THE ASYMMETRIC PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS

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Let \( L_i = \sum_{j=1}^{t} a_{ij} x_j \), \( i = 1, 2, 3 \), be three linear forms in the variables \( x_1, x_2, x_3 \) with real coefficients \( a_{ij} \). A theorem of Davenport asserts that, if \( | \det (a_{ij}) | = 7 \), then there exist integers \( u_1, u_2, u_3 \), not all zero, such that

\[
\prod_{i=1}^{3} L_i(u_1, u_2, u_3) \leq 1.
\]

Under the same hypothesis, W. H. Adams has asked whether, given a positive real number \( u \), there exist integers \( u_1, u_2, u_3 \), not all zero, such that

\[
-u^{-1} \leq L_1(u_1, u_2, u_3)L_2(u_1, u_2, u_3)L_3(u_1, u_2, u_3) \leq u.
\]

Our objective is to prove this conjecture.

Davenport gave several proofs of his theorem [3], and other proofs have been given by Chalk and Rogers [2] and Mordell [8]. Isolation results, notably those of Davenport [6] and Swinnerton-Dyer [10], show that Adams conjecture is true for real \( u \) in some open interval containing 1.

The set of points \((L_1, L_2, L_3)\) in \( R_3 \), formed as the variables range over all integral values, is a lattice \( \Lambda \) of determinant \( d(\Lambda) = | \det (a_{ij}) | \). In terms of \( \Lambda \), our result is as follows.

**Theorem.** If \( d(\Lambda) = 7 \), then there exists a point \((x_1, x_2, x_3)\) of \( \Lambda \), other than the origin, such that

\[
-u^{-1} \leq x_1 x_2 x_3 \leq u,
\]

with the equality sign being necessary only if \( u = 1 \).

The method of proof is the projective one due to Davenport [3]. We begin with three lemmas.

**Lemma 1.** If \( x, y, z, t \) are real numbers with \( 1 < t^2 \leq 1.9 \), such that the inequality

\[
-t^2 < (n + x)(n + y)|n + z| < 1
\]

is not solvable in integers \( n \), then

\[
\varphi = (x - y)^2 + (y - z)^2 + (z - x)^2 > 14t.
\]

We note that this is a generalization of a lemma due to
Davenport [3].

Proof. We may assume that none of $x$, $y$, $z$ is an integer, for otherwise inequality (1) is solvable for an integer $n$. We distinguish cases according to the comparative sizes of $[x]$, $[y]$, $[z]$.

Case 1. Two of $[x]$, $[y]$, $[z]$ are equal.

As $x$, $y$, $z$ may be replaced by $x + n$, $y + n$, $z + n$ respectively, for any integer $n$, without altering either the hypothesis or the conclusion of the lemma, we may assume that two of $[x]$, $[y]$, $[z]$ are zero. Inequality (1) implies that

$$|(n + x)(n + y)(n + z)| < 1$$

has no solution in integers $n$.

If $[x] = [y] = 0$, then $xy(1 - x)(1 - y) \leq 1/16$. If, further, $|xyz(x - 1)(y - 1)(z - 1)| < 1$, then (3) is solvable for one of the values $n = 0$, $-1$. Hence, we must have $|z(z - 1)| \geq 16$, whence $z(z - 1) \geq 16$, so that either $z < -3.5$ or $z > 4.5$. As $0 < x$, $y < 1$, it follows that $|x - z| > 3.5$ and $|y - x| > 3.5$ and therefore also $\varphi > 24.5$. Thus, if $\varphi \leq 14t$, then $t > 1.75$ and $t^2 > 1.9$, contrary to hypothesis. Hence $\varphi > 14t$.

As (3) is symmetric in $x$, $y$, $z$ the other two possibilities follow by the same argument.

Case 2. Two of $[x]$, $[y]$, $[z]$ differ by 1 and no two are equal.

Suppose first $[x]$, $[y]$ differ by 1. As we may replace $x$, $y$, $z$ by $x + n$, $y + n$, $z + n$ respectively, for any integer $n$, without altering either the hypothesis or the conclusion of the lemma, we may assume that $[x] + [y] = -1$. Again, we may replace $x$, $y$, $z$ by $-x$, $-y$, $-z$ respectively, without altering the lemma, so we may assume that $z > 0$. Finally, by the symmetry of $x$ and $y$ in the lemma, we may assume that $-1 < x < 0 < y < 1$.

If $z < 1$ then $-1 < xyz < 0$, contrary to inequality (1). Therefore $z > 1$. Putting $f(n) = (x + n)(y + n)(z + n)$, we have $f(1) \geq 1$, $f(0) \leq -t^2$ and $f(-1) \geq 1$, so that $f(1) = 1 + e_1$, $f(0) = -t^2 - e_2$, $f(-1) = 1 + e_3$, where $e_1$, $e_2$, $e_3$ are nonnegative real numbers. Introducing the new variables $\xi = xyz$, $\eta = xy + yz + zx$ and $\zeta = x + y + z$, these equations become

$$\xi + \eta + \zeta = e_1$$
$$\xi = -t^2 - e_2$$
$$\xi - \eta + \zeta = 2 + e_3,$$
from which it follows that

\[ \zeta = 1 + t^2 + \frac{1}{2}e_1 - e_2 + \frac{1}{2}e_3 \]

\[ \eta = -1 + \frac{1}{2}e_1 - \frac{1}{2}e_3 . \]

Hence

\[ \frac{1}{2} \varphi = \zeta^2 - 3\eta = (1 + t^2 + \frac{1}{2}e_1 + e_2 + \frac{1}{2}e_3)^2 + 3(1 - \frac{1}{2}e_1 + \frac{1}{2}e_3) \]

\[ \geq (1 + t^3)^2 + 3 \]

\[ > 7t , \]

since the last inequality may be written in the form

\[ (t - 1)(t^3 + t^2 + 3t - 4) > 0 , \]

which is true as \( t > 1 \). Thus \( \varphi > 14t \) as required.

We may therefore assume that \([x], [y] \) do not differ by 1. By the symmetry of \( x \) and \( y \) we may suppose that \([y], [z] \) differ by 1. As before, we may assume that \(-1 < z < 0 < y < 1 \). Since we are assuming that the previous cases do not arise, it follows that either \( x > 2 \) or \( x < -1 \).

Suppose first that \( x > 2 \). Then \( f(1) = 1 + e_1, f(0) = -1 - e_2 \) and \( f(-1) = t^2 + e_3 \) where \( e_1, e_2, e_3 \) are nonnegative real numbers. As before, solving these three equations for \( \zeta, \eta \) gives

\[ 2\varphi = (2\zeta)^2 - 6(2\eta) = (3 + t^2 + e_1 + 2e_2 + e_3)^2 + 6(1 + t^2 - e_1 + e_3) \]

\[ \geq (3 + t^3)^2 + 6(1 + t^3) \]

\[ > 28t , \]

since the last inequality may be written in the form

\[ (t - 1)(t^3 + t^2 + 13t - 15) > 0 . \]

Hence \( \varphi > 14t \), as required.

Now suppose that \( x < -1 \). Then \( f(1) = -t^2 - e_1, f(0) = t^2 + e_2, f(-1) = -1 - e_3 \) where \( e_1, e_2, e_3 \) are nonnegative real numbers. Proceeding as before, we obtain

\[ 2\varphi = (1 + 3t^2 + e_1 + 2e_2 + e_3)^2 + 6(1 + t^2 + e_1 - e_3) \]

\[ \geq (1 + 3t^3)^2 + 6(1 + t^3) \]

\[ > 28t , \]

since the last inequality may be written as

\[ (t - 1)(9t^3 + 9t^2 + 21t - 7) > 0 . \]
This completes Case 2.

The preceding two cases imply that each pair of \([x], [y], [z]\) differ by at least 2. If each pair differ by at least 3, then some two of \(x, y, z\) differ by at least 5, which implies that \(\varphi \geq 25 > 14t\) since \(t^2 \leq 1.9\). Therefore, we may assume from now on that some pair of \([x], [y], [z]\) differ by exactly 2. The symmetry of \(x\) and \(y\) yields three cases.

**Case 3.** \(-2 < x < -1, 0 < y < 1, 2 < z\).

We have \(f(1) \leq -t^2, f(0) \leq -t^2, f(-1) \geq 1\) and \(f(-2) \geq 1\), i.e.,

\[
\begin{align*}
(4) & \quad \zeta \leq -1 - t^2 - \eta - \xi \\
(5) & \quad \xi \leq -t^2 \\
(6) & \quad \zeta \geq 2 + \eta - \xi \\
(7) & \quad 4\zeta \geq 9 + 2\eta - \xi .
\end{align*}
\]

Inequalities (4) and (6) imply that

\[
(8) \quad \eta \leq -\frac{1}{2}(t^2 + 3)
\]

whereas (4) and (7) yield

\[
(9) \quad \eta \leq -\frac{1}{6}(13 + 4t^2 + 3\zeta).
\]

Assume first that

\[
(10) \quad 2\eta - 3\zeta \geq 1,
\]

so that (8) and (10) give

\[
(11) \quad \xi \leq -\frac{1}{3}(t^2 + 4).
\]

By (6) and (11),

\[
(12) \quad \zeta \geq \frac{1}{3}(t^2 + 10) + \eta.
\]

Now if \(\eta \leq -1/3(t^2 + 10)\), then

\[
\frac{1}{2}\varphi = \zeta^2 - 3\eta \geq t^2 + 10 > 11 > 7t .
\]

Therefore we may assume that

\[
(13) \quad \eta > -\frac{1}{3}(t^2 + 10) .
\]
Then (12) and (13) imply that
\[ \zeta^2 - 3\eta \geq \left( \eta + \frac{1}{3}(t^2 + 10) \right)^2 - 3\eta > 7t \]
provided that the quadratic in \( \eta \),
\[ \left( \eta + \frac{1}{3}(t^2 + 10) \right)^2 - 3\eta - 7t , \]
has nonreal roots, i.e., provided that \( 4t^2 - 28t + 31 > 0 \). This inequality holds if \( t < 1/2(7 - 3\sqrt{2}) \), which is true since \( t^2 < 1.9 \).
Hence we may suppose that (10) is false, i.e.,
\[ (14) \quad \eta < \frac{1}{2}(1 + 3\xi) . \]
We may further assume that
\[ 9 + 2\eta - \xi > 0 , \]
for otherwise, by (5),
\[ 2\eta \leq \xi - 9 \leq -t^2 - 9 < -10 , \]
and therefore also
\[ \zeta^2 - 3\eta > 15 > 7t . \]
Thus, by (7),
\[ \zeta^2 - 3\eta \geq \frac{1}{16}(9 + 2\eta - \xi)^2 - 3\eta = g(\eta) , \text{ say.} \]
The quadratic \( g(\eta) \) attains its minimum value at
\[ \eta = \frac{1}{2}(\xi + 3) > \frac{1}{2}(1 + 3\xi) \text{ by (5)} . \]
Hence, by (14),
\[ g(\eta) \geq \frac{1}{16}(10 + 2\xi)^2 - \frac{3}{2}(1 + 3\xi) = h(\xi) , \text{ say.} \]
The quadratic \( h(\xi) \) attains its minimum value at \( \xi = 4 \). Suppose first that \( \xi \leq -1/3(4 + t^2) \). Then
\[ g(\eta) \geq h(\xi) \geq \frac{1}{36}(11 - t^2)^2 + \frac{1}{2}(9 + 3t^2) > 7t \]
since
\[ t^4 + 32t^2 - 252t + 283 > 0 \]

when
\[ t^2 < 1.9 \, . \]

Thus we may assume that

\[ \xi > -\frac{1}{3}(4 + t^2) \, . \]

As \( g(\eta) \) is decreasing \( \eta \leq 1/2(\xi + 3) \), and (15) shows that
\[ -\frac{1}{6}(13 + 4t^2 + 3\xi) < \frac{1}{2}(\xi + 3) \, , \]

so (9) implies that

\[ g(\eta) \geq \frac{1}{36}(7 - 2t^2 - 3\xi)^2 + \frac{1}{2}(13 + 4t^2 + 3\xi) = j(\xi) \, , \text{ say}. \]

But \( j(\xi) \) has the minimum value \( 31/4 + t^2 \). Hence
\[ g(\eta) \geq \frac{31}{4} + t^2 > 7t \, , \]

since \( 4t^2 - 28t + 31 > 0 \), as we have already seen. This completes the proof for Case 3.

\textbf{Case 4.} \(-2 < x < -1, 0 < z < 1, 2 < y.\)

Here \( f(-1) \geq t^2, f(-2) \geq t^2, f(1) \leq -t^2, f(0) \leq -t^2 \) and these imply the four inequalities (4)-(7) of Case 3. Therefore the same argument applies here.

\textbf{Case 5.} \( y < -1, 0 < x < 1, 2 < z < 3.\)

Here \( f(1) \leq -t^2, f(0) \leq -t^2, f(-1) \geq 1, f(-2) \geq 1 \) which yield the four inequalities (4)-(7) of Case 3. Therefore the same argument applies here. This completes the proof of Lemma 2.

**Lemma 2.** With \( g(n) = (x + n)(y + n)|z + n| \), suppose that \(-t^2 < g(n) < 1\) has no solution in integers \( n \). If, further, \(-2 < z < -1 < x < 0, 1 < y < 2\) then \( t^2 \leq 2 \).

\textbf{Proof.} We have \( g(2) \geq 1, g(1) \geq 1, g(0) \leq -t^2, g(-1) \leq -t^2 \) and \( g(-2) \geq 1 \). Now
\[ -3g(0) + 2g(1) + g(-2) \geq 3(1 + t^2) \, , \]

i.e.,
\[ \zeta \leq \frac{1}{2} (1 - t^2). \]

Also
\[ 2g(1) - g(0) + g(2) \geq 3 + t^2, \]
i.e.,
\[ \zeta \geq \frac{1}{2} (t^2 - 3). \]

Hence \( \frac{1}{2}(t^2 - 3) \leq \frac{1}{2}(1 - t^2) \) or \( t^2 \leq 2 \), as required.

**Lemma 3.** With \( g(n) \) as defined in Lemma 2, suppose that \( -t^2 < g(n) < 1 \) has no solution in integers \( n \) when \( t^2 \geq 1.9 \). Then, with \( X = x - z \) and \( Y = y - z \), the point \( (X, Y) \) does not lie in the plane region given by the two inequalities
\[ XY > -2t^2 - \frac{1}{4}, \quad |X + Y| < \delta, \]
where \( \delta = 5 \) if \( t^2 > 2 \) and \( \delta = 4.81 \) if \( 1.9 \leq t^2 \leq 2 \).

**Proof.** Determine an integer \( n_0 \) such that \( [n_0 + z] = 0 \) and put \( \lambda = n_0 + z \), so that \( 0 < \lambda < 1 \). Put \( F(\lambda^i) = (X + \lambda^i)(Y + \lambda^i)|\lambda^i| \) so that the condition on \( g(n) \) becomes
\[ -t^2 < F(\lambda^i) < 1 \]
has no solutions in real numbers \( \lambda^i \equiv \lambda \pmod{1} \).

Put \( \zeta = XY \) and \( \eta = X + Y \) and \( \lambda^i = \lambda, \lambda - 1 \) successively in (16). It follows that the point \( (\zeta, \eta) \) does not lie in either of the two strips given by
\[ \frac{-t^2}{\lambda} < \zeta + \lambda \eta < \frac{1}{\lambda} \]
and
\[ \frac{-t^2}{1 - \lambda} < \zeta + (\lambda - 1) \eta < (\lambda - 1)^2 < \frac{1}{1 - \lambda}. \]

Hence the point \( (\zeta, \eta) \) lies in one of four regions, giving four cases, as follows.

**Case a.**

(ai) \[ \zeta + \lambda \eta + \lambda^2 \leq \frac{-t^2}{\lambda} \]
Multiplying (ai) by $1 - \lambda$ and (aii) by $\lambda$ and adding, we obtain

$$\zeta \leq -t^2\left(\frac{1-\lambda}{\lambda} + \frac{\lambda}{1-\lambda}\right) - \lambda + \lambda^2.$$ 

Hence if

$$-t^2\left(\frac{1-\lambda}{\lambda} + \frac{\lambda}{1-\lambda}\right) - \lambda + \lambda^2 \leq -2t^2 - \frac{1}{4}$$

the lemma holds. But this inequality may be written in the form

$$\left(\lambda - \frac{1}{2}\right)(\lambda^2 - \lambda + 4t^2) \geq 0,$$

which is true since $0 < \lambda < 1$ and $t > 1$.

**Case b.**

(bi) $$\zeta + \lambda \eta + \lambda^2 \leq \frac{-t^2}{\lambda}$$

(bii) $$\zeta + (\lambda - 1)\eta + (\lambda - 1)^2 \geq \frac{1}{1-\lambda}.$$ 

Subtracting (bii) from (bi), we obtain

$$\eta \leq +\frac{1}{1-\lambda} + \frac{t^2}{\lambda} + 2\lambda - 1.$$ 

Hence the lemma holds if

$$\delta \leq -\frac{1}{1-\lambda} - \frac{t^2}{\lambda} - 2\lambda + 1$$

i.e., if

(biii) $$2\lambda^3 - (3 + \delta)\lambda^2 + (t^2 + \delta)\lambda - t^2 < 0.$$ 

In case $1.9 \leq t^2 \leq 2$ and $\delta = 4.81$, (biii) becomes

$$2\lambda^3 - 7.81\lambda^2 + 6.71\lambda - 1.9 < 0,$$

which is true for $0 < \lambda < 1$.

In case $t^2 > 2$ and $\delta = 5$, (biii) becomes

$$2\lambda^3 - 8\lambda^2 + 7\lambda - 2 < 0,$$

which also holds for $0 < \lambda < 1$. This takes care of Case b.

**Case c.**
(ci) \[ \zeta + (\lambda - 1)\eta + (\lambda - 1)^2 \leq \frac{-t^2}{1 - \lambda} \]

(cii) \[ \zeta + \lambda \eta + \lambda^2 \geq \frac{1}{\lambda} . \]

If we replace \( \lambda \) by \( 1 - \lambda \) and \( \eta \) by \( -\eta \) in (ci) and (cii), we obtain (bi) and (bii). Hence, by symmetry, \( |\eta| > \delta \).

Case d.

(di) \[ \zeta + \lambda \eta + \lambda^2 \geq \frac{1}{\lambda} \]

(dii) \[ \zeta + (\lambda - 1)\eta + (\lambda - 1)^2 \geq \frac{1}{1 - \lambda} . \]

Multiplying (di) by \( 1 - \lambda \) and (dii) by \( \lambda \) and adding, we obtain
\[ \zeta \geq \frac{1 - \lambda}{\lambda} + \frac{\lambda}{1 - \lambda} + \lambda(\lambda - 1) \geq 1 . \]

Hence \( \zeta = XY > 0 \) and \( X, Y \) have the same sign. If \( X, Y \) are both negative we may change them into \( -X, -Y \) respectively, replace \( \lambda \) by \( 1 - \lambda \) and \( \eta \) by \( -\eta \) which leaves condition (16) unchanged and turns inequalities (di) and (dii) into each other. Therefore, there is no loss of generality in assuming that \( X, Y \) are both positive. Again by the symmetry of \( X, Y \) we may assume from now on that
\[ 0 < X \leq Y . \]

If \( X + \lambda \leq Y + \lambda < 2 \), then one of the values \( F(\lambda), F(\lambda - 1) \) contradicts (16). Further, if \( 0 < X + \lambda < 1 < Y + \lambda \), then \( F(\lambda - 1) < 0 \), contrary to (dii). Thus, we may assume from now on that \( 1 < X + \lambda \) and \( 2 < Y + \lambda \).

Assume first that \( 1 < X + \lambda < 2 < Y + \lambda \). Condition (16) with \( \lambda^1 = \lambda - 2 \) becomes

(diii) \[ -\zeta - (\lambda - 2)\eta - (\lambda - 2)^2 \geq \frac{t^2}{2 - \lambda} . \]

Addition of this inequality to (dii) yields
\[ \eta \geq \frac{1}{1 - \lambda} + \frac{t^2}{2 - \lambda} + 3 - 2\lambda \]

(div)
\[ \geq \frac{1}{1 - \lambda} + \frac{1.9}{2 - \lambda} + 3 - 2\lambda \]
\[ \geq 4.81 \]
if \( f(\lambda) = 2\lambda^3 - 4.19\lambda^2 + 1.47\lambda - .28 \leq 0 \). Now \( f(\lambda) \) has a local maximum at \( \lambda_0 \) where \( 0 < \lambda_0 < 1 \) and

\[
f'(\lambda_0) = 6\lambda_0^2 - 8.38\lambda_0 + 1.47 = 0.
\]

Hence \( 3f(\lambda_0) - f'(\lambda_0) = -4.19\lambda_0^3 + 2.94\lambda_0 - .84 < 0 \) since the discriminant is negative. Thus \( f(\lambda_0) < 0 \), and as \( f(0) < 0 \) and \( f(1) < 0 \), it follows that \( f(\lambda) < 0 \) and therefore also that \( \eta \geq 4.81 \). Hence, if \( 1.9 \leq t^2 \leq 2 \), the lemma holds. Now assume that \( t^2 > 2 \). Inequality (div) implies that

\[
\eta \geq \frac{1}{1 - \lambda} + \frac{2}{2 - \lambda} - 2\lambda + 3
\]
\[
\geq 5 \quad \text{if} \quad 2\lambda^3 - 4\lambda^2 + \lambda \leq 0,
\]

which is true if \( \lambda \geq 1 - 1/\sqrt{2} \). Thus we may assume that \( \lambda < 1 - 1/\sqrt{2} \). If \( 2 < Y + \lambda < 3 \), inequality (diii) may be written in the form

\[
(2 - \lambda)(X + \lambda - 2)(Y + \lambda - 2) \leq -t^2,
\]

which is clearly false since \( t^2 > 2 \). If \( 3 < Y + \lambda < 4 \) then, by Lemma 2, \( t^2 > 2 \). Therefore we may assume that \( Y + \lambda > 4 \). By (16) with \( \lambda_1 = \lambda - 4 \), it follows that

\[
-\xi - (\lambda - 4)\eta - (\lambda - 4)^2 \geq \frac{t^2}{4 - \lambda}.
\]

Adding this inequality to (dii), we obtain

\[
3\eta \geq \frac{2}{4 - \lambda} + \frac{1}{1 - \lambda} + 15 - 6\lambda.
\]

Hence

\[
\eta \geq 5 \quad \text{if} \quad \frac{2}{4 - \lambda} + \frac{1}{1 - \lambda} - 6\lambda \geq 0
\]
i.e., if

\[
-2\lambda^3 + 10\lambda^2 - 9\lambda + 2 \geq 0.
\]

The left hand side is monotone decreasing for \( 0 \leq \lambda \leq 1/3 \) and has the value 1/27 at \( \lambda = 1/3 \). As \( 1/3 > 1 - 1/\sqrt{2} \), so \( \eta \geq 5 \) if \( \lambda \leq 1 - 1/\sqrt{2} \). Therefore, the lemma is true if \( 1 < X + \lambda < 2 \), and we may assume from now on that \( X + \lambda > 2 \).

Assume next that \( 2 < X + \lambda < 3 \). In case \( 2 < Y + \lambda < 3 \), condition (16) with \( \lambda^1 \) taken successively as \( \lambda - 2 \) and \( \lambda - 3 \) yields

\[
(2 - \lambda)(X + \lambda - 2)(Y + \lambda - 2) \geq 1
\]
and

\[(3 - \lambda)(X + \lambda - 3)(Y + \lambda - 3) \geq 1\,.
\]

Multiplying these two inequalities together and observing that

\[-\frac{1}{4} \leq (X + \lambda - 2)(X + \lambda - 3), \quad (Y + \lambda - 2)(Y + \lambda - 3) < 0,\]

we obtain a contradiction. Thus we may assume that \(3 < Y + \lambda\).

Again condition (16) with \(\lambda^1\) taken as \(X - 2\) and \(\lambda - 3\) yields

\[\zeta + (\lambda - 2)\eta + (\lambda - 2)^2 \geq \frac{1}{2 - \lambda}\]

and

\[-\zeta - (\lambda - 3)\eta - (\lambda - 3)^2 \geq \frac{t^2}{3 - \lambda} .\]

Adding these two inequalities together gives

\[(dv) \quad \eta \geq \frac{1}{2 - \lambda} + \frac{t^2}{3 - \lambda} + 5 - 2\lambda .\]

If \(t^2 > 2\) then \(\eta \geq 5\) provided

\[\frac{1}{2 - \lambda} + \frac{2}{3 - \lambda} - 2\lambda \geq 0\]

i.e.,

\[(1 - \lambda)(7 - 8\lambda + 2\lambda^2) \geq 0 ,\]

which is true since \(0 < \lambda < 1\). On the other hand, if \(1.9 \leq t^2 \leq 2\), inequality (dv) implies \(\eta \geq 4.81\) provided

\[\frac{1}{2 - \lambda} + \frac{1.9}{3 - \lambda} + 5 - 2\lambda \geq 4.81\]

i.e.,

\[-2\lambda^3 + 10.19\lambda^2 - 15.85\lambda + 7.94 \geq 0 ,\]

which is true for \(0 < \lambda < 1\), since the left hand side is monotone decreasing in this range.

We are left with the case \(3 < X + \lambda, \ Y + \lambda\). Here, if \(\eta < 5\), then

\[X + Y + 2\lambda < 7\]

so
hence, by the arithmetic-geometric mean inequality,
\[(X + \lambda - 3)(Y + \lambda - 3) < \frac{1}{4}\]
and therefore also
\[(3 - \lambda)(X + \lambda - 3)(Y + \lambda - 3) < \frac{3}{4}\]
contrary to condition (16) with \(\lambda^i = \lambda - 3\). This proves Lemma 3.

**Proof of the theorem.** Denote by \(A^*\) the set of points of \(A\) other than 0. We may assume that \(u < 1\), for otherwise, apply the transformation \(T: x_i \to -x_i\) so that, if \(T(A^*)\) has a point in the region
\[-u \leq x_1x_2 | x_3 | \leq \frac{1}{u}\]
then \(A^*\) has a point in the region
\[-\frac{1}{u} \leq x_1x_2 | x_3 | \leq u\,.

Put \(\mu = \inf x_1x_2 | x_3 |\) extended over all points \((x_1, x_2, x_3)\) of \(A\) for which \(x_1x_2 | x_3 | > 0\). Then, either the theorem is true, or \(\mu \geq u\). If \(\mu \geq 1\), the theorem follows immediately from Davenport’s result. Hence, we may assume that \(\mu < 1\) and that \(A^*\) has no point in the region given by
\[-\frac{1}{\mu} < x_1x_2 | x_3 | < \mu\,.

Put \(\mu = \gamma^3\). By a classical argument, using Mahler’s compactness theorem (5), there is no loss of generality in assuming that \(A^*\) contains the point \((\gamma, \gamma, \gamma)\).

The projection of \(A^*\) onto the plane \(x_1 + x_2 + x_3 = 0\), parallel to the vector \((1, 1, 1)\) is a two-dimensional lattice, \(A'\) say, of determinant \(d(A') = 7/\sqrt{3}\,\gamma\). By the classical theory of quadratic forms, there is a point of \(A'\), other than 0, within a euclidean distance \(\sqrt{14/3}\,\gamma\) of 0. Hence there is a point \((x, y, z)\) of \(A^*\), linearly independent of \((\gamma, \gamma, \gamma)\), such that
\[(x - y)^2 + (y - z)^2 + (z - x)^2 \leq \frac{14}{\gamma}\,.

Taking \( t = 1/\gamma^3 \), if \( 1 < t^2 \leq 1.9 \), then by Lemma 1, there is an integer \( n \) such that

\[-t^2 < \left( n + \frac{x}{\gamma} \right) \left( n + \frac{y}{\gamma} \right) \left| n + \frac{z}{\gamma} \right| < 1 ,\]
i.e.

\[-\frac{1}{\mu} < \left( n\gamma + x \right) \left( n\gamma + y \right) \left| n\gamma + z \right| < \mu ,\]

which proves the theorem for the case when \( 1 < t^2 \leq 1.9 \).

If \( t^2 > 1.9 \), the projection of \( \Lambda^* \) onto the plane \( x_3 = 0 \), parallel to the vector \( (1, 1, 1) \), is a two-dimensional lattice \( \Lambda'' \) of determinant \( d(\Lambda'') = 7/\gamma \). Taking \( \delta = 5 \) if \( t^2 > 2 \), \( \delta = 4.81 \) if \( 1.9 < t^2 \leq 2 \), by Minkowski's theorem on linear forms, there is a point \((X, Y, 0)\) of \( \Lambda'' \), other than 0, such that

\[|X - Y| < 2\gamma \sqrt{2t^2 + 1/4}\]

and

\[|X + Y| < \delta \gamma ,\]

since

\[49t^2 < \delta^2 \left( 2t^2 + \frac{1}{4} \right) .\]

Therefore, by the arithmetic-geometric mean inequality, there is a point \((X, Y, 0)\) of \( \Lambda'' \), other than 0, such that

\[XY > -\gamma^2 \left( 2t^2 + \frac{1}{4} \right) \]

and

\[|X + Y| < \delta \gamma .\]

We have \( X = x - z \), \( Y = y - z \) for some point \((x, y, z)\) of \( \Lambda^* \), linearly independent of \((\gamma, \gamma, \gamma)\). Applying Lemma 3, there is an integer \( n \) such that

\[-t^2 < \left( n + \frac{x}{\gamma} \right) \left( n + \frac{y}{\gamma} \right) \left| n + \frac{z}{\gamma} \right| < 1 ,\]
i.e.,

\[-\frac{1}{\mu} < \left( n\gamma + x \right) \left( n\gamma + y \right) \left| n\gamma + z \right| < \mu ,\]

and the theorem is proved.
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