Pacific Journal of Mathematics

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Vol. 94, No. 1 May 1981

PACIFIC JOURNAL OF MATHEMATICS
Vol. 94, No. 1, 1981

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Let M be a simply-connected complete d-dimensional Riemannian manifold of nonpositive sectional curvature K. If $K \leqq -k^2 < 0$, then the infimum of the L^2 spectrum of the negative Laplacian is greater than or equal to $(d-1)^2k^2/4$ with equality in case $K \to -k^2$ sufficiently fast at infinity. This general result is obtained by analyzing a system of ordinary differential equations. If either $d=2$ or the manifold possesses appropriate symmetry, the result is obtained under weaker conditions by analyzing a Riccati equation. Finally the case $k=0$ is treated separately.

1. Description of results. The infimum of the L^2 spectrum is defined by

$$\lambda_1 = \inf_{\phi \neq 0} \frac{\int_M |d\phi|^2}{\int_M \phi^2} \tag{1.0}$$

when the infinum is taken over H_0^1, the closure of $C_0^\infty(M)$ in the norm $\int_M (\phi^2 + |d\phi|^2)$. Let $K_x(P)$ be the sectional curvature of the two-plane $P \subseteq M_x$, the tangent space at x. Let $\gamma(t) = \gamma(t; 0, \xi)$ be the unit-speed geodesic emanating from $0 \in M$ and having initial velocity $\xi \in M_0$. Let

$$\varepsilon(t) = \sup_{|\xi|=1} \sup_{P \subseteqq M_{\gamma(t)}} |K_{\gamma(t)}(P) + k^2|$$

where k is a positive constant. Our main result is the following upper bound.

THEOREM. *Suppose that*

$$\int_1^\infty \varepsilon(t)dt < \infty . \tag{1.1}$$

Then

$$0 < \lambda_1 \leqq (d-1)^2k^2/4 .$$

This immediately implies

COROLLARY 1. *Suppose that outside of some compact set M has constant sectional curvature $K = -k^2 < 0$. Then* $0 < \lambda_1 \leqq (d-1)^2k^2/4$.

Finally, we have the result stated in the first paragraph.

COROLLARY 2. *Suppose that* (1.1) *holds and that* $K \leqq -k^2 < 0$ *everywhere on* M. *Then* $\lambda_1 = (d-1)^2k^2/4$.

2. **Proofs.** We will study Jacobi fields $J(t)$ along a geodesic $\{\gamma(t),\ t \geqq 0\}$ where $J(0) = 0$, $J(t) \not\equiv 0$, $(J(t), \gamma') = 0$. For this purpose, let $\{E_i(t),\ 2 \leqq i \leqq d\}$ be a parallel field of orthonormal vectors along γ with $(E_i, \gamma') = 0$. Write

$$J(t) = \sum_{i=2}^{d} f_i(t)E_i(t)\ . \tag{2.0}$$

From the Jacobi equation we have the following system of equations [2]

$$f_i''(t) + \sum_{j=2}^{d} (R(E_i, \gamma')\gamma', E_j)f_j(t) = 0 \quad (2 \leqq i \leqq d)\ . \tag{2.1}$$

By the representation of R in terms of sectional curvature, we have

$$(R(E_i, \gamma')\gamma', E_j) = -k^2\delta_{ij} + \varepsilon_{ij}$$

where $|\varepsilon_{ij}| \leqq \varepsilon(t)$.

We use the following result from ordinary differential equations.

PROPOSITION. *Consider the system*

$$f_i''(t) - k^2f_i(t) = \sum_{j=2}^{d} \varepsilon_{ij}(t)f_j(t) \quad (2 \leqq i \leqq d) \tag{2.2}$$

where $\int_1^\infty |\varepsilon_{ij}(t)|\,dt < \infty$. *Then* (2.2) *has solutions* $f_i^{(1)}, f_i^{(2)}$ *with*

$$\begin{aligned} f_i^{(1)} &\sim e^{kt}, & f_i^{(1)\prime} &\sim ke^{kt} & (t &\longrightarrow \infty) \\ f_i^{(2)} &\sim e^{-kt}, & f_i^{(2)\prime} &\sim -ke^{-kt} & (t &\longrightarrow \infty)\ . \end{aligned}$$

For the proof see Hartman [5, p. 381] for the case $d = 2$. To apply this to (2.1) we recall that from the Rauch comparison theorem [2] $|J(t)| \to \infty$ when $t \to \infty$. Now let

$$f_i(t) = \sum_{j=2}^{d} [c_{ij}f_j^{(1)}(t) + d_{ij}f_j^{(2)}(t)]\ . \tag{2.3}$$

We claim that $c_{ij} \neq 0$ for at least one value of (i, j). Indeed, if $c_{ij} \equiv 0$, then $f_i(t) = \mathcal{O}(e^{-kt})$, $t \to \infty$ which implies that $|J(t)| \to 0$, a contradiction. Now

$$
\begin{aligned}
\frac{(J'(t),\, J(t))}{(J(t),\, J(t))} &= \frac{\sum_{i=2}^{d} f_i(t) f_i'(t)}{\sum_{i=2}^{d} f_i(t)^2} \\
&= \frac{\sum_{i,j} c_{ij}^2 f_i^{(1)} f_i^{(1)\prime}}{\sum_{i,j} c_{ij}^2 f_i^{(1)2}} (1 + o(1)) \quad (t \longrightarrow \infty) \\
&= k(1 + o(1)) \qquad\qquad (t \longrightarrow \infty) .
\end{aligned} \tag{2.4}
$$

Thus we have proved the following proposition.

LEMMA 1. *Let $J(t)$ be a Jacobi field along γ with $J(0) = 0$, $(J(t), \gamma') = 0$, $J(t) \not\equiv 0$. If (1.1) is satisfied, then*

$$
\frac{(J'(t),\, J(t))}{(J(t),\, J(t))} \longrightarrow k\,, \qquad t \longrightarrow \infty\,. \tag{2.5}
$$

LEMMA 2. *Let r be the geodesic distance from $0 \in M$. Then (1.1) implies that*

$$
\Delta r(\gamma(t)) \longrightarrow (d-1)k \qquad (t \longrightarrow \infty) \tag{2.6}
$$

where the convergence is uniform over S^{d-1}.

Proof. Let $\gamma(t; 0, \xi)$ be the geodesic emanating from $0 \in M$ with initial velocity ξ. Let $\{J_i(t),\, 2 \leqq i \leqq d\}$ be Jacobi fields along γ with $J_i(0) = 0$, $J_i'(0) = E_i$ where $(\gamma'(0), E_2, \cdots, E_d)$ is an orthonormal basis of M_0. Then from the second variation of arclength [1], we have

$$
\Delta r(\gamma(t)) = \sum_{k=2}^{d} \frac{(J_k'(t),\, J_k(t))}{(J_k(t),\, J_k(t))}\,. \tag{2.7}
$$

Using Lemma 1 the result follows.

LEMMA 3. *Let $m = (d-1)k$, $0 < R_0 < R_1 < \infty$,*

$$
\phi(r) = \begin{cases} e^{-mr/2} \sin \pi \dfrac{(r - R_0)}{R_1 - R_0} & R_0 \leqq r \leqq R_1 \\ 0 & \textit{otherwise.} \end{cases} \tag{2.8}
$$

Then

$$
\Delta\phi + \left|\frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2}\right| \phi = (\Delta r - m)\phi'(r)\,. \tag{2.9}
$$

Proof. Calculus and the formula $\Delta\phi = \phi'' + (\Delta r)\phi'$ [1, p. 134]. Now let B be the annular domain $R_0 \leqq r \leqq R_1$.

LEMMA 4.

$$-\int_B (\phi')^2 + \left|\frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2}\right| \int_B \phi^2 = \int_B (\Delta r - m)\phi\phi' . \tag{2.10}$$

Proof. Multiply equation (2.9) by ϕ, integrate by parts and use the boundary condition $\phi = 0$.

Proof of the theorem. Let $X = \|\phi'\|^2_{L^2(B)}$, $I = \|\phi\|^2_{L^2(B)}$, $c = m^2/4 + \pi^2/(R_1 - R_0)^2$. Applying Schwarz's inequality we have

$$\left|\int_B (\Delta r - m)\phi\phi'\right| \leqq \varepsilon_1(R_0) \|\phi\|_{L_2(B)} \|\phi'\|_{L_2(B)}$$

where $\varepsilon_1(R_0) \to 0$ when $R_0 \to \infty$.

Applying this to (2.10), we have

$$|X - cI| \leqq \varepsilon_1(R_0)\sqrt{IX} . \tag{2.11}$$

But this implies that X is smaller than the largest root of the corresponding equation, i.e.,

$$\sqrt{X} \leqq \sqrt{I}\left\{\frac{\varepsilon_1(R_0)}{2} + \sqrt{c + \frac{\varepsilon_1(R_0)^2}{4}}\right\} .$$

A glance at the definition (1.0) shows that $\lambda_1 \leqq X/I$. This holds for all $R_1 > R_0$; letting $R_1 \to \infty$, we have

$$\sqrt{\lambda_1} \leqq \frac{\varepsilon_1(R_0)}{2} + \sqrt{\frac{m^2}{4} + \frac{\varepsilon_1^2(R_0)}{4}} .$$

Finally letting $R_0 \to \infty$, we have the result $\lambda_1 \leqq m^2/4$.

To prove the lower bound, we first note that for some δ

$$\Delta r \geqq \delta < 0 . \tag{2.12}$$

Indeed, outside of some sufficiently large compact set we can use Lemma 2. On the other hand, the proof of the Rauch comparison theorem implies that for any Jacobi field along γ with $J(0) = 0$, $(J(t), \gamma') = 0$, $J(t) \not\equiv 0$, we have $(J'(t), J(t))/(J(t), J(t)) \geqq 1/r$. Hence

$$\Delta r \geqq \frac{d-1}{r} > 0 \qquad (0 < r < \infty)$$

$$\Delta r \longrightarrow (d-1)k \qquad (r \longrightarrow \infty) .$$

Having proved (2.12), we can use the method of McKean. For this purpose let $G(t, \xi) = |J_2(t) \wedge \cdots \wedge J_d(t)|$. From (2.12) we see that $G_t/G \geqq \delta$. Now M is the image of R^d under $\exp_0$. Integrals over

M can be computed over R^d according to the following:

For any $\phi \in H_0^1$, $f \in L^1$

$$\int_M f = \int_{S^{d-1}} d\omega \int_0^\infty f(\exp_0 t\omega) G(t, \omega) dt \tag{2.13}$$

$$\int_M |d\phi|^2 \geqq \int_M |d\phi(\partial/\partial r)|^2 . \tag{2.14}$$

But

$$\begin{aligned}\int_0 \phi^2 G(t, \omega) dt &\leqq \frac{1}{\delta}\int_0^\infty \phi^2 G_t dt \\ &= -\frac{2}{\delta}\int_0^\infty \phi\phi_t G dt \\ &\leqq \frac{2}{\delta}\left(\int_0^\infty \phi^2 G dt\right)^{1/2}\left(\int_0^\infty \phi_t^2 G dt\right)^{1/2} .\end{aligned}$$

Thus

$$\int_0^\infty \phi_t^2 G dt \geqq \frac{\delta^2}{4}\int_0^\infty \phi^2 G dt .$$

Integrating this inequality on S^{d-1} and referring to (2.13)-(2.14), it is clear that we have proved

$$\int_M |d\phi|^2 \geqq \frac{\delta^2}{4}\int_M \phi^2 \quad (\phi \in H_0^1) .$$

Thus $\lambda_1 \geqq \delta^2/4 > 0$, as required.

3. On condition (1.1). In certain cases one may relax the technical condition (1.1). These are the following

$$d = 2 \tag{3.1}$$

$$M \text{ is a model [4]} . \tag{3.2}$$

The latter means that for every orthogonal transformation ϕ in M_0, there exists an isometry Φ: $M \to M$ such that $\Phi(0) = 0$, $\Phi^*(0) = \phi$.

PROPOSITION. *Suppose that the* CH *manifold* M *satisfies either* (3.1) *or* (3.2) *and in addition*

$$\varepsilon(t) \longrightarrow 0 \qquad (t \longrightarrow \infty) . \tag{3.3}$$

Then

$$0 < \lambda_1 \leqq (d-1)^2 k^2/4 .$$

Proof. Following the proof of the theorem, the result will follow once we prove Lemma 1. In case (3.1), the Jacobi equation is a single scalar equation

$$J''(t) + K(t)J(t) = 0 \tag{3.4}$$

where $K(t)$ is the Gaussian curvature. Let $h(t) = J'(t)/J(t)$. Then

$$h'(t) + h(t)^2 = -K(t) . \tag{3.5}$$

Recall the following asymptotic result [7] concerning solutions of (3.5).

$$\liminf_{t\to\infty} \sqrt{-K(t)} \leqq \liminf_{t\to\infty} h(t) \leqq \limsup_{t\to\infty} h(t) \leqq \limsup_{t\to\infty} \sqrt{-K(t)} . \tag{3.5a}$$

Thus (3.3) implies that $h(t) \to k$, which proves Lemma 1 in this case.

To treat the case (3.2), we use the following result of Greene-Wu [4, p. 25]: every proper Jacobi field $J(t)$ along a geodesic γ which is orthogonal to γ' and vanishes at 0 has the form

$$J(t) = f(t)E(t)$$

when $E(t)$ is a parallel vector field along γ and $f(t)$ is a real-valued function. The Jacobi equation then takes the form

$$f''(t) + K(t)f(t) = 0 \tag{3.6}$$

where $K(t)$ is the sectional curvature of the 2-plane spanned by $(\gamma'(t), E(t))$. Observing that (3.6) is of the same form as (3.4), we can copy the above proof for $d = 2$ to conclude Lemma 1 in this case also, thus completing the proof of the proposition.

Finally, using the method of Gage [3], we can obtain results using only Ricci curvature. Indeed, Gage has proved that

$$G_{rr} + \frac{R_{11}}{d-1}G = -\frac{G}{2(d-1)^2}\Sigma(\mu_i - \mu_j)^2 \tag{3.7}$$

where $G = |J_2 \wedge \cdots \wedge J_d|^{1/(d-1)}$, R_{11} is the Ricci curvature in the direction $\gamma(t)$ and $(\mu_2, \cdots, \mu_d)$ are the eigenvalues of the second fundamental form relative to the geodesic sphere. Ignoring the right hand member of (3.7) gives an inequality. Letting $h = G'/G$, we have the Riccati inequality

$$h'(t) + h(t)^2 \leqq -\frac{R_{11}}{d-1} .$$

Let $h_1(t)$ be the solution of the corresponding equation, with the same initial behavior. Then standard comparison methods yield

$$h(t) \leqq h_1(t) .$$

But the asymptotic result (3.5a) now applies to the $h_1(t)$. Combining all of the above, we have the following

PROPOSITION. *Suppose that for the* CH *manifold* M

$$R_{11}(\gamma(t)) \longrightarrow -(d-1)k^2 \qquad (t \longrightarrow \infty)$$

then

$$\lambda_1 \leqq \frac{(d-1)^2k^2}{4} .$$

4. Asymptotic flatness. The previous results are all formulated under the hypothesis $k \neq 0$, which we now remove.

DEFINITION. The CH manifold M is *asymptotically flat* if $k=0$ and either (1.1) holds or (3.3) holds with $d = 2$ or (3.3) holds where M is a model.

PROPOSITION 4.1. *Suppose that the* CH *manifold* M *is asymptotically flat. Then* $\lambda_1 = 0$.

Proof. In this case $\Delta r \to 0$ when $r \to \infty$. Using the trial function $f = \sin \pi(r - R_0)/(R_1 - R_0)$ in the definition of λ_1, the previous proof remains unchanged, with the conclusion $\lambda_1 = 0$.

Conversely, we have the following negative result.

PROPOSITION 4.2. *There exists a* CH *manifold with* $\lambda_1 = 0$ *and curvature function* K *which satisfies* $\liminf_{r\to\infty} K < 0$.

For the proof we will construct a 2-dimensional CH manifold M with metric

$$ds^2 = dr^2 + G(r)^2 d\theta^2$$

where $G'' + KG = 0$, $G(0) = 0$, $G'(0) = 1$. The curvature function $K(r)$ is

$$K(r) = \begin{cases} 0 & r \notin (a_k, a_k + \varepsilon_k) \\ -1 & r \in (a_k, a_k + \varepsilon_k) \end{cases}$$

where a_k, ε_k are to be specified below.

Let $h = G'/G$. Then h satisfies the Riccati equation $h' + h^2 = -K$, with $h(r) = 1/r$ for $0 < r < a_1$. Note the following facts:

(i) On any interval $(a_k, a_k + \varepsilon_k)$, $h' = 1 - h^2 \leqq 1$ and thus $h(a_k + \varepsilon_k) \leqq h(a_k) + \varepsilon_k$.

(ii) On any interval $(a_k + \varepsilon_k, a_{k+1})$, the Riccati equation has the explicit solution $h(r) = (a_k + \varepsilon_k)h(a_k + \varepsilon_k)/r$.

Now let $\varepsilon_k = 1/2^{k+1} (k \geqq 1)$, $a_1 = 4$, $a_{k+1} > 4(a_k + \varepsilon_k)$. Such a choice is clearly possible, we will show that $h(r) \to 0$. First we show inductively that $h(a_k + \varepsilon_k) < 1/2^k$.

On the interval $0 < r < a_1$, $h(r) = 1/r$ and thus $h(a_1) < 1/4$. Using (i) above, we have $h(a_1 + \varepsilon_1) \leqq h(a_1) + \varepsilon_1 < 1/4 + \varepsilon_1 = 1/2$. Now if $h(a_k + \varepsilon_k) < 1/2^k$, then on the interval $a_k + \varepsilon_k < r < a_{k+1}$, $h(r) = h(a_k + \varepsilon_k)(a_k + \varepsilon_k)/r$ and thus $h(a_{k+1}) < (1/4)h(a_k + \varepsilon_k) < 1/2^{k+2}$. Using (i) again, $h(a_{k+1} + \varepsilon_{k+1}) \leqq h(a_{k+1}) + \varepsilon_{k+1} < 1/2^{k+1}$.

Finally, we check that $h(r) \to 0$ as $r \to \infty$. Indeed, on the interval $(a_k + \varepsilon_k, a_{k+1})h$ is decreasing, and thus $h(r) \leqq h(a_k + \varepsilon_k) < 1/2^k$. On the interval $(a_{k+1}, a_{k+1} + \varepsilon_{k+1})$ we have $h' \leqq 1$ and thus $h(r) \leqq h(a_{k+1}) + (r - a_{k+1}) \leqq 1/2^k + 1/2^{k+2}$.

We can now prove that $\lambda_1 = 0$. Indeed, from earlier work [8] we know that $(4\lambda_1)^{1/2} \leqq \lim_{r\to\infty} G_r/G$. Thus $\lambda_1 = 0$, as required.

REMARKS 1. By modifying the above example, it is possible to find a metric for which $\lambda_1 = 0$ and $\liminf_{r\to\infty} K(r) = -\infty$. Indeed, it suffices to replace -1 by a sequence going to $-\infty$ and choose $\varepsilon_k \to 0$ sufficiently fast.

2. It would be interesting to find a necessary condition for $\lambda_1 = 0$, expressed in terms of the curvature function. From our previous paper [8] we know that $\lambda_1 = 0$ implies $\liminf_{r\to\infty} h(r) = 0$. But we do not know what this says about $K(r)$.

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Received January 23, 1980 and in revised form April 25, 1980. Work done while the author was Visiting Professor at the University of Chicago. Also supported by NSF Grant MCS 78-02144.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 94, No. 1 May, 1981