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SPECTRAL SYNTHESIS IN PRODUCTS AND QUOTIENTS OF HYPERGROUPS

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In this paper we first discuss the coset spaces K/H and $K//H$ of left cosets and double cosets respectively of a hypergroup K by a compact subhypergroup H. This development is then used to obtain some results connecting spectral synthesis for $L^1(K/H)$ to that for $L^1(K)$ when K is commutative. We also indicate that some of the results for quotient group carry over to K/H when H is a subgroup of the center $Z(K)$ of K. Finally we discuss how Malliavin's theorem fails in a strong way in many hypergroups and further show that for certain closed sets of the form $E_1 \times E_2$ in $\hat{K}_1 \times \hat{K}_2$, where $K_1 = \boldsymbol{R}^+$, Z^+ etc. and K_2 is a locally compact commutative hypergroup such that the dual $\hat{K}_2$ of K_2 is a σ-compact hypergroup, $E_1 \times E_2$ can inherit various properties of E_1 such as being nonspectral, non ultra-strong Ditkin for the respective hypergroup algebras.

1. Introduction. The propose of this paper is to discuss spectral synthesis for quotients and products of hypergroups. The basic development of harmonic analysis for hypergroups can be found in ([10], [11], [13], [20] and [21]). Spectral synthesis for hypergroups has been developed in ([9], [7] and [8]). The motivation for all this has been the fact that hypergroups arise in a natural way as a double coset space, the space of conjugacy classes of a compact group and harmonic analysis on them is closely related to that of the groups, a survey of this can be found in ([18] and [6]). Our main reference for the basic theory of hypergroups will be [13] and most of the further notation and terminology is as in ([12], [9] and [7]). Throughout this paper K will denote a locally compact hypergroup (same as 'convo' in [13]) possessing a Haar measure m, $L^1(K) = L^1(m)$ the convolution algebra and H a compact subhypergroup of K. In § 2 we discuss the quotient hypergroup $K//H$ of double cosets and the Weil's formula for $L^1(K//H)$, also we briefly describe the situation for the left coset space K/H as well. In the next section we confine our attention to those commutative hypergroups K for which the dual $\hat{K}$ is a hypergroup under pointwise operations. We first identify the dual of K/H as the subhypergroup $H^\perp$ of $\hat{K}$ and relate spectral synthesis of $L^1(K/H)$ with that of $L^1(K)$. In particular, we show that $\Delta \subset H^\perp$ is spectral for $L^1(K/H)$ if it is so for $L^1(K)$ and Δ is strong Ditkin for $L^1(K/H)$ if and only if it is so for $L^1(K)$. In § 4 we again take K to be a commutative hypergroup

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whose dual $\hat{K}$ is a hypergroup and prove similar results for the coset space K/H, when H is a subgroup of the center $Z(K)$ of K [19]. In the last section we discuss spectral synthesis for the hypergroup algebra $L^1(K_1 \times K_2)$, where K_1 can be one of the $\boldsymbol{R}^+$, Z^+, Naimark example ([13], 9.5) and the dual $\hat{F}$ of the hypergroup F of conjugacy classes of the compact group $SU(2)$ ([13], 15.4 and [9], 4.7) and K_2 is any commutative hypergroup whose dual $\hat{K}_2$ is a σ-compact hypergroup under pointwise operations. In particular we obtain methods to find various nonspectral sets, non ultra-strong Ditkin sets in $\hat{K}_1 \times \hat{K}_2$. In this process we also show how Malliavin's theorem fails in a strong way in many hypergroups. We just remark that the analogues for Segal algebras of some of the results on spectral synthesis in this paper can be easily formulated and proved based on [7].[1]

2. Quotient hypergroups. In this section K will be a locally compact hypergroup with Haar measure m and H a compact subhypergroup of K with normalized Haar measure σ. As shown in ([13], § 14) the quotient map π on K onto the double coset space $K//H = \{H*x*H: x \in K\}$ gives rise to maps π_*, on $M(K)$ to $M(K//H)$ and π^* on $M(K//H)$ to $M(K)$ defined by

$$\pi_*(\mu) = \mu\circ\pi^{-1} \text{ and } \pi^*(p_{H*x*H}) = \sigma * p_x * \sigma\ .$$

Further $K//H$ can be made into a hypergroup with convolution defined by

$$\begin{aligned}\int_{K//H} f\,dp_{H*x*H}*p_{H*y*H} &= \int_{K//H} f\,d\pi_*(\pi^*(P_{H*x*H})*\pi^*(p_{H*y*H})) \\ &= \int_K f\circ\pi\,dp_x*\sigma*p_y\end{aligned}$$

for all $f \in B^\infty(K//H)$. A Haar measure on $K//H$ is given by $\dot{m} = \int_K p_{H*x*H}m(dx)$. Since the mapping $\dot{x} = H*x*H \to \sigma*p_x*\sigma$ is the recomposition of π consistent with m([13], 14.2H), we have the following result.

THEOREM 2.1. (*Weil's formula*) *The Haar measure $\dot{m}$ on $K//H$ can be so chosen that*

$$\int_{K//H}\int_H (\sigma*f)(x*\xi)\,d\sigma(\xi)\,d\dot{m}(\dot{x}) = \int_K f\,dm, \text{ for } f \text{ in } L^1(K)$$

or equivalently

$$\text{(A)} \qquad \int_{K//H}\int_K f\,d\sigma*p_x*\sigma\,d\dot{m}(\dot{x}) = \int_K f\,dm\ .$$

[1] The present paper along with [7] and [8] forms a major part of [23].

REMARK 2.2. The reason why we call the above theorem the Weil's formula is that in case H is *strongly normal* in the sense that $p_x*\sigma = \sigma*p_x$ for all x in K we have that $x*H = H*x = H*x*H$ for each x in K and (A) takes the form

$$\int_{K/H}\int_{H} f(x*\xi)d\sigma(\xi)d\dot{m}(\dot{x}) = \int_{K} f(x)dm(x)$$

which is the natural analogue of Weil's formula in the group case.

LEMMA 2.3. *Let F_1 be a compact subset of $K//H$ then there exists a compact subset F_2 of K such that $\pi(F_2) = F_1$.*

Proof. Follows from ([13], 13.2A).

THEOREM 2.4. *Let T_H be the mapping defined by $T_H f(H*x*H) = \int_K f d\sigma*p_x*\sigma$, $f \in L^1(K)$. Then*

(i) *for $g \in L^1(K//H)$ we have* (a) $f = g\circ\pi \in L^1(K)$.

(b) $f(\eta*x*\xi) = f(x)$ *for all $\eta, \xi \in H$ and $x \in K$.*

(c) $\int_{K//H} g(\dot{x})d\dot{m}(\dot{x}) = \int_K g\circ\pi(x)dm(x)$.

(ii) *T_H is a bounded linear map of $L^1(K)$ onto $L^1(K//H)$ with norm* 1.

(iii) *T_H induces the isometric isomorphism π^{*-1} on the subspace of H-invariant functions in $L^1(K)$ onto $L^1(K//H)$.*

(iv) *In case H is strongly normal, T_H is an algebra homomorphism.*

Proof. For $g \in L^1(K//H)$, $g\circ\pi$ is Borel measurable ([13], 13.2G). Using Theorem 2.1 simple computations show that $g\circ\pi \in L^1(K)$. Also $g\circ\pi(\eta*x*\xi) = g\circ\pi(x)$ for all $x \in K$ and $\eta, \xi \in H$ and $T_H(g\circ\pi) = g$ with $\|g\circ\pi\|_1 = \|g\|_1$. Thus we have (i) (a), (b) and (iii) and (i)(c) follows from ([13], 13.2H). (ii) can be proved using Theorem 2.1. For (iv) let us assume that $p_x*\sigma = \sigma*p_x$ for all $x \in K$. Simple computations show that $(T_H f)_{\dot{x}} = T_H f_x$ for all $x \in K$, further this fact can be used to prove that T_H is an algebra homomorphism using ([13], 5.1D) and the fact that $(f*g)_x = f_x*g$.

REMARK 2.5. Mackey [14] and Bruhat [5] generalized Weil's formula for quotient space of a locally compact group G by a closed subgroup H (not necessarily normal) which is also discussed as the formula of Mackey-Bruhat for quasi-invariant measures m ([16], VIII). In case H is compact the quotient space G/H can also be thought of as the semi-convo of cosets with convolution given by

$p_{x*H}*p_{y*H} = \int_H p_{xtyH} d\sigma(t)$, where σ is the Haar measure of H. It can be shown that the measure $\dot{m}$ in ([16], VIII, 1·1) is left invariant and therefore is the Haar measure on G/H ([13], 5·2, 8.2). Thus Weil's formula of Mackey and Bruhat fits well as Weil's formula in the hypergroup setting.

Now if K is a hypergroup with Haar measure m and π_H is the quotient map on K onto K/H, then we can show that K/H is a semi-convo (using 2.3H, 10.3B, 13.2A, B, C and analogues of 14.1A, 14.1D of [13]) with Haar measure $\dot{m} = \int_K p_{x*H} m(dx)$. Since H is compact using 5.3C of [13] we have that $\Delta_H(H)$ and $\Delta_K(H)$ are compact subgroups of the multiplicative group of positive real numbers and therefore $\Delta_H(H) = \{1\} = \Delta_K(H)$. This helps us in showing that the map $x*H \to p_x*\sigma$ is a recomposition of π_H consistent with m or equivalently we have the Weil's formula in the following form

$$\int_{/HK}\int_H f(x*\xi)d\sigma(\xi)d\dot{m}(\dot{x}) = \int_K fdm .$$

3. Spectral synthesis for $L^1(K/H)$. In this section we consider only those K which are commutative and for which $\hat{K}$ is a hypergroup with pointwise operations. Then every compact subhypergroup H of K is strongly normal and thus gives rise to the quotient hypergroup K/H which is commutative and therefore has a Haar measure by ([21] or [13], 14.2). As in § 2 above the Haar measure $\dot{m}$ of K/H can be so chosen as to satisfy Weil's formula. In this case the convolution algebra $L^1(K/H)$ is commutative and we can discuss spectral synthesis for this algebra. For the sake of uniformity we further assume that $\hat{K} = \mathfrak{X}_b(K)$, eventhough for some of the results it can be replaced by weaker conditions such as the regularity of $L^1(K)$ on $\mathfrak{X}_b(K)$ ([9], [7]).

LEMMA 3.1. *Let $x, y \in K$, $x*H = y*H$ implies that*

$$\int_K fdp_x*\sigma = \int_K fdp_y*\sigma \text{ for all } f \in C_b(K) .$$

Proof. By ([13], 3.2G), $\int_K fdp_x*\sigma = \int_{x*H} fdp_x*\sigma$ and now modify 14.1B of [13] using compactness of $(x*H) \cup (y*H)$ and ([9], 2.5).

THEOREM 3.2. *For φ in $H^\perp$, $\varphi'(x*H) = \varphi(x)$ defines an element of $(K/H)^\wedge$, $\varphi \to \varphi'$ is a one-to-one map of $H^\perp$ onto $(K/H)^\wedge$.*

Proof. We first note a fact often used in the proof namely for

φ in $H^{\perp}$

$$\int_K \varphi d\sigma = \int_H \varphi d\sigma = 1 \ .$$

Let $x \in K$ and $y \in x*H$. Then ([13], 10.3A) gives $x*H = y*H$. By Lemma 3.1 $\int_K \varphi dp_x*\sigma = \int_K \varphi dp_y*\sigma$, so using ([13], 3.1E) and the homomorphism property of φ, we have that $\varphi(x) = \varphi(y)$. Hence φ' is well defined. Now using ([13], 4.2H) at appropriate places it is easy to show that $\varphi'((x*H)*(y*H)) = \varphi'(x*H)\varphi'(y*H)$. Clearly φ' is hermitian and ([13], 10.3B) implies that φ' is continuous and therefore $\varphi' \in (K/H)^{\wedge}$. Obviously $\varphi \to \varphi'$ is a well defined map. Using ([13], 3.1E, 10.3B) again we have that if $\psi \varepsilon (K/H)^{\wedge}$ then $\varphi(x) = \psi(x*H)$ defines an element in $H^{\perp}$ such that $\varphi' = \psi$. Thus η given by $\eta(\varphi) = \varphi'$ maps $H^{\perp}$ onto $(K/H)^{\wedge}$.

If F_1 is a compact subset of K/H then by Lemma 2.3 there exists a compact subset F_2 of K such that $\pi(F_2) = F_1$. Consider $W(F_1, \chi, \varepsilon) = \{\varphi \in (K/H)^{\wedge}: |\chi(x*H) - \varphi(x*H)| < \varepsilon$ for all $x*H \in F_1\}$ and $W(F_2, \chi', \varepsilon) = \{\gamma \in \hat{K}: |\chi'(x) - \gamma(x)| < \varepsilon$ for all $x \in F_2\} \cap H^{\perp}$. Clearly $\eta(W(F_2, \chi', \varepsilon)) = W(F_1, \chi, \varepsilon)$ and hence the result.

COROLLARY 3.3. *$(K/H)^{\wedge}$ is a commutative hypergroup under pointwise operations.*

COROLLARY 3.4. *For each $x \in K \sim H$ there exists $\psi \in H^{\perp}$ such that $\psi(x) \neq 1$.*

Proof. Apply Lemma 2.1 [19] to K/H.

LEMMA 3.5. *The Fourier transform of $T_H f \in L^1(K/H)$ is the restriction of the Fourier transform $\hat{f}$ of f to $H^{\perp}$.*

Proof. Apply Theorem 2.4 and ([13], 5.1D).

REMARK 3.6. We just note the algebra homomorphism property of T_H can be deduced from the above lemma using the uniqueness of Fourier transform for the special K considered in this section.

REMARK 3.7. It is clear that from the proof of Theorem 3.2, that $\chi_b(K/H) = (K/H)^{\wedge} = H^{\perp}$.

LEMMA 3.8. *If H is compact or open then $H^{\perp}$ is open or compact respectively.*

Proof. If H is compact then $\hat{H}$ is discrete. Since the restriction map $\lambda: \hat{K} \to \hat{H}$ ([19], [10]) is continuous and $H^{\perp} = \lambda^{-1}\{1\}$ we have that $H^{\perp}$ is open in $\hat{K}$. The other part follows using Theorem 3.2 since K/H is discrete and therefore $(K/H)^{\wedge}$ is compact in this case.

LEMMA 3.9. *The space of H-invariant functions in $L^1(K)$ forms an ideal in $L^1(K)$.*

Proof. Follows using ([9], 2.4) and ([13], 5.5A).

LEMMA 3.10. *$E \subset \hat{K}$ is a strong Ditkin (respectively, ultra-strong Ditkin) set if and only if there exists a net $\{\mu_\alpha\}_{\alpha \in \Lambda}$ of finite measures in $M(K)$ such that*

(a) (i) *for each α, $\hat{\mu}_\alpha$ has compact support and equals 1 in a neighborhood of E.*

(ii) *for each $f \in k(E)$, $\|\mu_\alpha * f\|_1 \to 0$.*

(b) *There is a $C \geqq 0$ such that for all $f \in J(E)$ (in $L^1(K)$) with $\|f\|_1 \leqq 1$ we have $\|f * \mu_\alpha\|_1 \leqq C$ for all α.*
(Respectively, we have an $M \geqq 0$ with $\|\mu_\alpha\| \leqq M$ for all α.)

Proof. Modify ([17], 2.2(b)).

THEOREM 3.11. *Let H be a compact subhypergroup of K and $\Delta \subset H^{\perp}$. Then*

(i) *Δ is spectral for $L^1(K/H)$ if it is so for $L^1(K)$.*

(ii) *Δ is ultra-strong Ditkin (respectively, strong Ditkin, Calderon) for $L^1(K/H)$ if and only if it is so for $L^1(K)$.*

(iii) *Δ is sequentially strong Ditkin for $L^1(K/H)$ if it is so for $L^1(K)$. If K is first countable and Δ is sequentially strong Ditkin for $L^1(K/H)$ then it is so for $L^1(K)$.*

Proof. We shall only prove the ultra-strong Ditkin part of (ii) and indicate a proof for the strong Ditkin case. It will be clear how to formulate proof of the rest of the theorem.

Let Δ be ultra-strong Ditkin for $L^1(K)$. Then there is a net $\{f_\alpha\}_{\alpha \in \Lambda}$ in $L^1(K)$ such that for each α, $\hat{f}_\alpha = 0$ on a neighborhood V_α of Δ in $\hat{K}$ and $\hat{f}_\alpha$ has compact support; further $M = \sup\{\|f_\alpha\|_1 : \alpha \in \Lambda\} < \infty$ and for all $f \in L^1(K)$ with $\hat{f} = 0$ on Δ we have $\|f_\alpha * f - f\|_1 \to 0$. Let $\dot{f} \in L^1(K/H)$ be such that $\hat{\dot{f}} = 0$ on Δ. Then by Theorem 2.4 there exists f in $L^1(K)$ such that $T_H f = \dot{f}$, so for $\gamma \in \Delta$, $\hat{f}(\gamma) = \hat{\dot{f}}(\gamma) = 0$ by Lemma 3.5. Let $g_\alpha = T_H f_\alpha (\alpha \in \Delta)$. Then by Lemma 3.5 again for each α, $\hat{g}_\alpha$ vanishes on the neighborhood $V_\alpha \cap H^{\perp}$ of Δ in $H^{\perp}$

and has compact support. Applying Theorem 2.4 we have that $\{g_\alpha\}$ serves as the required net.

Conversely, let Δ be ultra-strong Ditkin for $L^1(K/H)$, applying Lemma 3.10 there exists a net $\{\dot{\mu}_\alpha\}_{\alpha\in A}$ in $M(K/H)$ such that for each α, $\hat{\dot{\mu}}_\alpha = 1$ on a neighborhood V_α of Δ in $H^\perp$ and $\hat{\dot{\mu}}_\alpha$ has compact support, further $M = \sup\{\|\dot{\mu}_\alpha\|: \alpha\in A\} < \infty$ and for $\dot{f}\in L^1(K/H)$ with $\hat{\dot{f}}$ zero on Δ, $\|\dot{f}*\dot{\mu}_\alpha\|_1 \to 0$. Let $\{u_\beta\}_{\beta\in B}$ be an approximate unit for $L^1(K)$ such that for each β, $\hat{u}_\beta$ has compact support U_β and $\|u_\beta\|_1 = 1$ ([9], 2.8). Let $W_\beta = U_\beta \cap H^\perp$. Then W_β is compact so by ([9], 2.5) there exists $\dot{g}_\beta \in L^1(K/H)$ such that $\hat{\dot{g}}_\beta = 1$ on W_β. Let $\dot{f}_{\alpha,\beta} = \dot{\mu}_\alpha * \dot{g}_\beta$. Then there exists $f_{\alpha,\beta}\in L^1(K)$ such that $T_H f_{\alpha,\beta} = \dot{f}_{\alpha,\beta}$ and $f_{\alpha,\beta}(x*\xi) = f_{\alpha,\beta}(x)$ for all $x\in K$ and $\xi\in H$. Let D be the directed set $B\times A$. For $\rho = (\beta,\alpha)\in B\times A$, let $h_\rho = u_\beta - u_\beta * f_{\alpha,\beta}$ so that $\hat{h}_\rho = \hat{u}_\beta(1-\hat{f}_{\alpha,\beta})$ and therefore $\hat{h}_\rho$ is zero on V_α and has compact support contained in U_β. Now

$$\|h_\rho\|_1 \leqq \|u_\beta\|_1 + \|u_\beta * f_{\alpha,\beta}\|_1 = 1 + \|u_\beta * f_{\alpha,\beta}\|_1 .$$

To estimate $\|u_\beta^* f_{\alpha,\beta}\|_1$ we use $T_H u_\beta^* \dot{g}_\beta = T_H u_\beta$ which follows easily using uniqueness of Fourier transform and the fact that $\hat{\dot{g}}_\beta(\gamma) = 1$ for $\gamma\in W_\beta$ and $(T_H u_\beta)\hat{\ }(\gamma) = 0$ for $\gamma \in H^\perp \sim U_\beta = H^\perp \sim W_\beta$. This equality will be used in further computations as well. Now using Lemma 3.9 and Theorem 2.4 and ([13], 6.2B).

$$\begin{aligned}\|u_\beta * f_{\alpha,\beta}\|_1 &= \|T_H(u_\beta * f_{\alpha,\beta})\|_1 = \|T_H u_\beta * T_H f_{\alpha,\beta}\|_1 \\ &= \|T_H u_\beta * \dot{g}_\beta * \dot{\mu}_\alpha\|_1 \\ &= \|T_H u_\beta * \dot{\mu}_\alpha\|_1 \leqq \|T_H u_\beta\|_1 \|\dot{\mu}_\alpha\| \leqq \|u_\beta\|_1 \|\dot{\mu}_\alpha\| \leqq M .\end{aligned}$$

So $\|h_\rho\|_1 \leqq 1 + M$ for all ρ. Fix any $f\in L^1(K)$ such that $\hat{f}$ vanishes on Δ. Then for each $n\in N$ there exists $\beta_n \in B$ such that $\|f - f*u_\beta\|_1 < 1/n$ for all $\beta \geqq \beta_n$. Also for all n there exists α_n such that

$$\|T_H f * \dot{\mu}_\alpha\|_1 < \frac{1}{n} \text{ for all } \alpha \geqq \alpha_n .$$

Let $\varepsilon > 0$ be arbitrary. Let $n_0\in N$ be such that $1/n_0 < \varepsilon/2$. Then for $\rho \geqq \rho_0 = (\beta_{n_0}, \alpha_{n_0})$

$$\begin{aligned}\|f*h_\rho - f\|_1 &\leqq \|f*u_\beta - f\|_1 + \|f*u_\beta * f_{\alpha,\beta}\|_1 \\ &< \frac{1}{n_0} + \|f*u_\beta * f_{\alpha,\beta}\|_1 .\end{aligned}$$

Further using Lemma 3.9 and Theorem 2.4 again

$$\begin{aligned}\|f*u_\beta*f_{\alpha,\beta}\|_1 &= \|T_H f*T_H u_\beta*\dot{g}_\beta*\dot{\mu}_\alpha\|_1\\ &= \|T_H f*T_H u_\beta*\dot{\mu}_\alpha\|_1\\ &\leqq \|T_H u_\beta\|_1\|T_H f*\dot{\mu}_\alpha\|_1 < \frac{1}{n_0}\,.\end{aligned}$$

So $\|f*h_\rho - f\|_1 < 2/n_0 < \varepsilon$.

Hence $\{h_\rho\}_{\rho\in D}$ serves as a required net. Now we note that for $\dot{h}\in L^1(K/H)$.

$$\begin{aligned}&\sup\{\|\dot{h}*\dot{g}\|_1: \dot{g}\in J(\Delta) \text{ in } L^1(K/H) \text{ and } \|\dot{g}\|_1\leqq 1\}\\ &\quad = \sup\{\|\dot{h}*T_H(\dot{g}\circ\pi)\|_1: \dot{g}\in J(\Delta) \text{ in } L^1(K/H) \text{ and } \|\dot{g}\|_1\leqq 1\}\\ &\quad \leqq \sup\{\|\dot{h}\circ\pi*\dot{g}\circ\pi\|_1: \dot{g}\in J(\Delta) \text{ in } L^1(K/H) \text{ and } \|\dot{g}\|_1\leqq 1\}\\ &\quad \leqq \sup\{\|\dot{h}\circ\pi*g\|_1: g\in J(\Delta) \text{ in } L^1(K) \text{ and } \|g\|_1\leqq 1\}\,.\end{aligned}$$

So the proof for the ultra-strong Ditkin can be appropriately changed to show that if Δ is strong Ditkin for $L^1(K)$ then it is so for $L^1(K/H)$. For the converse part we choose h_ρ as above using strong Ditkin version of Lemma 3.9 and note that for $f\in J(\Delta)$ in $L^1(K)$ with $\|f\|_1\leqq 1$ we have $T_H f\in J(\Delta)$ (in $L^1(K/H)$) and $\|T_H f\|_1\leqq 1$ by Theorem 2.4 and Lemma 3.7; further we perform obvious computations to show that $\|f*h_\rho\|_1\leqq C$ for all ρ.

COROLLARY 3.12. *Let $E\subset H^\perp$ and $\gamma\in Z(\hat{K})$. Then E is Calderon (respectively, strong Ditkin, ultra-strong Ditkin) for $L^1(K/H)$ if and only if γE is so for $L^1(K)$.*

Proof. It follows immediately from ([9], 3.7).

4. Quotient by subgroups of the center $Z(K)$. In this section H will be a closed subgroup of the center $Z(K)$ of K where K is a commutative hypergroup such that its dual $\hat{K}$ is also a hypergroup under pointwise operations. As shown in ([9], 4.1) $K/H = \{xH: x\in K\}$ is a commutative hypergroup with convolution defined by

$$\int_{K/H} f\,dp_{xH}*p_{yH} = \int_K f\circ\pi\,dp_x*p_y\,.$$

Let σ be the Haar measure of H.

THEOREM 4.1. *The Haar measure $\dot{m}$ on K/H can be chosen so that*

$$\int_{K/H}\int_H f(x\xi)d\sigma(\xi)d\dot{m}(\dot{x}) = \int_K f\,dm \text{ for all } f\in L^1(K)\,.$$

Proof. Define T_H on $C_{00}(K)$ by $T_H f(\dot{x}) = \int_H f(x\xi)d\sigma(\xi)$. Then proof for the group case ([16], III, 3.2, 4.7) or ([12], 28.54) can be modified to have the required result.

THEOREM 4.2. *Let T_H be as in proof of Theorem* 4.1.

(i) *Every function $\dot{f}$ in $L^1(K/H)$ has the form $T_H f$ for some $f \in L^1(K)$ with $\|\dot{f}\|_1 = \|f\|_1$.*

(ii) *$T_H: L^1(K) \to L^1(K/H)$ is an algebra homomorphism with norm* 1.

Proof. (i) for $g \in C_{00}(K/H)$ select $f \in C_{00}(K)$ as in ([16], III, 4.2), (ii) involves simple computations.

THEOREM 4.3. *For $\varphi \in H^{\perp}$, $\varphi'(xH) = \varphi(x)$ defines an element of $(K/H)\hat{}$, $\varphi \to \varphi'$ is a one-to-one map of $H^{\perp}$ onto $(K/H)\hat{}$. Hence $(K/H)\hat{}$ is also a hypergroup under pointwise operations.*

Proof. ([12], 5.24(b)) can be modified using ([13], 10.3B, 3.2B) again to give an analogue of Lemma 2.3 in this case also. Hence the techniques of ([19], 4.4) can be used to have the required result.

LEMMA 4.4. *The Fourier transform of $T_H f \in L^1(K/H)$ is the restriction of the Fourier transform $\hat{f}$ of f to $H^{\perp}$.*

THEOREM 4.5. *Let $\Delta \subset H^{\perp}$. If Δ is spectral (respectively, Calderon, strong Ditkin, sequentially strong Ditkin, ultra-strong Ditkin) for $L^1(K)$ then Δ is so for $L^1(K/H)$.*

Proof. The proof is similar to that of the corresponding parts of Theorem 3.11. We just note that analogues of other parts of Theorem 3.11 are not available since we do not have an ideal I in $L^1(K)$ (corresponding to the ideal of invariant functions in $L^1(K)$) such that T_H is an isometric algebraic homomorphism on I onto $L^1(K/H)$.

5. **Spectral synthesis in products of hypergroups.** In [15] Reiter gave an example of a function f in $L^1(\boldsymbol{R}^n)(n \geqq 3)$ and a ψ in $L^\infty(\boldsymbol{R}^n)$ such that $\hat{f} = 0$ exactly on the unit sphere S^{n-1} in $\boldsymbol{R}^n$ and $\langle f_y, \psi\rangle \neq 0$ for some $y \in \boldsymbol{R}^n$ but $\langle (f*f)_z, \psi\rangle = 0$ for all $z \in \boldsymbol{R}^n$ which in turn gives that f and $f*f$ generate different closed ideals in $L^1(\boldsymbol{R}^n)$ but have the same zero set S^{n-1}. Malliavim proved that every nondiscrete locally compact abelian group $\hat{G}$ contains a non-spectral set for $A(\hat{G})$. It is well known that Malliavin's theorem is not true for hypergroups because every closed subset in $\hat{K}$ is even strong Ditkin for the hypergroup $K = \boldsymbol{Z}_+$ related to p-adic

numbers ([11], [9] and [7]). However it was shown in [8] that not every closed subset of Z_+^* is ultra-strong Ditkin thus showing that we have a weaker form of Malliavin's theorem for Z_+. In this section we first show that Malliavin's theorem fails even for $Z_+ \times K_2$, K_2 being a compact abelian group with countable dual and on the other hand we develop methods to find nonspectral and non ultra-strong Ditkin sets for hypergroups of the type $K = \boldsymbol{R}^+ \times K_2$ or $K = Z_+ \times K_2$ respectively, where K_2 is a locally compact hypergroup such that its dual $\hat{K}_2$ is a σ-compact hypergroup. We begin with a few lemmas. As in ([13], 10.5) $K_1 \times K_2$ can be made into a hypergroup in the following way, if (s, t) and (x, y) are in $K_1 \times K_2$ then

$$p_{(s,t)} * p_{(x,y)} = (p_s * p_x) \times (p_t * p_y), \ (s, t)^{\check{}} = (\check{s}, \check{t})$$

where K_1, K_2 are locally compact hypergroups.

LEMMA 5.1. *Let K_1, K_2 be locally compact commutative hypergroups. For every $(\chi_1, \chi_2) \in \hat{K}_1 \times \hat{K}_2$, let $[\chi_1, \chi_2]$ denote the function $(x_1, x_2) \to \chi_1(x_1)\chi_2(x_2)$ defined on $K_1 \times K_2$. Then the mapping $\theta: (\chi_1, \chi_2) \to [\chi_1, \chi_2]$ is a one-to-one map of $\hat{K}_1 \times \hat{K}_2$ onto $(K_1 \times K_2)^\wedge$. If $\hat{K}_1$, $\hat{K}_2$ are hypergroups then so is $(K_1 \times K_2)^\wedge$.*

LEMMA 5.2. *Let K_1, K_2 be commutative hypergroups such that $\hat{K}_1$, $\hat{K}_2$ are also hypergroups, $f_1 \in L^1(K_1)$ and $f_2 \in L^1(K_2)$. If $f(x, y) = f_1(x)f_2(y)$ for $(x, y) \in K_1 \times K_2$ then*

(i) $\hat{f}((\gamma_1, \gamma_2)) = \hat{f}_1(\gamma_1)\hat{f}_2(\gamma_2)$ *for all* $(\gamma_1, \gamma_2) \in \hat{K}_1 \times \hat{K}_2$.

(ii) $f * f(x, y) = f_1 * f_1(x) f_2 * f_2(y)$.

LEMMA 5.3. *Let K_1, K_2 be as in Lemma 5.2, $f \in L^1(K_1 \times K_2)$ and $\gamma_2 \in \hat{K}_2$. There exists $f_{\gamma_2} \in L^1(K_1)$ such that $\hat{f}(\gamma_1, \gamma_2) = \hat{f}_{\gamma_2}(\gamma_1)$ for each $\gamma_1 \in \hat{K}_1$.*

Proof. Repeated applications of Fubini's theorem give that f_{γ_2} can be defined as $f_{\gamma_2}(x_1) = \int_{K_2} f(x_1, x_2)\gamma_2(x_2)dm(x_2)$, m being the Haar measure of K_2.

The following results can be deduced from Theorem 3.11 and 4.5 above and ([9], 3.7 and 3.8).

THEOREM 5.4. *Let $K = K_1 \times K_2$, K_1 and K_2 be as in Lemma 5.2.*

(i) *If K_2 is compact or a locally compact abelian group then if $E \subset \hat{K}_1$ is not a spectral (respectively, Calderon sequentially* *strong Ditkin, ultra-strong Ditkin) set for $L^1(K_1)$ then $E\times\{1_2\}$ is not a spectral (respectively, Calderon, sequentially strong Ditkin, ultra-strong Ditkin) set for $L^1(K)$ where 1_2 is the identity of $\hat{K}_2$.*

(ii) *If K_2 is compact, $(\gamma_1, \gamma_2) \in Z(\hat{K}_1) \times Z(\hat{K}_2) = Z(\hat{K}_1 \times \hat{K}_2)$ then $E \subset \hat{K}_1$ is Calderon (respectively, strong Ditkin, ultra-strong Ditkin) for $L^1(K_1)$ if and only if $\gamma_1 E \times \{\gamma_2\}$ is Calderon (respectively strong Ditkin, ultra-strong Ditkin) for $L^1(K)$.*

(iii) *If K_2 is a compact abelian group with $\hat{K}_2$ countable, $\hat{K}_1$ is discrete except at a countable subset S_1 and every point of S_1 is Calderon (in particular when $S_1 \subset Z(\hat{K}_1)$) then every closed subset of $\hat{K}_1 \times \hat{K}_2$ is Calderon for $L^1(K)$.*

REMARK 5.5. Theorem 5.4 (iii) gives us that Malliavin's theorem fails for more hypergroups such as $Z_+ \times T^n$, T being the circle group and n any positive integer.

The basic lemmas of this section, ([9], 2.5) and the techniques of ([22], p. 419) can be further used to modify the proof of ([14], 3.8) and hence have the following generalizations to hypergroups for $p = 1$* see also ([3], 2.5.5 (f)) ([1], p. 307) and ([2], p. 240).

THEOREM 5.6. *Let K_1, K_2 be as in Lemma 5.2 with further assumption that $\hat{K}_2$ is σ-compact. If $E_1 \subset \hat{K}_1$ is a nonspectral set, then $E_1 \times \hat{K}_2$ is a nonspectral set for $L^1(K_1 \times K_2)$.*

THEOREM 5.7. *Let $K = \boldsymbol{R}^n \times K_2 (n \geqq 3)$ where K_2 is such that $\hat{K}_2$ is a σ-compact hypergroup and let E be a nonempty closed open subset of $\hat{K}_2$. Then there is an $f \in L^1(K)$ such that the closed ideal generated by $f*f$ does not contain f and $Z(\hat{f}) = S^{n-1} \times E$.*

Proof. Let f_1 in $L^1(\boldsymbol{R}^n)$ and ψ_1 in $L^\infty(\boldsymbol{R}^n)$ be the same as f, ψ in ([15], § 2, Theorem 1) and g_1 be a rotationally invariant function in $L^1(\boldsymbol{R}^n)$ such that $\hat{g}_1 > 0$. Using ([9], 2.5) and techniques of ([22], p. 419) there are $f_2, g_2 \in L^1(K_2)$ such that $Z(\hat{g}_2) = E$, $Z(\hat{f}_2) = \hat{K}_2 \sim E$ and $\hat{g}_2 \geqq 0$, $\hat{f}_2 \geqq 0$. Since $E \neq \Phi$ and the zero set of I_{g_2}, the closed ideal in $L^1(K_2)$ generated by g_2 is E, we have that $f_2 \notin I_{g_2}$. So there exists $\psi_2 \in L^\infty(K_2)$ such that $\langle f_2, \psi_2\rangle \neq 0$ but $\langle h, \psi_2\rangle = 0$ for all $h \in I_{g_2}$. Define f on $\boldsymbol{R}^n \times K_2$ by

$$f(x_1, x_2) = f_1(x_1)f_2(x_2) + g_1(x_1)g_2(x_2)\ .$$

Let ψ be defined on $\boldsymbol{R}^n \times K_2$ by $\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$. Then $\psi \in L^\infty(\boldsymbol{R}^n \times K_2)$. Clearly $S^{n-1} \times E$ is the zero set of $\hat{f}$. Consider $(y_1,$

* For p-spectral synthesis in hypergroups see [23] and [24].

$y_2) \in \boldsymbol{R}^n \times K_2$ where y_2 is the identity of K_2 and y_1 is an element of $\boldsymbol{R}^n$ with the first coordinate between 0 and 1/2 and with other coordinates zero. Then $f_y = (f_1)_{y_1} f_2 + (g_1)_{y_1} g_2$. Since $\langle f_2, \psi_2 \rangle \neq 0$ and $\langle g_2, \psi_2 \rangle = 0$, Reiter's computations in ([15], § 2, Theorem 1) can be modified to give that $\langle f_y, \psi \rangle \neq 0$. But for any $z = (z_1, z_2) \in \boldsymbol{R}^n \times K_2$ using Lemma 5.2 and ([9], 2.2, 2.4),

$$\begin{aligned}\langle (f*f)_z, \psi \rangle &= \langle (f_1*f_1)_{z_1}, \psi_1 \rangle \langle (f_2*f_2)_{z_2}, \psi_2 \rangle \\ &+ 2\langle (f_1*g_1)_{z_1}, \psi_1 \rangle \langle (f_2*g_2)_{z_2}, \psi_2 \rangle \\ &+ \langle (g_1*g_1)_{z_1}, \psi_1 \rangle \langle (g_2*g_2)_{z_2}, \psi_2 \rangle = 0 ,\end{aligned}$$

because $\langle h, \psi_2 \rangle = 0$ for all h in I_{g_2}, I_{g_2} contains $(f_2*g_2)_{z_2} = (f_2)_{z_2}*g_2$ and $(g_2*g_2)_{z_2} = (g_2)_{z_2}*g_2$ using ([9], 2.4, 2.11) and $\langle (f_1*f_1)_{z_1}, \psi_1 \rangle$ is zero. Hence f and $f*f$ generate different closed ideals in $L^1(\boldsymbol{R}^n \times K_2)$.

The proof of ([15], § 2, Theorem 2) can be used in the same way to obtain the following generalization of the above theorem.

THEOREM 5.8. *If T is an arbitrary closed set in $\boldsymbol{R}^n$ such that $S^{n-1} \sim T \neq \Phi$ and E is as in Theorem 5.7, then for $n \geqq 3$ there exists f in $L^1(\boldsymbol{R}^n \times K_2)$ whose zero set is $(S^{n-1} \cup T) \times E$ and f, $f*f$ generate different closed ideals in $L^1(\boldsymbol{R}^n \times K_2)$.*

THEOREM 5.9. *Let K_2 be as in Theorem 5.7. For $n \geqq 3$ $(S^{n-1} \cup T) \times E$ is a nonspectral set in $\boldsymbol{R}^n \times \hat{K}_2$ for $L^1(\boldsymbol{R}^n \times K_2)$ where E is a nonempty closed open set in $\hat{K}_2$ and T is as in Theorem 5.8. In particular $S^{n-1} \times E$ is nonspectral.*

Proof. Follows from Theorem 5.8.

Now we consider $\boldsymbol{R}^+ = [0, \infty)$ viewed as the hypergroup $\boldsymbol{R}^+ = G_B$ of B-orbits in $G = \boldsymbol{R}^n (n \geqq 3)$ where B is the compact group of rotations in G. See the discussion in ([18], § 3) and ([9], 4.3).

THEOREM 5.10. *$\{x_0\} \times E$ is a nonspectral set in $\boldsymbol{R}^+ \times \hat{K}_2$ where $x_0 \neq 0$ is any point of $\boldsymbol{R}^+$ and K_2 and E are as in Theorem 5.7.*

Proof. It follows immediately from Theorem 5.7 since f_1 and g_1 are rotationally invariant.

REMARK 5.11. In the above theorems E can be a subset of $\chi_b(K_2)$ if we assume the regularity of $L^1(K_2)$ on $\chi_b(K_2)$, the hypergroup $\boldsymbol{R}^+$ can be replaced by a hypergroup K_1 whose dual $\hat{K}_1$ is a σ-compact hypergroup and the set $\{x_0\}$ by a set E_1 in $\hat{K}_1$ if there exists a function f_1 in $L^1(K_1)$ such that f_1 and f_1*f_1 generate different closed ideals in $L^1(K_1)$ and the zero set of $\hat{f}_1$ is exactly E_1. Now we discuss another method to obtain nonspectral sets for certain hypergroups whose obvious generalizations can be easily formulated in view of discussion in ([9], 4.3).

THEOREM 5.12. *Let $K = K_1 \times K_2$ where K_2 is as in Theorem 5.7 and K_1 is the dual $\hat{F}$ of the hypergroup F of conjugacy classes of the compact group $SU(2)$ ([13], 15.4) and ([9], 4.7). Then $\{\chi_\theta\} \times E$ is a nonspectral set in $\hat{K}_1 \times \hat{K}_2$ where $\theta \in (0, \pi)$ and E is as in Theorem 5.7.*

Proof. Let $f_2, g_2 \in L^1(K_2)$ be as in Theorem 5.7 and $g_1 \in L^1(K_1)$ be such that $\hat{g}_1 > 0$. Let $m(\{n\}) = \alpha_n$ for $n \in K_1$ so that $\alpha_n > 0$ for all $n \in K_1$. For $\theta \in (0, \pi)$ let g_θ be as in ([9], 4.7). Then $g_\theta \in L^\infty(K_1)$. For a function h on K vanishing outside $\{0, 1, 2\}$,

$$\hat{h}(\chi_\theta) = h(0)\alpha_0 + h(1)\alpha_1 \cos\theta + h(2)\alpha_2\left(1 - \frac{4}{3}\sin^2\theta\right)$$

and $\langle h, g_\theta\rangle = -h(1)\alpha_1 \sin\theta - (8/3)h(2)\alpha_2 \sin\theta\cos\theta$. Taking $f_1(0)\alpha_0 = \sin 2\theta$, $f_1(1)\alpha_1 = -2\sin\theta$, $f_1(2)\alpha_2 = 0$ we have $\hat{f}_1(\chi_\theta) = 0$ but $\langle f_1, g_\theta\rangle \neq 0$ and $\hat{f}_1(\chi_{\theta'}) = 2\sin\theta(\cos\theta - \cos\theta') \neq 0$ for $\theta \neq \theta'$. Also taking

$$\varphi_1(0)\alpha_0 = \frac{1}{3} + \frac{4}{3}\cos^2\theta,\ \varphi_1(1)\alpha_1 = -\frac{8}{3}\cos\theta \quad \text{and}$$

$$\varphi_1(2)\alpha_2 = 1, \text{ we have } \hat{\varphi}_1(\chi_\theta) = 0 = \langle\varphi_1, g_\theta\rangle$$

but $\hat{\varphi}_1(\chi_{\theta'}) = 4/3(\cos\theta - \cos\theta')^2 \neq 0$ for $\theta \neq \theta'$. Let

$$f(x, y) = f_1(x)f_2(y) + g_1(x)g_2(y)$$

and

$$\varphi(x, y) = \varphi_1(x)f_2(y) + g_1(x)g_2(y)\ .$$

Clearly the zero set of $\hat{f}$ as well as that of $\hat{\varphi}$ is $\{\chi_\theta\} \times E$. Define $\chi_\theta\gamma_2(x, y) = \chi_\theta(x)\gamma_2(y)$ and $g_\theta\gamma_2(x, y) = g_\theta(x)\gamma_2(y)$ and let $E_\theta = \{\chi_\theta\gamma_2 \colon \gamma_2 \in E\}$ and $F_\theta = \{g_\theta\gamma_2 \colon \gamma_2 \in E\}$. Then in view of discussion in ([9], 4.3 and 4.7) $I = \{h \colon \langle h, \psi\rangle = 0,\ \psi \in E_\theta\}$ and $J = \{h \colon \langle h, \psi\rangle = 0,\ \psi \in E_\theta \cup F_\theta\}$ are distinct closed ideals with the same zero set E_θ since $\varphi \in J$ and $f \in I \sim J$. Hence $\{\chi_\theta\} \times E$ is a nonspectral set for $L^1(K_1 \times K_2)$. In view of ([9], 4.3 and 4.7) the same conclusion also follows when we take $\varphi(x, y) = (f_1 * f_1)(x)f_2(y) + g_1(x)g_2(y)$.

REMARK 5.13. Let $K_1 = [0, \infty)$ be the hypergroup named as Naimark's example ([13], 9.5)([9], 4.8) and K_2, E be as in Theorem 5.7. Fix any $b > 0$ and let $a = b^2$ then $\{\chi_a\} \times E$ is a nonspectral set for $L^1(K_1 \times K_2)$. Let f_2 and g_2 be as in Theorem 5.7 and for any fixed $c > 1$ take $g_1(x) = f_c(x) = e^{-cx}/\sinh x$ ([13], 9.5), we have $\hat{g}_1 > 0$. Select f_1 and φ_1 in $L^1(K_1)$ as follows

$$\begin{aligned} f_1(x) &= \frac{1}{\sinh x} \quad \text{for } 0 < x \leqq \frac{2\pi}{b} \\ &= 0 \qquad\quad \text{otherwise} , \\ \varphi_1(x) &= \frac{1 - bx}{\sinh x} \quad \text{for } 0 < x \leqq \frac{\pi}{2b} \\ &= 0 \qquad\quad \text{otherwise} \end{aligned}$$

and define

$$\begin{aligned} f(x, y) &= f_1(x) f_2(y) + g_1(x) g_2(y) \\ \varphi(x, y) &= \varphi_1(x) f_2(y) + g_1(x) g_2(y) . \end{aligned}$$

The zero set of $\hat{f}$ and $\hat{\varphi}$ is $\{\chi_a\} \times E$. Let

$$\psi_1(x) = \frac{x \cos bx}{b \sinh x} \quad \text{for all } x \in K .$$

For $\gamma_2 \in E$ let $\psi(x, y) = \psi_1(x)\gamma_2(y)$. Then $\psi \in L^\infty(K_1 \times K_2)$. Also $\langle f_1, \psi_1 \rangle = 0$ and $\langle \varphi_1, \psi_1 \rangle \neq 0$ and therefore $\langle f, \psi \rangle = 0$ and $\langle \varphi, \psi \rangle \neq 0$. As argued in Theorem 5.12 in view of discussion in ([9], 4.8) we have that f and φ generate different closed ideals.

REMARK 5.14. Let K_1 be $\boldsymbol{R}^+$, $\hat{F}$ or Naimark's example and K_2 be the hypergroup Z_+([11], [8]). Then $\hat{K}_2$ is a σ-compact hypergroup. If E_2 is a closed set in $\hat{K}_2$, which is finite and does not contain the identity or is the complement of a finite set (and therefore contains the identity) then E_2 is open. Even though by ([8], Example 10) it is ultra-strong Ditkin for $L^1(K_2)$, the above results can be applied to have that $\{\gamma\} \times E_2$ is nonspectral for $L^1(K_1 \times K_2)$ where $\gamma \in \hat{K}_1 \sim Z(\hat{K}_1)$. Our next result further shows that if E_2 is an infinite closed set in Z_+^* which is not open (and hence has an infinite complement and therefore by ([8], Example 10) it is not ultra-strong Ditkin for $L^1(K_2)$), then $\{\gamma\} \times E_2$ is not ultra-strong Ditkin for $L^1(K_1 \times K_2)$.

THEOREM 5.15. *Let $K = Z_+ \times K_2$, where K_2 is as in Lemma* 5.2 *and Z_+ is the hypergroup considered in* ([11], [8]). *Then $E_1 \times E_2$ is a non-ultra-strong Ditkin set for $L^1(K)$ where E_1 is* (*as in* [8]) *an infinite closed subset of Z_+^* such that its complement is also infinite and E_2 is a nonempty closed subset of $\hat{K}_2$.*

Proof. Let $\varphi = \hat{g}_1$ in $A(Z_+^*)$ be as in ([8], Example 10(ii)(c)).

Also there exists $g_2 \in L^1(K_2)$ such that $\hat{g}_2 \neq 0$ on E_2. Define f on $Z_+ \times K_2$ by $f(x, y) = g_1(x)g_2(y)$. Clearly f belongs to $k(E_1 \times E_2)$. Suppose $E_1 \times E_2$ is an ultra-strong Ditkin set then as in [8] the Banach algebra $k(E_1 \times E_2)$ has a bounded approximate identity and therefore it has factorization by Cohen's factorization theorem. So there exist $g, h \in k(E_1 \times E_2)$ such that $f = g*h$. Fix $\gamma_2 \in E_2$ such that $\hat{g}_2(\gamma_2) \neq 0$ and without loss of generality we may take $\hat{g}_2(\gamma_2)=1$. So for all $\gamma_1 \in Z_+^*$ using, Lemma 5.3,

$$\begin{aligned}\hat{g}_1(\gamma_1) &= \hat{f}(\gamma_1, \gamma_2) = \hat{g}(\gamma_1, \gamma_2)\hat{h}(\gamma_1, \gamma_2)\\ &= \hat{g}_{\gamma_2}(\gamma_1)\hat{h}_{\gamma_2}(\gamma_1)\end{aligned}$$

that is $g_1 = g_{\gamma_2}*h_{\gamma_2}$. Since $g, h \in k(E_1 \times E_2)$, g_{γ_2} and h_{γ_2} belongs to $k(E_1)$ which is a contradiction to the proof of ([8], Example 10(ii)(c)). Hence $E_1 \times E_2$ is a non ultra-strong Ditkin set for $L^1(Z_+ \times K_2)$.

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