

# Pacific Journal of Mathematics

**EXTENDING WHITNEY MAPS**

L. E. WARD

## EXTENDING WHITNEY MAPS

L. E. WARD, JR.

The following theorems are proved. (1) If  $X$  is a continuum then any Whitney map for  $C(X)$ , the space of subcontinua of  $X$ , can be extended to a Whitney map for  $2^X$ , the space of nonempty closed subsets of  $X$ . (2) If  $Y$  is a continuum and  $X$  is a subcontinuum of  $Y$  then any Whitney map for  $C(X)$  (resp.,  $2^X$ ) can be extended to a Whitney map for  $C(Y)$  (resp.,  $2^Y$ ). The proofs entail recasting these problems in the more inclusive setting of partially ordered spaces and then employing results of Nachbin.

1. Introduction. In this paper a *continuum* is a compact connected metric space. If  $X$  is a continuum then  $2^X$  (respectively,  $C(X)$ ) is the hyperspace of nonempty closed subsets (respectively, subcontinua) of  $X$ , endowed with the Hausdorff metric. If  $A \subset 2^X$  and if  $A$  contains all of the singleton subsets of  $X$ , then a *Whitney map* for  $A$  is a continuous function  $\omega: A \rightarrow [0, +\infty)$  such that  $\omega(\{x\}) = 0$  for each  $x \in X$  and  $\omega(A) < \omega(B)$  whenever  $A$  and  $B$  are members of  $A$  and  $A$  is properly contained in  $B$ .

Among the many interesting and heretofore unsolved problems in the theory of hyperspaces are the following. Nadler ([4], 14.71.5) has asked if every Whitney map for  $C(X)$  can be extended to a Whitney map for  $2^X$ . A related question (due to Bruce Hughes and communicated to me by Professor Carl Eberhart) asks whether a Whitney map for  $C(X)$  can always be extended to a Whitney map for  $C(Y)$  if  $X$  is a subcontinuum of  $Y$ . We shall answer these questions in the affirmative. The keystone of our approach is to recast the problem in the more general setting of partially ordered spaces, whereupon Nachbin's order-theoretic analog of Tietze's theorem [3] provides an essential ingredient of the proof.

At this point it is worth recalling that in [6] we promoted the notion—certainly not new—that some problems concerning hyperspaces become more tractable if the hyperspace is regarded as a special type of partially ordered space. There is a substantial literature dealing with the latter which can then be utilized. The present paper constitutes further evidence in support of this view.

2. Definitions and known results. A *partially ordered space* is a topological space  $P$  endowed with a partial order  $\leq$  whose graph is a closed subset of  $P \times P$ . It is known (see, for example, [2], p. 167) that if  $X$  is a regular Hausdorff space then  $2^X$  is a partially

ordered space with respect to inclusion. If  $P$  is a partially ordered space and  $x \in P$ , we write  $L(x) = \{p \in P: p \leq x\}$  and  $M(x) = \{p \in P: x \leq p\}$ , and if  $A \subset P$  then

$$L(A) = \cup \{L(x): x \in A\},$$

$$M(A) = \cup \{M(x): x \in A\}.$$

An element  $m$  of a partially ordered space  $P$  is *minimal* (*maximal*) if, whenever  $x \in P$  and  $x \leq m$  ( $m \leq x$ ), it follows that  $x = m$ . The set of minimal elements of  $P$  is denoted  $\text{Min } P$ ; the set of maximal elements is denoted  $\text{Max } P$ . It is known [5] that if  $P$  is compact and  $x \in P$  then  $L(x)$  meets  $\text{Min } P$  and  $M(x)$  meets  $\text{Max } P$ .

A *Whitney map* for a partially ordered space  $P$  is a continuous function  $\omega: P \rightarrow [0, 1]$  which satisfies

- (i) if  $x \in \text{Min } P$  then  $\omega(x) = 0$ ,
- (ii) if  $x \in \text{Max } P$  then  $\omega(x) = 1$ ,
- (iii) if  $x < y$  in  $P$  then  $\omega(x) < \omega(y)$ .

It is obvious that if  $P = 2^X$  for some continuum  $X$  and if  $\omega$  satisfies (i), (ii) and (iii), then  $\omega$  is a Whitney map in the hyperspace sense. Moreover, if  $X$  is a continuum then a Whitney map for  $2^X$  is, up to a constant factor, a Whitney map in the sense of partially ordered spaces.

It is well-known (for example, see the discussion in [4], pp. 24-27) that  $2^X$  admits a Whitney map whenever  $X$  is a continuum. In a recent note [6] the author generalized this result to an appropriate class of partially ordered spaces, as follows.

**THEOREM 2.1.** *If  $P$  is a compact metric partially ordered space such that  $\text{Min } P$  and  $\text{Max } P$  are disjoint closed sets, then  $P$  admits a Whitney map.*

At this point it is helpful to take cognizance of several results of Nachbin [3] for partially ordered spaces. The statements given here for Nachbin's results differ slightly from those in [3], but they follow easily. In particular, Nachbin's order-theoretic version of Tietze's Theorem (2.4) is stated here only for compact partially ordered spaces, whereas the original result was established in the more general setting of "normally ordered" spaces.

**THEOREM 2.2.** *If  $K$  is a compact subset of a partially ordered space, then  $L(K)$  and  $M(K)$  are closed sets.*

**THEOREM 2.3.** *If  $x$  and  $y$  are elements of a compact partially ordered space and if  $M(x) \cap L(y) = \emptyset$ , then there are disjoint open sets  $U$  and  $V$  such that  $x \in U = M(U)$  and  $y \in V = L(V)$ .*

If  $A$  and  $B$  are partially ordered sets then a function  $f: A \rightarrow B$  is said to be *order-preserving* if, whenever  $x \leq y$  in  $A$ , it follows that  $f(x) \leq f(y)$ .

**THEOREM 2.4.** *If  $Q$  is a closed subset of the compact partially ordered space  $P$  and if  $f: Q \rightarrow [0, 1]$  is a continuous order-preserving function, then there exists a continuous order-preserving function  $g: P \rightarrow [0, 1]$  such that  $g|_Q = f$ .*

**3. Extending Whitney maps for partially ordered spaces.** Our main result is the following theorem.

**THEOREM 3.1.** *Let  $P$  be a compact metric partially ordered space such that  $\text{Min } P$  and  $\text{Max } P$  are disjoint closed sets and let  $Q$  be a closed subset of  $P$  such that  $\text{Min } Q \subset \text{Min } P$  and  $\text{Max } Q \subset \text{Max } P$ . Then a Whitney map for  $Q$  can be extended to a Whitney map for  $P$ .*

The proof of (3.1) depends on a delicate application of (2.4). To facilitate this we first obtain a lemma.

**LEMMA 3.2.** *Suppose  $P$  is a compact partially ordered space such that  $\text{Min } P$  and  $\text{Max } P$  are disjoint closed sets,  $Q$  is a closed subset containing  $(\text{Min } P) \cup (\text{Max } P)$ , and suppose  $A$  and  $B$  are disjoint nonempty closed subsets such that  $A = M(A)$  and  $B = L(B)$ . If  $f: Q \rightarrow [0, 1]$  is a continuous order-preserving function such that  $f|(\text{Min } P) \equiv 0$  and  $f|(\text{Max } P) \equiv 1$ , then  $f$  admits a continuous order-preserving extension  $\bar{f}: P \rightarrow [0, 1]$  such that  $\bar{f}(a) \geq \inf f|(A \cap Q)$  for each  $a \in A$  and  $\bar{f}(b) \leq \sup f|(B \cap Q)$  for each  $b \in B$ .*

*Proof.* By (2.4) the function  $f|(A \cap Q)$  admits a continuous order-preserving extension

$$f_1: A \longrightarrow [\inf f|(A \cap Q), 1],$$

and the function  $f|(B \cap Q)$  admits a continuous order-preserving extension

$$f_0: B \longrightarrow [0, \sup f|(B \cap Q)].$$

The mapping  $f \cup f_0 \cup f_1$  is a continuous order-preserving function defined on the closed set  $Q \cup A \cup B$ , and another application of (2.4) yields the desired function  $\bar{f}: P \rightarrow [0, 1]$ .

We turn now to proof of (3.1). Let  $\omega_Q$  be a Whitney map for  $Q$ . We may extend  $\omega_Q$  at once to  $(\text{Min } P) \cup (\text{Max } P)$  by letting  $\omega_Q|(\text{Min } P) \equiv 0$  and  $\omega_Q|(\text{Max } P) \equiv 1$ , so there is no loss of generality

if we assume  $Q$  contains  $(\text{Min } P) \cup (\text{Max } P)$ , and hence (3.2) may be applied. We employ a variation on an argument due to Carruth [1]. Suppose  $\mathcal{U}$  is a countable base for the topology of  $P$ , and let  $\mathcal{B}$  denote the family of all pairs  $(U, V)$  of members of  $\mathcal{U}$  such that  $M(\bar{U}) \cap L(\bar{V}) = \emptyset$ . Then  $\mathcal{B}$  is also countable and we may enumerate its elements:

$$\mathcal{B} = \{(U_n, V_n) : n = 1, 2, \dots\}.$$

By (2.2) the sets  $M(\bar{U}_n)$  and  $L(\bar{V}_n)$  are closed, so by (3.2), for each positive integer  $n$  there is a continuous order-preserving extension  $f_n : P \rightarrow [0, 1]$  of  $\omega_Q$  such that

$$\begin{aligned} f_n(a) &\geq \inf \omega_Q | (M(\bar{U}_n) \cap Q) & \text{if } a \in M(\bar{U}_n), \\ f_n(b) &\leq \sup \omega_Q | (L(\bar{V}_n) \cap Q) & \text{if } b \in L(\bar{V}_n). \end{aligned}$$

Define  $\omega_P : P \rightarrow [0, 1]$  by  $\omega_P = \sum 2^{-n} f_n$ . Obviously  $\omega_P$  is continuous and  $\omega_P$  is an extension of  $\omega_Q$ . Since each  $f_n$  is order-preserving, so is  $\omega_P$ . Thus it remains to show that if  $x < y$  in  $P$  then  $\omega_P(x) < \omega_P(y)$ . Clearly, it is sufficient to verify the existence of a positive integer  $n$  such that  $f_n(x) < f_n(y)$ .

Let  $t_x = \sup \omega_Q | (L(x) \cap Q)$  and  $t_y = \inf \omega_Q | (M(y) \cap Q)$ . Since  $\omega_Q$  is a Whitney map it follows that  $t_x < t_y$ . Let  $0 < \varepsilon < (t_y - t_x)/2$ . By (2.3) there are disjoint open sets  $U$  and  $V$  such that  $x \in V = L(V)$  and  $y \in U = M(U)$ , and by a straightforward compactness argument we may assume that  $\omega_Q(V \cap Q) \subset [0, t_x + \varepsilon)$  and  $\omega_Q(U \cap Q) \subset (t_y - \varepsilon, 1]$ . It follows that there is a positive integer  $n$  such that  $x \in V_n \subset \bar{V}_n \subset V$  and  $y \in U_n \subset \bar{U}_n \subset U$ , from which we conclude that

$$f_n(x) \leq t_x + \varepsilon < t_y - \varepsilon \leq f_n(y).$$

The proof is complete.

**COROLLARY 3.3.** *If  $X$  is a continuum then any Whitney map for  $C(X)$  can be extended to a Whitney map for  $2^X$ .*

**COROLLARY 3.4.** *If  $Y$  is a continuum and  $X$  is a subcontinuum of  $Y$ , then any Whitney map for  $C(X)$  (resp.,  $2^X$ ) can be extended to a Whitney map for  $C(Y)$  (resp.,  $2^Y$ ).*

*Proof.* We give the proof for  $C(X)$ ; the proof for  $2^X$  follows similarly. Clearly  $C(X)$  is a closed subset of  $C(Y)$  and  $\text{Min } C(X) \subset \text{Min } C(Y)$ . However,  $\text{Max } C(X) = \{X\}$  is not a subset of  $\text{Max } C(Y)$ . This deficiency is readily corrected by defining  $Q = C(X) \cup \{Y\}$  so that  $\text{Max } Q = \{Y\} = \text{Max } C(Y)$ . If  $\omega_X$  is a Whitney map for  $C(X)$  with  $\omega_X(X) = 1$ , let  $\omega_Q : Q \rightarrow [0, 1]$  be defined by

$$\omega_q|C(X) = \frac{1}{2}\omega_x,$$

$$\omega_q(Y) = 1.$$

Theorem 3.1 now applies and  $\omega_q$  extends to a Whitney map  $\omega_Y$  for  $C(Y)$ . Clearly,  $2\omega_Y$  is the desired extension of  $\omega_x$ .

It is worth remarking that the family of mappings  $f_n: P \rightarrow [0, 1]$  does not, in general, distinguish points of  $P$  and hence does not generate an order homeomorphism of  $P$  into the Hilbert cube. However, Carruth [1] has shown that such order homeomorphisms exist for all compact metric partially ordered spaces. The following question arises naturally.

*Problem 3.5. Let  $\omega_P$  be a Whitney map for the compact metric partially ordered space  $P$ . Under what conditions does there exist an order-homeomorphism  $\varphi: P \rightarrow H$ , the Hilbert cube, so that  $\omega|\varphi(P) = \omega_P\varphi^{-1}$ , where  $\omega: H \rightarrow [0, 1]$  is the Whitney map defined by  $\omega(x) = \sum 2^{-n}x_n$ ?*

#### REFERENCES

1. J. H. Carruth, *A note on partially ordered compacta*, Pacific J. Math., **24** (1968), 229-231.
2. K. Kuratowski, *Topology I*, Academic Press, 1966.
3. L. Nachbin, *Topology and Order*, Van Nostrand, 1965.
4. S. B. Nadler, Jr., *Hyperspaces of Sets*, Dekker, 1978.
5. L. E. Ward, Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc., **5** (1954), 144-161.
6. ———, *A note on Whitney maps*, Canad. Math. Bull., to appear.

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