THE SHEAF OF $H^p$-FUNCTIONS IN PRODUCT DOMAINS

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Let $W=W_1 \times W_2 \times \cdots \times W_n$ be a bounded polydomain in $\mathbb{C}^n$ such that the boundary of each $W_i$ consists of finitely many disjoint Jordan curves. The correspondence that assigns to every relatively open polydomain $V$ in $\overline{W}$ (the closure of $W$) the Hardy space $H^p(V \cap W)$, defines a sheaf $\mathcal{H}^p$ over $\overline{W}$. This sheaf is locally determined in the sense that $\Gamma(\overline{W}, \mathcal{H}^p)$ is canonically isomorphic to $H^p(W)$. In this paper we prove, for any $0<p<\infty$ and all integers $q \geq 1$, that the cohomology groups $H^q(\overline{W}, \mathcal{H}^p)$ are trivial.

I. Introduction. The Hardy spaces $H^p(U^n)$, $0<p<\infty$, for the unit polydisc $U^n$, consist of all functions $F$ which are holomorphic in $U^n$ and satisfy

$$\sup_{0<r<1} \int_0^{2\pi} \cdots \int_0^{2\pi} |F(re^{i\theta_1}, \cdots, re^{i\theta_n})|^p d\theta_1 \cdots d\theta_n < +\infty.$$ 

The observation ([9, Exercise 3.4.4(b), p. 52]) that $F \in H^p(U^n)$ if and only if $F$ is holomorphic and $|F|^p$ has an $n$-harmonic majorant in $U^n$, leads to a definition of Hardy spaces for arbitrary product domains; the requirement now being that $F$ be holomorphic and $|F|^p$ have an $n$-harmonic majorant in the polydomain in question.

The symbol $H^p$ can thus be regarded as a presheaf on the polydomains in $\mathbb{C}^n$. In this paper we concern ourselves with the sheaf induced by $H^p$ on the closure of a polydomain, and prove, under certain topological restrictions, that the corresponding cohomology groups are trivial.

Specifically, let $W=W_1 \times W_2 \times \cdots \times W_n$ be a bounded polydomain in $\mathbb{C}^n$, and suppose each $W_i$ is bounded by finitely many disjoint Jordan curves. The correspondence that assigns to each relatively open product domain $V$ in $\overline{W}$ (the closure of $W$) the linear space $H^p(V \cap W)$, defines a sheaf $\mathcal{H}^p$ over $\overline{W}$. This sheaf is locally determined, i.e., $\Gamma(\overline{W}, \mathcal{H}^p)$ is canonically isomorphic to $H^p(W)$. Our goal is to prove, for any such $W$, for $0<p<\infty$, and for all integers $q \geq 1$, that the cohomology groups $H^q(\overline{W}, \mathcal{H}^p)$ are trivial.

In [8] A. Nagel proved similar results for a wide class of sheaves of holomorphic functions satisfying boundary conditions in polydomains. Although Nagel's methods can be applied to the sheaves $\mathcal{H}^p$ when $1<p<\infty$, the cases $0<p \leq 1$ present difficulties. Instead, as in the earlier papers [12], [13], we follow the approach.
of E. L. Stout in [11]. In this respect Theorem 3.3, which is central to our study, is the analogue of Lemma 1.2 of [11].

The crux of our work is Theorem 3.3 (the Decomposition Theorem); the proof, together with the necessary groundwork, appears in § III which is essentially self-contained. The basic definitions are listed in § II. In § IV we consider the Čech cohomology with coefficients in $\mathcal{H}_w^p$, and prove our main result, Theorem 4.9.

We mention in closing that although most of our results are proven for the case $n > 1$, they are also verified if $n = 1$ (the modifications in the proofs required for this case are always straightforward).

II. Preliminaries. A polydomain in $C^n$ is a cartesian product $W_1 \times W_2 \times \cdots \times W_n$ of $n$ open connected subsets (domains) of $C$. If each $W_i$ is a bounded domain, bounded by finitely many disjoint Jordan curves (a Jordan domain) we say that $W$ is a Jordan polydomain.

Possessing an $n$-harmonic majorant in a Jordan polydomain is a local property (see also [5]):

**THEOREM 2.1.** [12, Th. 2.10, p. 301]. Let $W$ be a Jordan polydomain and let $\{U_a\}$ be a relatively open covering of $\bar{W}$. If $s$ is a positive $n$-subharmonic function in $W$ with "local" $n$-harmonic majorants $u_a$ in each intersection $U_a \cap W$, then $s$ has an $n$-harmonic majorant in $W$.

**DEFINITION 2.2.** Let $V$ be a polydomain, and let $0 < p < \infty$. We define the Hardy space $\mathcal{H}^p(V)$ to be the linear space of all functions $F$ which are holomorphic in $V$ and for which $|F|^p$ has an $n$-harmonic majorant in $V$. We establish the convention $\mathcal{H}^p(\emptyset) = \{0\}$.

**DEFINITION 2.3.** Let $W$ be a fixed polydomain in $C^n$. We define the sheaf $\mathcal{H}_w^p$ (the sheaf of germs of $\mathcal{H}^p$-functions on $\bar{W}$) as the sheaf over $\bar{W}$ which is induced by the correspondence between the relatively open polydomains $V \subset \bar{W}$ and the linear spaces $\mathcal{H}^p(V \cap W)$.

If $W$ is a Jordan polydomain, it is a direct consequence of Theorem 2.1 that the linear spaces $\Gamma(\bar{W}, \mathcal{H}_w^p)$ and $\mathcal{H}^p(W)$ are canonically isomorphic.

If $W$ and $V$ are Jordan polydomains in $C^n$, with correspondingly conformally equivalent coordinate domains, the sheaves $\mathcal{H}_v^p$ and $\mathcal{H}_w^p$ are isomorphic; consequently, the cohomology groups of $V$ and $W$ with coefficients in $\mathcal{H}_v^p$ and $\mathcal{H}_w^p$, respectively, are isomorphic.
This follows from the invariance of the $H^p$-spaces under $n$-conformal transformations, and the well known fact that a conformal equivalence between Jordan domains extends to a homeomorphism between their closures.

III. A decomposition theorem. In what follows, $U$ will be the open unit disc $\{z \in C: |z| < 1\}$ and $T$ its boundary, the unit circle. The cartesian product of $n$ copies of $U$ will be denoted by $U^n$. Similarly, $T^n$ will be the cartesian product of $n$ copies of $T$. We will denote the normalized Haar measure on $T^n$ by $m_n$ (by $m$ in the particular case $n = 1$); the corresponding $L^p$-spaces will be indicated by $L^p(T^n)$, and the $L^p$-norm by $\| \cdot \|_{L^p(T^n)}$. The extended complex plane will be denoted by $\mathbb{C}$.

Let $F$ be a holomorphic function in $U^n$ and let $0 < r < 1$. We denote by $F_r$ the function defined on $T^n$ by the equation

$$F_r(w) = F(rw);$$

and define, for each $0 < p < \infty$,

$$\|F\|_{L^p(U^n)} = \lim_{r \to 1} \|F_r\|_{L^p(T^n)}.\]

An alternative characterization of the Hardy space $H^p(U^n)$ is that it consists of all holomorphic $F$ for which

$$\|F\|_{L^p(U^n)} < +\infty.$$

Moreover, if $H$ is the least $n$-harmonic majorant of $|F|^p$ in $U^n$, then

$$\|F\|_{L^p(U^n)} = H(0),$$

where we denote the $n$-tuple $(0, 0, \cdots, 0)$ by $0$.

We define $H^p((S^2 - \bar{U}) \times U^{n-1})$ to be the class of all functions $F$ for which the function $F^*$, defined for $(x, y) \in U \times U^{n-1}$ by

$$F^*(x, y) = F\left(\frac{1}{\bar{x}}, y\right),$$

is in $H^p(U^n)$. If $F$ and $F^*$ are related as above, we write

$$\|F\|_{L^p((S^2 - \bar{U}) \times U^{n-1})} = \|F^*\|_{L^p(U^n)}.\]

The space of test functions on $T$ will be represented by $\mathcal{C}^\infty(T)$, the space of distributions on $T$ by $\mathcal{D}(T)$, and the bilinear pairing between $h \in \mathcal{C}^\infty(T)$ and $f \in \mathcal{D}(T)$ by

$$\langle h(\cdot), f(\cdot) \rangle.$$

Let $\mathbb{Z}$ be the set of integers. For each $j \in \mathbb{Z}$ and $w \in T$, we define

$$e_j(w) = w^j.$$
The Fourier coefficients of \( f \in \mathcal{D}(T) \) are the numbers

\[
\hat{f}(j) = \langle e_{-j}(\cdot), f(\cdot) \rangle,
\]

where \( j \) ranges over \( \mathbb{Z} \).

Given \( F \in \mathcal{H}^p(U) \), \( 0 < p < \infty \), there exists a unique \( f \in \mathcal{D}(T) \) such that the Fourier coefficients \( \hat{f}(j) \), with \( j \geq 0 \), are the Taylor coefficients of \( F \), and such that \( \hat{f}(j) = 0 \) whenever \( j < 0 \). This can be derived, for example, from [3, Th. 6.4, p. 98]. We refer to \( f \) as the boundary distribution of \( F \).

Let \( w \in T \) and \( z \in S^1 - T \). The Cauchy kernel \( C(z, w) \) is defined by the equation

\[
C(z, w) = \frac{1}{1 - wz}.
\]

If we fix \( z \) and allow \( w \) to vary, we obtain a test function which we denote by \( C(z, \cdot) \). If \( F \in \mathcal{H}^p(U) \) has the boundary distribution \( f \), then, for all \( z \in U \),

\[
F(z) = \langle C(z, \cdot), f(\cdot) \rangle.
\]

On the other hand, if \( z \notin \bar{U} \),

\[
0 = \langle C(z, \cdot), f(\cdot) \rangle.
\]

The first part of the next lemma states that the Toeplitz operators induced by the functions in \( C^\infty(T) \) extend or restrict to bounded operators on \( \mathcal{H}^p(U) \) for \( 0 < p < \infty \). This was proven in an earlier paper ([14, Th. 3.2]). A straightforward modification of the proof yields part (2).

**Lemma 3.2.** [14, Th. 3.2]. Let \( h \in C^\infty(T) \), let \( F \in \mathcal{H}^p(U) \), \( 0 < p < \infty \), and let \( f \) be the boundary distribution of \( F \). Define

\[
\mathcal{T}_h F(z) = \langle h(\cdot)C(z, \cdot), f(\cdot) \rangle.
\]

There are constants \( B = B(p, h) \) and \( B^* = B^*(p, h) \), independent of \( F \), such that

\[
(1) \quad \| \mathcal{T}_h F \|_{\mathcal{H}^p(U)} \leq B \| F \|_{\mathcal{H}^p(U)},
\]

and

\[
(2) \quad \| \mathcal{T}_h F \|_{\mathcal{H}^p(S^2 - \bar{U})} \leq B^* \| F \|_{\mathcal{H}^p(U)}.
\]

For the next theorem, let \( L_1 \) and \( L_2 \) be disjoint closed arcs on the unit circle \( T \), and define \( V_j \), for \( j = 1, 2 \), to be the union of the unit disc \( U \), its exterior \( S^2 - \bar{U} \), and the interior (relative to \( T \)) of \( L_j \).
THEOREM 3.3. (Decomposition Theorem). Let \( n > 1 \), and let \( Y \) be a Jordan polydomain in \( \mathbb{C}^{n-1} \). If \( F \in H^p(U \times Y) \), \( 0 < p < \infty \), there exist holomorphic functions \( F_1 \) in \( V_1 \times Y \) and \( F_2 \) in \( V_2 \times Y \) such that

1. \( F(z) = F_1(z) + F_2(z) \) if \( z \in U \times Y \),
2. \( 0 = F_1(z) + F_2(z) \) if \( z \in (S^2 - \bar{U}) \times Y \),

and, for \( j = 1, 2 \),

3. \( F_j \in \mathcal{H}^p(U \times Y) \),
4. \( F_j \in \mathcal{H}^p((S^2 - \bar{U}) \times Y) \),
5. \( F_j \in \mathcal{H}^p(D_j \times Y) \) for some open set \( D_j \subset \mathbb{C} \) that contains \( L_j \).

Proof. Choose functions \( h_j \in \mathcal{C}^\infty(T) \) such that \( h_j \) is identically zero on a neighborhood of \( L_j \) in \( T \), and such that \( h_1(\xi) + h_2(\xi) = 1 \) for all \( \xi \in T \). If \( (x, y) \in U \times Y \) we write \( F^y(x) = F(x, y) \). For each \( y \in Y \), the function \( F^y \) is in \( \mathcal{H}^p(U) \); denote its boundary distribution by \( f^y \) and define

\[
F_j(x, y) = \mathcal{F}_j F^y(x) = \langle h_j(\cdot), C(x, \cdot), F^y(\cdot) \rangle.
\]

We observe that \( F_j \) is separately holomorphic in \( x \) and \( y \), and hence holomorphic, at all \( z = (x, y) \) such that \( y \in Y \) and \( x \) is not in the closed support of \( h_j \). In particular, \( F_j \) is holomorphic in \( V_j \times Y \).

Since \( h_1 + h_2 \equiv 1 \), we have

\[
F_1(x, y) + F_2(x, y) = \langle C(x, \cdot), f^y(\cdot) \rangle.
\]

Fix \( y \in Y \). The right-hand term above, the Cauchy representation formula for \( F^y \), is 0 if \( x \in S^2 - \bar{U} \) and \( F^y(x) = F(x, y) \) if \( x \in U \). This establishes (1) and (2).

To prove the remainder of the theorem, we assume first that \( Y \) is the cartesian product of \( n - 1 \) simply connected domains.

Without loss of generality set \( Y = U^{n-1} \). Let \( H \) be the least \( w \)-harmonic majorant of \( |F|^{p} \) in \( U^n \), and write \( H^y(x) = H(x, y) \) for \( (x, y) \in U \times U^{n-1} \). The relations

\[
F_j(x, y) = \mathcal{F}_j F^y(x),
\]

\[
||\mathcal{F}_j F^y||_{\mathcal{W}^p(U)} \leq B||F^y||_{\mathcal{W}^p(U)}
\]

(part (1) of Lemma 3.2), and

\[
||F^y||_{\mathcal{W}^p(U)} \leq H^y(0),
\]

imply

\[
\int_T |F_j(\tau_2, \tau_1)|^p dm(\xi) \leq B^p H(0, \tau_1)
\]

for all \( 0 < r < 1 \) and \( w = (\xi, \eta) \in T \times T^{n-1} \). Integrating the above with respect to \( \eta \), we get
\[
\int_{\mathbb{T}} |F_j(rw)|^p dm_n(w) \leq B^p H(0) = B^p \|F\|_{L^p(U^n)}^p.
\]

Hence \( F_j \in \mathcal{H}^p(U^n) \).

By part (2) of Lemma 3.2 we have
\[
\|\mathcal{T}_{h_j} F^p \|_{L^p(S^2 - \bar{U})} \leq B^* \|F^p\|_{L^p(U)}.
\]

A similar argument to the one used above then establishes \( F_j \in \mathcal{H}^p(S^2 - \bar{U}) \times U^{n-1} \).

Finally, for the case \( Y = U^{n-1} \), we prove part (5) of the theorem.

Fix \( j = 1, 2 \). The function \( h_j \) will be identically zero on some open connected subset \( O_j \) of \( T \) which contains the arc \( L_j \). Let \( H_U \) and \( H_{S^2 - \bar{U}} \) be \( n \)-harmonic majorants of \( |F_j|^p \) in \( U^n \) and \( (S^2 - \bar{U}) \times U^{n-1} \) respectively. Considered as functions of the single complex variable \( x \), \( H_U(x, 0) \) and \( H_{S^2 - \bar{U}}(x, 0) \) (where 0 is the zero element in \( C^{n-1} \)), are positive harmonic functions (in \( U \), and in \( S^2 - \bar{U} \)). As is well known, they must have nontangential boundary values at almost all points of \( T \). Choose in each of the two connected components of \( O_j - L_j \) a point where both \( H_U(x, 0) \) and \( H_{S^2 - \bar{U}}(x, 0) \) simultaneously have a nontangential boundary value. Call these points \( \zeta' \) and \( \zeta'' \), and let \( C \) be a circle that intersects \( T \) precisely at \( \zeta' \) and \( \zeta'' \). Let \( a \) be the center and \( \rho \) the radius of \( C \), we write \( C = a + \rho T \) and let \( D_j \) be the disc bounded by \( a + \rho T \). The function \( F_j \) is holomorphic in a neighborhood of \( D_j \times U^{n-1} \); we proceed to show that \( F_j \in \mathcal{H}^p(D_j \times U^{n-1}) \), or equivalently, that the function \( G \), defined by
\[
G(x, y) = F_j(a + \rho x, y),
\]
is in \( \mathcal{H}^p(U^n) \).

Since the circle \( a + \rho T \) intersects \( T \) nontangentially at \( \zeta' \) and \( \zeta'' \), there is a constant \( K \) such that
\[
H_U(x, 0) \leq K
\]
for \( x \in (a + \rho T) \cap U \), and
\[
H_{S^2 - \bar{U}}(x, 0) \leq K
\]
for \( x \in (a + \rho T) \cap (S^2 - \bar{U}) \). Hence, for all \( 0 < r < 1 \), we have
\[
(3.3.1) \quad \int_{\mathbb{T}^{n-1}} |F_j(a + \rho \xi, r\eta)|^p dm_{n-1}(\eta) \leq \int_{\mathbb{T}^{n-1}} H_U(a + \rho \xi, r\eta) dm_{n-1}(\eta)
= H_U(a + \rho \xi, 0) \leq K
\]
whenever \( \xi \in T \) is such that \( a + \xi \in U \), and
\[
(3.3.2) \quad \int_{\mathbb{T}^{n-1}} |F_j(a + \rho \xi, r\eta)|^p dm_{n-1}(\eta) \leq \int_{\mathbb{T}^{n-1}} H_{S^2 - \bar{U}}(a + \rho \xi, r\eta) dm_{n-1}(\eta)
= H_{S^2 - \bar{U}}(a + \rho \xi, 0) \leq K
\]
whenever $\xi \in T$ is such that $a + \rho \xi \in S^2 - \bar{U}$.

The inequalities (3.3.1) and (3.3.2) yield, for all $0 < r < 1$,

$$\int_{T^n} |F_j(a + \rho \xi, r\eta)|^p dm_{n-1}(\eta)dm(\xi) \leq K.$$ 

Recalling the definition $G(x, y) = F_j(a + \rho x, y)$, and writing $w = (\xi, \eta)$, we obtain

$$\int_{T^n} |G(rw)|^p dm_n(w) = \int_{T^{n-1}} \int_T |F_j(a + \rho r \xi, r\eta)|^p dm(\xi)dm_{n-1}(\eta) \leq K.$$ 

It follows that $G \in \mathcal{H}^p(U^n)$, or equivalently that $F_j \in \mathcal{H}^p(D_j \times U^{n-1})$.

We next assume that $Y = Y_1 \times Y_2 \times \cdots \times Y_n$ is an arbitrary Jordan polydomain in $C^{n-1}$.

Decompose each $Y_i$ as a finite union $Y_i = \bigcup_k U_i^{(k)}$, where the sets $U_i^{(k)}$ are simply connected domains in $C$, and where every boundary point of $Y_i$ has a neighborhood that intersects inside some $U_i^{(k)}$. Let $\mathcal{U}$ be the class of all cartesian products $U_i^{(k_1)} \times U_i^{(k_2)} \times \cdots \times U_i^{(k_{n-1})}$.

The members of $\mathcal{U}$ are cartesian products of simply connected domains in $C$; accordingly, as was proven earlier, for each $Q \in \mathcal{U}$ we have $F_j \in \mathcal{H}^p(U \times Q)$, $F_j \in \mathcal{H}^p((S^2 - \bar{U}) \times Q)$, and $F_j \in \mathcal{H}^p(D_j^Q \times Q)$, where $D_j^Q$ is a disc, depending on $Q$, which contains $L_j$. From our construction of $\mathcal{U}$ it follows that $\{U \times Q\}_{Q \in \mathcal{U}}$ is a covering of $U \times Y$ that satisfies the requirements of Theorem 2.1; the same is the case for the coverings $\{(S^2 - \bar{U}) \times Q\}_{Q \in \mathcal{U}}$ of $(S^2 - \bar{U}) \times Y$, and $\{D_j \times Q\}_{Q \in \mathcal{U}}$ of $D_j \times Y$, where $D_j$ is the intersection of the (finitely many) discs $D_j^Q$. If we apply Theorem 2.1 to the $n$-subharmonic function $|F_j|^p$, we conclude that $F_j \in \mathcal{H}^p(U \times Y)$, $F_j \in \mathcal{H}^p((S^2 - \bar{U}) \times Y)$, and $F_j \in \mathcal{H}^p(D_j \times Y)$. This completes the proof of the theorem.

IV. The Čech cohomology with coefficients in $\mathcal{H}^p_w$. Throughout this section $0 < p < \infty$ will be fixed. We assume $n > 1$. Our goal is to prove, for any Jordan polydomain $W$ in $C^n$ and all integers $q \geq 1$, that $H^q(W, \mathcal{H}^p_w) = 0$.

It simplifies matters if we take our coefficients in the presheaf $\mathcal{H}^p$ rather than in its completion, the sheaf $\mathcal{H}^p$. We specify below what we mean by the Čech cohomology theory with coefficients in $\mathcal{H}^p$.

Let $W$ be a polydomain in $C^n$. We define a class $\Omega_w$ of open coverings of $W$ as follows.

An open covering $\mathcal{U}$ of $W$ belongs to $\Omega_w$ if and only if:
(1) Each member of \( \mathcal{U} \) is a polydomain.

(2) For every point \( b \) on the boundary of \( W \) there exists a neighborhood \( N(b) \) and a set \( U \in \mathcal{U} \) such that \( N(b) \cap W \subseteq U \).

Equivalently, \( \mathcal{U} \in \Omega_w \) if and only if \( \mathcal{U} \) is the restriction to \( W \) of a family of polydomains that covers \( W \).

Let \( \mathcal{U} \in \Omega_w \). A \( q \)-simplex \( \sigma \) of \( \mathcal{U} \) is a \( q + 1 \)-tuple \((U_0, U_1, \ldots, U_q)\) of members of \( \mathcal{U} \); its support \( |\sigma| \) is the set \( U_0 \cap U_1 \cap \cdots \cap U_q \). We denote by \( S_q(\mathcal{U}) \) the collection of all \( q \)-simplices of \( \mathcal{U} \), and by \( C^q(\mathcal{U}, \mathcal{H}^p) \) the group of all functions \( \gamma \) (\( q \)-cochains) that assign to each \( \sigma \in S_q(\mathcal{U}) \) an element \( \gamma(\sigma) \) of \( \mathcal{H}^p(|\sigma|) \).

The graded group \( C^q(\mathcal{U}, \mathcal{H}^p) \), together with the obvious coboundary operator \( \delta: C^q(\mathcal{U}, \mathcal{H}^p) \to C^{q+1}(\mathcal{U}, \mathcal{H}^p) \), constitutes a cochain complex with cocycles \( Z^q(\mathcal{U}, \mathcal{H}^p) \), coboundaries \( B^q(\mathcal{U}, \mathcal{H}^p) \), and cohomology group \( H^q(\mathcal{U}, \mathcal{H}^p) \). The relation of refinement induces a partial ordering on \( \Omega_w \). The corresponding direct limit groups

\[
H^q(W, \mathcal{H}^p) = \lim_{\mathcal{U} \in \Omega_w} H^q(\mathcal{U}, \mathcal{H}^p)
\]

are the cohomology groups of \( W \) with coefficients in the presheaf \( \mathcal{H}^p \).

As can be easily verified ([10, Cor. 18, p. 329]):

**Lemma 4.1.** The groups \( H^q(\bar{W}, \mathcal{H}^p) \) and \( H^q(W, \mathcal{H}^p) \) are isomorphic for all integers \( q \geq 0 \).

If \( V \subseteq W \) are polydomains, and if \( \mathcal{U} \in \Omega_w \), we denote by \( \mathcal{U}(V) \) the restriction of \( \mathcal{U} \) to \( V \) (in particular \( \mathcal{U} = \mathcal{U}(W) \)). We then have restriction homomorphisms \( C^q(\mathcal{U}(W), \mathcal{H}^p) \to C^q(\mathcal{U}(V), \mathcal{H}^p) \), which as can be easily verified, commute with the coboundary operators. If \( \gamma \in C^q(\mathcal{U}(W), \mathcal{H}^p) \) we denote its restriction to \( \mathcal{U}(V) \) by the same symbol \( \gamma \).

**Lemma 4.2.** Let \( W \) be a polydomain in \( C^n \), and let \( W = \{W^{(1)}, W^{(2)}\} \) be a covering in \( \Omega_w \).

If \( \mathcal{U} \in \Omega_w \) satisfies the conditions:

(1) For every simplex \( \sigma \in S^q(\mathcal{U}) \), the support \( |\sigma| \) is either a Jordan polydomain or the empty set.

(2) For every \( \sigma \in S^q(\mathcal{U}) \), the homomorphism

\[
\mathcal{H}^p(|\sigma| \cap W^{(1)}) \oplus \mathcal{H}^p(|\sigma| \cap W^{(2)}) \xrightarrow{\psi} \mathcal{H}^p(|\sigma| \cap W^{(1)} \cap W^{(2)}),
\]

defined by \( \psi(g^{(1)}, g^{(2)}) = g^{(1)} + g^{(2)} \), is onto.

Then there is an exact sequence of groups and homomorphisms
0 \to \cdots \to H^q(H(U(W), \mathcal{A}^p) \to H^q(H(U(W^{(1)}), \mathcal{A}^p) \oplus H^q(H(U(W^{(2)}), \mathcal{A}^p) \\
\phi \to H^q(H(U(W^{(1)} \cap W^{(2)}), \mathcal{A}^p) \to H^{q+1}(U(W), \mathcal{A}^p) \to \cdots .
(Such a sequence will be called a Mayer-Vietoris sequence.)

Proof. For each \( \sigma \in S^q(U) \) define

\[ \mathcal{A}^p(|\sigma|) \xrightarrow{\phi} \mathcal{A}^p(|\sigma| \cap W^{(1)}) \oplus \mathcal{A}^p(|\sigma| \cap W^{(2)}) \]

by the equation \( \phi(g) = (g, -g) \), with suitable restrictions.

By hypothesis \( |\sigma| \) is a Jordan polydomain (or the empty set). We can then invoke Theorem 2.1, and conclude that the image of \( \phi \) and the kernel of \( \psi \) are the same. Since also \( \phi \) is one-one, we have, for each \( \sigma \in S^q(U) \), a short exact sequence

\[ 0 \to \mathcal{A}^p(|\sigma|) \xrightarrow{\phi} \mathcal{A}^p(|\sigma| \cap W^{(1)}) \oplus \mathcal{A}^p(|\sigma| \cap W^{(2)}) \]

\[ \xrightarrow{\psi} \mathcal{A}^p(|\sigma| \cap W^{(1)} \cap W^{(2)}) \to 0 ,\]

which in turn induces a short exact sequence of graded groups

\[ 0 \to C^n(U(W), \mathcal{A}^p) \xrightarrow{\phi} C^n(U(W^{(1)}), \mathcal{A}^p) \oplus C^n(U(W^{(2)}), \mathcal{A}^p) \]

\[ \xrightarrow{\psi} C^n(U(W^{(1)} \cap W^{(2)}), \mathcal{A}^p) \to 0 ; \]

for if \( V \) is a polydomain in \( W \), then

\[ C^n(U(V), \mathcal{A}^p) = \Pi \mathcal{A}^p(|\sigma| \cap V) \]

\[ \sigma \in S^q(U) . \]

Since the homomorphisms \( \phi \) and \( \psi \) of (4.2.1) commute with the coboundary operators, the sequence (4.2.1) is a short exact sequence of cochain complexes. As is well known ([4, Th. 3.7, p. 128]) there is then an associated exact cohomology sequence. This completes the proof.

Our next lemma is a direct consequence of Theorem 2.1.

**Lemma 4.3.** If \( W \) is a Jordan polydomain, and if \( U \in \Omega_W \), then \( H^0(U, \mathcal{A}^p) \) and \( \mathcal{A}^p(W) \) are canonically isomorphic.

Henceforth, unless otherwise indicated, \( W = W_1 \times W_2 \times \cdots \times W_n \) will be a Jordan polydomain.

Towards our goal of establishing \( H^q(W, \mathcal{A}^p) = 0 \) we consider two cases.
1. The Simply Connected Case. We follow the argument of [13]. The proofs are identical (replacing the symbol $P$ by $\mathcal{H}^p$, and using Theorem 3.3 instead of [13, Lemma 3.1, p. 269]). We outline the procedure. Without loss of generality we take $W$ to be a poly-rectangle; this will allow a systematic partitioning into smaller polyrectangles.

Let $I, I_1,$ and $I_2$ be the open intervals $(-1, 1), (-1, \frac{1}{2}),$ and $(-\frac{1}{2}, 1)$, respectively. Consider the rectangles $R = I + iI, R_1 = I_1 + iI, R_2 = I_2 + iI$. For Lemmas 4.4 and 4.5 we write $W = R^n, W_i^{(1)} = R_1 \times R^{n-1}, W_i^{(2)} = R_2 \times R^{n-1}$; and let $\mathcal{U}$ be a finite open covering of $W$ consisting of polyrectangles with edges parallel to the real and imaginary axes of $C$.

**Lemma 4.4.** If $\sigma \in \mathcal{S}(\mathcal{U})$ and $g \in \mathcal{H}^p(|\sigma| \cap W_i^{(1)} \cap W_i^{(2)})$, there exist $g^{(1)} \in \mathcal{H}^p(|\sigma| \cap W_i^{(1)}), g^{(2)} \in \mathcal{H}^p(|\sigma| \cap W_i^{(2)}),$ such that $g = g^{(1)} + g^{(2)}$.

**Lemma 4.5.** For all integers $q \geq 1$, the cohomology groups $H^q(\mathcal{U}, \mathcal{H}^p)$ are trivial.

**Theorem 4.6.** If $W$ is a simply connected Jordan polydomain in $C$, then $H^q(W, \mathcal{H}^p) = 0$ for all integers $q \geq 1$.

2. The Multiply Connected Case. We first observe that Theorem 3.3 remains valid if we substitute the unit disc by a suitable doubly connected domain.

Let $0 < r_1 < r_2,$ and $r_2 - r_1/2 < \rho < r_2 + r_1/2$. Write

$$A = \{z \in C: r_1 < |z| < r_2\},$$

$$\Omega_1 = \left\{z \in C: \left|z - \frac{r_1 + r_2}{2}\right| < \rho\right\},$$

$$\Omega_2 = \left\{z \in C: \left|z + \frac{r_1 + r_2}{2}\right| < \rho\right\},$$

and define $B(r_1, r_2; \rho) = A \cup \Omega_1 \cup \Omega_2$. The set $B = B(r_1, r_2; \rho)$ is the union of the annulus $A$ with the symmetric discs $\Omega_1$ and $\Omega_2$. Any such region will be called a buleed annulus.

We write

$$C^+ = \{z \in C: \text{Im } z > 0\},$$

and set $A^{(1)} = A \cap C^+, A^{(2)} = A \cap (-C^+), B^{(1)} = A^{(1)} \cup \Omega_1 \cup \Omega_2,$ and $B^{(2)} = A^{(2)} \cup \Omega_1 \cup \Omega_2$.

**Lemma 4.7.** Let $Y$ be a Jordan polydomain in $C^{n-1}$. If $g \in \mathcal{H}^p((\Omega_1 \cup \Omega_2) \times Y)$, there exist $g^{(1)} \in \mathcal{H}^p(B^{(1)} \times Y)$ and $g^{(2)} \in \mathcal{H}^p(B^{(2)} \times Y)$ such that $g(z) = g^{(1)}(z) + g^{(2)}(z)$ whenever $z \in (\Omega_1 \cup \Omega_2) \times Y$. 

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Proof. Let $C_1$ and $C_2$ be the boundaries of $\Omega_1$ and $\Omega_2$ respectively. Consider the disjoint closed arcs $L_i^{(j)} = C_i \cap A^{(j)}$, for $i, j = 1, 2$.

It is clear that Theorem 3.3 remains valid if we replace the unit disc $U$ by the disc $\Omega$. We apply Theorem 3.3 to $\Omega \times Y$, the restriction of $g$ to $\Omega_1 \times Y$, and the closed arcs $L_1^{(i)}$, $L_2^{(j)}$, to obtain holomorphic functions $g_1^{(i)}$ and $g_2^{(j)}$, which by Theorem 2.1 are in $A^{(1)}(A \times Y)$ and in $A^{(2)}(A \times Y)$ respectively, such that

$$g(z) = g_1^{(i)}(z) + g_1^{(2)}(z),$$

if $z \in \Omega_1 \times Y$, and

$$0 = g_1^{(1)}(z) + g_1^{(2)}(z),$$

if $z \notin \Omega_1 \times Y$.

Similarly, by applying Theorem 3.3 to $\Omega_2 \times Y$, the restriction of $g$ to $\Omega_2 \times Y$, and the closed arcs $L_2^{(i)}$, $L_2^{(j)}$, we obtain $g_2^{(i)} \in A^{(1)}(A \times Y)$ and $g_2^{(j)} \in A^{(2)}(A \times Y)$, such that

$$g(z) = g_2^{(i)}(z) + g_2^{(j)}(z),$$

if $z \in \Omega_2 \times Y$, and

$$0 = g_2^{(1)}(z) + g_2^{(2)}(z),$$

if $z \notin \Omega_2 \times Y$.

If we define $g^{(j)} = g_1^{(j)} + g_2^{(j)}$, for $j = 1, 2$, the lemma is verified.

We next prove that the set of bulged annuli is a canonical class for the doubly connected domains in $C$.

**Lemma 4.8.** Let $A$ be a doubly connected domain in $C$. There exists a bulged annulus which is conformally equivalent to $A$. If $A$ is bounded by two Jorden curves, the conformal equivalence extends to a homeomorphism between the closures.

Proof. Without loss of generality let $A$ be an annulus centered at the origin. To prove the lemma it suffices to show that there exists a bulged annulus with the same modulus as $A$.

The modulus $M(D)$ of a doubly connected domain $D$, we recall, is a conformal invariant which in the special case of an annulus of radii $a < b$ reduces to $1/2\pi \log b/a$. Moreover, two doubly connected regions with the same modulus are necessarily equivalent ([6, Th. 2, p. 208]).

Let $B = B(r_1, r_2; \rho)$ be a bulged annulus contained in $A$. Since $B$ separates the boundaries of $A$, we must have ([6, Th. 3, p. 209])

$$M(B) \leq M(A). \tag{4.8.1}$$

For each $0 \leq t < \infty$ define $B_t = B(r_1, r_2 + t; \rho + t/2)$. Given any
for we can always find an annulus of inner radius $r_1$ and modulus $\lambda$ contained in $B_t$ if we choose $t$ sufficiently large.

A direct calculation (using the extremal length characterization of the modulus $M(B_t)$) shows that $M(B_t)$ varies continuously with $t$. The function $f(t) = M(B_t)$ is therefore continuous on $[0, \infty)$. By (4.9.1) and (4.8.2), we have $f(0) \leq M(A)$ and $\lim_{t \to \infty} f(t) = +\infty$, respectively. Consequently, for some $t_0$ we must have $M(B_{t_0}) = f(t_0) = M(A)$. This proves the first assertion of the lemma.

As is well known ([6, Th. 1, p. 208]), if two conformally equivalent doubly connected domains are bounded by Jordan curves, any conformal equivalence between them extends to a homeomorphism between their closures.

**THEOREM 4.9.** If $W$ is a Jordan polydomain in $C^n$, then $H^q(W, \mathcal{H}^p) = 0$ for all integers $q \geq 1$.

**Proof.** Denote by $Z^+_n$ the set of all $n$-tuples of positive integers. If $\mu$ and $\nu$ are in $Z^+_n$, and if $\mu_i \leq \nu_i$ for all $1 \leq i \leq n$, we write $\mu \leq \nu$. We say that a polydomain $W = W_1 \times W_2 \times \cdots \times W_n$ is $\mu$-connected (for some $\mu \in Z^+_n$) if each $W_i$ has connectively $\mu_i$.

For each $\nu \in Z^+_n$ let $P(\nu)$ be the proposition:

$P(\nu)$: For all $\mu$-connected polydomains $W$, $\mu \leq \nu$, and all integers $q \geq 1$, the cohomology groups $H^q(W, \mathcal{H}^p)$ are trivial.

Since a Jordan polydomain is necessarily finitely connected, the theorem will be proven if we verify $P(\nu)$ for all $\nu \in Z^+_n$.

Suppose $P(\nu)$ is true for some $\nu \in Z^+_n$. Fix $1 \leq k \leq n$, and denote by $\nu'$ the $n$-tuple define by $\nu'_i = \nu_i$, if $i \neq k$, and $\nu'_k = \nu_k + 1$. We claim that $P(\nu')$ is true. Without loss of generality take $k = 1$.

We first consider the case $\nu_1 = 1$. Let $W$ be an arbitrary $\nu'$-connected Jordan polydomain, and write $W = B \times Y$, where $B$ is a doubly connected Jordan domain in $C$ and where $Y \subset C^{n-1}$. By Lemma 4.8 there is no loss of generality if we let $B$ a bulged annulus. As in Lemma 4.7, decompose $B = B^{(1)} \cup B^{(2)}$, with $B^{(1)} \cap B^{(2)} = \Omega_1 \cup \Omega_2$. Define $W^{(1)} = B^{(1)} \times Y$, and $W^{(2)} = B^{(2)} \times Y$.

We consider the coverings in $\Omega_w$ that satisfy the following condition: the support $|\sigma|$ of any simplex $\sigma$ is a Jordan polydomain (or the empty set) contained in either $W^{(1)}$ or $W^{(2)}$. Such coverings satisfy the hypotheses of Lemma 4.2; and the collection of them constitutes a cofinal subclass of $\Omega_w$. By taking the direct limit of the corresponding Mayer-Vietoris sequences, we obtain the exact sequence
0 \longrightarrow H^p(W) \xrightarrow{\phi} H^p(W^{(1)}) \oplus H^p(W^{(2)}) \xrightarrow{\psi} H^p(W^{(1)} \cap W^{(2)})

\xrightarrow{\mathcal{F}} \cdots \xrightarrow{\mathcal{F}} H^q(W, H^p) \xrightarrow{\phi^*} H^q(W^{(1)}, H^p) \oplus H^q(W^{(2)}, H^p)

\xrightarrow{\psi^*} H^q(W^{(1)} \cap W^{(2)}, H^p) \xrightarrow{\mathcal{F}} \cdots \longrightarrow H^{q+1}(W, H^p) \longrightarrow \cdots.

By Lemma 4.7, the first row above is a short exact sequence; we disregard it, and retain the exact sequence

\begin{align*}
0 \longrightarrow H^1(W, H^p) & \xrightarrow{\phi^*} H^1(W^{(1)}, H^p) \oplus H^1(W^{(2)}, H^p) \\
\xrightarrow{\mathcal{F}} \cdots \xrightarrow{\mathcal{F}} H^q(W, H^p) & \xrightarrow{\phi^*} H^q(W^{(1)}, H^p) \oplus H^q(W^{(2)}, H^p) \xrightarrow{\psi^*} \cdots
\end{align*}

(4.9.1)

Since $W^{(1)}$ and $W^{(2)}$ are Jordan polydomains of connectivity $\leq \nu$, and since $W^{(1)} \cap W^{(2)}$ is the disjoint union of two Jordan polydomains of connectivity $\leq \nu$, the inductive hypothesis implies $H^q(W^{(1)}, H^p) = 0$, $H^q(W^{(2)}, H^p) = 0$, and $H^q(W^{(1)} \cap W^{(2)}, H^p) = 0$. The exactness of (4.9.1) then establishes $H^q(W, H^p) = 0$ for all $q \geq 1$.

We next consider the case $\nu_1 > 1$. As before, let $W$ be an arbitrary $\nu'$-connected polydomain. Write $W = X \times Y$, where $Y \subset C^{n-1}$, and where $X$ is a domain in $C$ of connectivity $k = \nu_1 + 1$ which is bounded by an outer contour $C_k$ and $k$ inner contours $C_0, C_1, \ldots, C_{k-1}$.

Let $B$ be the doubly connected domain bounded by $C_0$ and $C_k$, and let $A^{(1)}$ and $A^{(2)}$ be simply connected Jordan domains such that

- $(1)$ $A^{(1)} \cup A^{(2)} = B$,
- $(2)$ $A^{(1)} \cap A^{(2)}$ is the disjoint union of two simply connected domains,
- $(3)$ each contour $C_i, C_2, \ldots, C_{k-1}$ is entirely contained in either $A^{(1)} - A^{(2)}$ or $A^{(2)} - A^{(1)}$.

We define $X^{(1)} = A^{(1)} \cap X$, $X^{(2)} = A^{(2)} \cap X$; and consider the Jordan polydomain $V^{(1)} = A^{(1)} \times Y$, $V^{(2)} = A^{(2)} \times Y$, $V = B \times Y$, $W^{(1)} = X^{(1)} \times Y$, and $W^{(2)} = X^{(2)} \times Y$.

As in the previous case of the theorem, by taking suitable coverings, applying Lemma 4.2, and taking the direct limit of the Mayer-Vietoris sequences that correspond to such coverings, we obtain the exact sequences

\begin{align*}
0 \longrightarrow H^p(V) & \xrightarrow{\phi} H^p(V^{(1)}) \oplus H^p(V^{(2)}) \xrightarrow{\psi} H^p(V^{(1)} \cap V^{(2)}) \\
\xrightarrow{\mathcal{F}} H^1(V, H^p) & \longrightarrow \cdots
\end{align*}

and

\begin{align*}
0 \longrightarrow H^p(W) & \xrightarrow{\phi} H^p(W^{(1)}) \oplus H^p(W^{(2)}) \xrightarrow{\psi} H^p(W^{(1)} \cap W^{(2)}) \\
\xrightarrow{\mathcal{F}} \cdots \xrightarrow{\mathcal{F}} H^q(W, H^p) & \xrightarrow{\phi^*} H^q(W^{(1)}, H^p) \oplus H^q(W^{(2)}, H^p) \xrightarrow{\psi^*} \cdots
\end{align*}

(4.9.2)
The polydomain $V$ has connectivity $\mu$, with $\mu_1 = 2$, and $\mu_i = \nu_i$ for $i = 2, 3, \ldots, n$. Consequently, as was established earlier, $H^q(V, \mathcal{H}_p) = 0$. In particular

$$0 \to \mathcal{H}_p(V) \xrightarrow{\phi} \mathcal{H}_p(V^{(1)}) \oplus \mathcal{H}_p(V^{(2)}) \xrightarrow{\psi} \mathcal{H}_p(V^{(1)} \cap V^{(2)}) \to 0$$

is exact. Since $W^{(1)} \cap W^{(2)} = V^{(1)} \cap V^{(2)}$, and since $W^{(1)} \subset V^{(1)}$, $W^{(2)} \subset V^{(2)}$, it follows that

$$0 \to \mathcal{H}_p(W) \xrightarrow{\phi} \mathcal{H}_p(W^{(1)}) \oplus \mathcal{H}_p(W^{(2)}) \xrightarrow{\psi} \mathcal{H}_p(W^{(1)} \cap W^{(2)}) \to 0$$

is also exact. We can then disregard the first row of (4.9.2) and retain exactness in

$$0 \to H^q(W, \mathcal{H}_p) \xrightarrow{\phi} H^q(W^{(1)}, \mathcal{H}_p) \oplus H^q(W^{(2)}, \mathcal{H}_p) \xrightarrow{\psi} H^q(W^{(1)} \cap W^{(2)}, \mathcal{H}_p) \to 0$$ (4.9.3)

The inductive hypothesis, together with the exactness of (4.9.3), implies $H^q(W, \mathcal{H}_p) = 0$ for all $q \geq 1$; for $W^{(1)}$ and $W^{(2)}$ are Jordan polydomains of connectivity $\leq \nu$, and $W^{(1)} \cap W^{(2)}$ is the disjoint union of two Jordan polydomains of connectivity $\leq \nu$.

We have thus established $P(\nu')$ in all cases. Since, as was proven in Theorem 4.6, $P(\nu)$ is true for $\nu = (1, 1, \ldots, 1)$, by the principal of mathematical induction $P(\nu)$ must also be true for all $\nu \in \mathbb{Z}^n_+$. This concludes the proof.

V. Remarks.

1. The Gleason Problem for $\mathcal{H}_p(W)$. Let $F \in \mathcal{H}_p(W)$, let $a \in W$, and suppose $F(a) = 0$. The problem asks if there exist $F_1, \ldots, F_n \in \mathcal{H}_p(W)$ such that $F(z) = (z_1 - a_1)F_1(z) + \cdots + (z_n - a_n)F_n(z)$ for all $z \in W$. The method of [7], together with the vanishing of the cohomology of $\mathcal{H}_p$, gives an affirmative answer when $W$ is a Jordan polydomain. A non cohomological treatment of the Gleason problem for various other functions spaces is given in [1].

2. The extension of $\mathcal{H}_p$-functions from hypersurfaces in $W$. Let $S$ be the zero set of a bounded holomorphic function in $U^n$. In [2] Andreotti and Stoll defined a strictly $\mathcal{H}_\infty$-function to be a function $f: S \to C$ for which there exists a covering $\{U_a\}$ of $\bar{U}^n$, and functions $f_a \in \mathcal{H}_\infty(U_a \cap U^n)$ and $g_{a\beta} \in \mathcal{H}_\infty(U_a \cap U_\beta \cap U^n)$ such that

(i) $f = f_a$ on $S \cap U_a$
(ii) \( f_\beta - f_\alpha = h g_{\alpha \beta} \) on \( U_\alpha \cap U_\beta \cap U^n \);
and proved, as a direct consequence of the vanishing of \( H^1(\bar{U}^n, \mathbb{H}^\infty) \),
that any such function has an extension in \( \mathbb{H}^\infty(U^n) \).

If \( W \) is a Jordan polydomain, \( S \) is the zero set of an \( \mathbb{H}^\infty \)-
function in \( W \), and \( f: S \to \mathbb{C} \) is a strictly \( \mathcal{H}^p \)-function (defined as
above, but requiring now that \( f_\alpha \) and \( g_{\alpha \beta} \) be in the corresponding
\( \mathcal{H}^p \)-spaces), the vanishing of the cohomology of \( \mathcal{H}^p \) establishes the
existence of an extension \( F \in \mathcal{H}^p(W) \) of \( f \).

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