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Let M be a von Neumann algebra with a faithful, normal, tracial state τ and H^∞ a finite, maximal, subdiagonal algebra in M. If $1 \leqq p < s \leqq \infty$, then there is a one-to-one correspondence between left-(resp. right-) invariant subspaces of the noncommutative Lebesgue space $L^p(M, \tau)$ and those of $L^s(M, \tau)$.

1. Introduction. Let M be a von Neumann algebra with a faithful, normal, tracial state τ and let H^∞ be a finite, maximal, subdiagonal algebra in M. A number of authors have investigated the structure of the invariant subspaces for H^∞ acting on the noncommutative Lebesgue space $L^p(M, \tau)$ (cf. [3], [4], [5] and [6]). In [6], we showed that, if $\mathfrak{M}$ is a left-(resp. right-) invariant subspace of $L^p(M, \tau)$, $1 \leqq p < \infty$, then $\mathfrak{M}$ is the closure of the space of bounded elements it contains.

In this paper, we shall show that, if $1 \leqq p < s \leqq \infty$, then there is a one-to-one correspondence between left- (resp. right-) invariant subspaces $\mathfrak{M}_p$ of $L^p(M, \tau)$ and left- (resp. right-) invariant subspaces $\mathfrak{M}_s$ of $L^s(M, \tau)$, such that $\mathfrak{M}_s = \mathfrak{M}_p \cap L^s(M, \tau)$ and $\mathfrak{M}_p$ is the closure in $L^p(M, \tau)$ of $\mathfrak{M}_s$. This is of course true in the weak*-Dirichlet algebras setting (cf. [2, p. 131]) and this is attractive to study the structure of the invariant subspaces of $L^p(M, \tau)$.

2. Let M be a von Neumann algebra with a faithful, normal, tracial state τ. We shall denote the noncommutative Lebesgue spaces associated with M and τ by $L^p(M, \tau)$, $1 \leqq p < \infty$ (cf. [7]). As is customary, M will be identified with $L^\infty(M, \tau)$. The closure of a subset S of $L^p(M, \tau)$ in the L^p-norm will be denoted by $[S]_p$; $[S]_\infty$ will denote the closure of S in the σ-weak topology on $L^\infty(M, \tau)$.

DEFINITION 1. Let H^∞ be a σ-weakly closed subalgebra of M containing the identity operator 1 and let Φ be a faithful, normal expectation from M onto $D = H^\infty \cap H^{\infty *}(H^{\infty *} = \{x^* : x \in H^\infty\})$. Then H^∞ is called a finite, maximal, subdiagonal algebra in M with respect to Φ and τ in case the following conditions are satisfied: (1) $H^\infty + H^{\infty *}$ is σ-weakly dense in M; (2) $\Phi(xy) = \Phi(x)\Phi(y)$, for all $x, y \in H^\infty$; (3) H^∞ is maximal among those subalgebras of M satisfying (1) and (2); and (4) $\tau \circ \Phi = \tau$.

For $1 \leqq p < \infty$, the closure of H^∞ in $L^p(M, \tau)$ is denoted by H^p and the closure of $H_0^\infty = \{x \in H^\infty: \Phi(x) = 0\}$ is denoted by H_0^p.

DEFINITION 2. Let $\mathfrak{M}$ be a closed (resp. σ-weakly closed) subspace of $L^p(M, \tau)$ (resp. $L^\infty(M, \tau)$). We shall say that $\mathfrak{M}$ is left- (resp. right-) invariant if $H^\infty \mathfrak{M} \subseteqq \mathfrak{M}$ (resp. $\mathfrak{M} H^\infty \subseteqq \mathfrak{M}$).

The following theorem shows that, in considering left- (resp. right-) invariant subspaces, it sufficies to consider left- (resp. right-) invariant subspaces of $L^2(M, \tau)$, or alternatively, σ-weakly closed left- (resp. right-) invariant subspaces of $L^\infty(M, \tau)$. The method in the proof is based on a facterization theorem, that is, if k is in M with inverse lying in $L^2(M, \tau)$, then there are unitary operators u_1, u_2 in M and operators a_1, a_2 in H^∞ with inverses lying in H^2 such that $k = u_1 a_1 = a_2 u_2$ ([6, Proposition 1]).

THEOREM 1. *Suppose* $1 \leqq p < s \leqq \infty$.

(1) *If* $\mathfrak{M}$ *is a left- (resp. right-) invariant subspacee of* $L^p(N, \tau)$, *then* $\mathfrak{M} \cap L^s(M, \tau)$ *is a left- (resp. right-) invariant subspace of* $L^s(M, \tau)$ *and* $\mathfrak{M} = [\mathfrak{M} \cap L^s(M, \tau)]_p$.

(2) *If* $\mathfrak{M}$ *is a left- (resp. right-) invariant subspace of* $L^s(M, \tau)$, *then* $[\mathfrak{M}]_p$ *is a left- (resp. right-) invariant subspace of* $L^p(M, \tau)$ *and* $\mathfrak{M} = [\mathfrak{M}]_p \cap L^s(M, \tau)$.

Proof. It sufficies to consider the assertion for left- invariant subspaces.

(1) Let $\mathfrak{M}$ be a left-invariant subspace of $L^p(M, \tau)$. It is clear that $\mathfrak{M} \cap L^s(M, \tau)$ is a left-invariant subspace of $L^s(M, \tau)$. By [6, Theorem], we have $\mathfrak{M} = [\mathfrak{M} \cap L^\infty(M, \tau)]_p$ and so

$$\mathfrak{M} = [\mathfrak{M} \cap L^\infty(M, \tau)]_p \subseteqq [\mathfrak{M} \cap L^s(M, \tau)]_p \subseteqq \mathfrak{M} .$$

Therefore $\mathfrak{M} = [\mathfrak{M} \cap L^s(M, \tau)]_p$. This completes the proof of (1).

(2) Let $\mathfrak{M}$ be a left-invariant subspace of $L^s(M, \tau)$. It is clear that $[\mathfrak{M}]_p$ is a left-invariant subspace of $L^p(M, \tau)$. Now, if the assertion (2) in the case $s = \infty$ is proved, then $[\mathfrak{M} \cap L^\infty(M, \tau]_p \cap L^\infty(M, \tau) = \mathfrak{M} \cap L^\infty(M, \tau)$. By (1),

$$\begin{aligned}[\mathfrak{M}]_p \cap L^s(M, \tau) &= [[\mathfrak{M}]_p \cap L^\infty(M, \tau)]_s = [[\mathfrak{M} \cap L^\infty(M, \tau)]_p \cap L^\infty(M, \tau)]_s \\ &= [\mathfrak{M} \cap L^\infty(M, \tau)]_s = \mathfrak{M} .\end{aligned}$$

Therefore, suppose that $s = \infty$. Let $\mathfrak{M}$ be a left-invariant subspace of $L^\infty(M, \tau)$ and put $\widetilde{\mathfrak{M}} = [\mathfrak{M}]_p \cap L^\infty(M, \tau)$. It is clear that $\mathfrak{M} \subseteqq \widetilde{\mathfrak{M}}$. If $\mathfrak{M} \subsetneqq \widetilde{\mathfrak{M}}$, then there exist $x \in \widetilde{\mathfrak{M}}/\mathfrak{M}$ and $a \in L^1(M, \tau)$ such that $\tau(ax) = 1$ and $\tau(ay) = 0$ for every $y \in \mathfrak{M}$.

(i) Case $2 \leqq p < \infty$. Define the number q by the equation $1/p + 1/q = 1$. Let $a = v|a|$ be the polar decomposition of a. Let f be the function on $[0, \infty)$ defined by the formula $f(t)=1$, $0 \leqq t \leqq 1$, $f(t) = 1$, $t > 1$, and define k to be $f(|a|^{1/p})$ through the functional calculus. Then note that $k \in L^{\infty}(M, \tau)$ and $k^{-1} \in L^{p}(M, \tau) \subset L^{2}(M, \tau)$. By [6, Proposition 1], we may choose a unitary operator u in $L^{\infty}(M, \tau)$ and an operator $b \in H^{\infty}$ such that $k = bu$ and $b^{-1} \in H^{2}$. Since $k^{-1} \in L^{p}(M, \tau)$, by [6, Proposition 2], $b^{-1} \in L^{p}(M, \tau) \cap H^{2} = H^{p}$ and note that $ab = v|a|^{1/q}|a|^{1/p}ku^{*} \in L^{q}(M, \tau)$, because $|a|^{1/p}k \in L^{\infty}(M, \tau)$. Since $\mathfrak{M}$ is left-invariant, $\tau(aby) = 0$ for every $y \in \mathfrak{M}$ and so $\tau(aby) = 0$ for every $y \in [\mathfrak{M}]_{p}$. On the other hand, $b^{-1}x \in H^{p}\widetilde{\mathfrak{M}} \subset [\widetilde{\mathfrak{M}}]_{p} = [\widetilde{\mathfrak{M}}]_{p}$ and so $\tau(ax) = \tau(abb^{-1}x) = 0$. This is a contradiction. Thus $\mathfrak{M} = \widetilde{\mathfrak{M}}$.

(ii) Case $1 \leqq p < 2$. Define the numbers q and r by the equations $1/p + 1/q = 1$ and $1/r + 1/2 = 1/p$. Put $k = f(|a|^{1/2})$, where f is the function in (i). By [6, Proposition 1], there are a unitary operator u in $L^{\infty}(M, \tau)$ and an operator $b \in H^{\infty}$ with inverse lying in H^{2} such that $k = bu$ and note that ab is a nonzero element in $L^{2}(M, \tau)$. Also, let $ab = v'|ab|$ be the polar decomposition of ab. Put $k' = f(|ab|^{2/r})$, where f is the function in (i). Since $|ab|^{2/r} \in L^{r}(M, \tau) \subset L^{2}(M, \tau)$, by [6, Proposition 1], there exists an operator c in H^{∞} with inverse lying in H^{r} such that abc is a nonzero element in $L^{q}(M, \tau)$. Since $\mathfrak{M}$ is left-invariant, we have $\tau((abc)y) = \tau(a(bcy)) = 0$, for every $y \in \mathfrak{M}$, and so $\tau(abcy) = 0$ for every $y \in [\mathfrak{M}]_{p}$. On the other hand, since $(bc)^{-1} = c^{-1}b^{-1} \in H^{r}H^{2} \subset H^{p}$, $(bc)^{-1}x \in H^{p}\widetilde{\mathfrak{M}} \subset [\widetilde{\mathfrak{M}}]_{p} = [\mathfrak{M}]_{p}$ and so $\tau(ax) = \tau(abc(bc)^{-1}x) = 0$. This is a contradiction. Thus $\mathfrak{M} = \widetilde{\mathfrak{M}}$.

This completes the proof of (2).

Next we shall consider the structure of doubly invariant subspaces and simply invariant subspaces of $L^{p}(M, \tau)$, $1 \leqq p \leqq \infty$.

DEFINITION 3. Let $\mathfrak{M}$ be a closed subspace of $L^{p}(M, \tau)$, $1 \leqq p \leqq \infty$.

(1) $\mathfrak{M}$ is said to be left (resp. right) doubly invariant if $(H^{\infty} + H^{\infty *})\mathfrak{M} \subseteqq \mathfrak{M}$ (resp. $\mathfrak{M}(H^{\infty} + H^{\infty *}) \subseteqq \mathfrak{M}$).

(2) $\mathfrak{M}$ is said to be left (resp. right) simply invariant if $[H_{0}^{\infty}\mathfrak{M}]_{p} \subsetneqq \mathfrak{M}$ (resp. $[\mathfrak{M}H_{0}^{\infty}]_{p} \subsetneqq \mathfrak{M}$).

By [5, Theorem 4.1] and Theorem 1, we have the following theorem.

THEOREM 2. *Let $\mathfrak{M}$ be a closed subspace of $L^{p}(M, \tau)$, $1 \leqq p \leqq \infty$. Then $\mathfrak{M}$ is a left (resp. right) doubly invariant subspace of $L^{p}(M, \tau)$ if and only if there exists a projection e in M such that $\mathfrak{M} = L^{p}(M, \tau)e$ (resp. $eL^{p}(M, \tau)$).*

In [3], Kamei has shown the simply invariant subspace theorem for antisymmetric finite subdiagonal algebras in case $p = 1, 2$. Also, in [5], we characterized the simply invariant subspace for H^∞ in $L^p(M, \tau)$, $1 \leqq p \leqq \infty$, when H^∞ is determined by a trace preserving ergodic flow. However, by Theorem 1 and [3], we have the following theorem.

THEOREM 3. *Let* $\mathfrak{M}$ *be a closed subspace of* $L^p(M, \tau)$, $1 \leqq p \leqq \infty$. *If* H^∞ *is antisymmetric, that is,* $D = C1$, *then* $\mathfrak{M}$ *is a left (resp. right) simply invariant subspace of* $L^p(M, \tau)$ *if and only if there is a unitary operator* u *in* M *such that* $\mathfrak{M} = H^p u$ *(resp.* uH^p*).*

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