b \)). Thus \( M = \bigcap_{n>2N} \text{cl}(M_n^*) \) is a proper subcontinuum of \( P_A \) containing both \( a \) and \( p \). Similarly, there is a proper subcontinuum of \( P_A \) containing both \( a \) and \( q \).

By the above claim, the fact that all of the \( y_j \)'s are in different composants, and Theorem 3.1 of [5], we get that \( p \) and \( q \) are in different composants of \( P_A \) if \( h(\bar{p}) \) and \( h(\bar{q}) \) are in different components of \( S^1 - C_A \).

If we use the above procedure to construct pseudo-arcs, but stop introducing new \( z_n \)'s and \( L_n \)'s at some point, we can obtain for each positive integer \( i \) a pseudo-arc in the plane with exactly \( i \) composants accessible.
3. **Questions.** Though our $P_s$'s are all embedded differently in $E^2$, any two contain equivalently embedded subcontinua (e.g., ones containing $z_t$). This leads us to the following question.

*Question 1.* Do every two pseudo-arcs in the plane contain equivalently embedded subcontinua? (A comparison of subcontinua of $P_s$ with subcontinua of $P_t$ might be useful here.)

The following is also of interest.

*Question 2.* Are there $c = 2^\omega$ distinct embeddings of the pseudo-circle in $E^2$? of every hereditarily indecomposable plane continuum?

We know that, though there are embeddings of the pseudo-arc with $c = 2^\omega$ distinct accessible composants, there are also always inaccessible composants [8]. Of the embeddings we have described, $P_s$ is the only one with the property that any two accessible points are in distinct composants. Is there any other embedding with this property?

Michel Smith has recently announced results analogous to these.

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