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ON THE SPECTRUM OF CARTAN-HADAMARD MANIFOLDS

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Let M be a simply-connected complete d-dimensional Riemannian manifold of nonpositive sectional curvature K. If $K \leq -k^2 < 0$, then the infimum of the L^2 spectrum of the negative Laplacian is greater than or equal to $(d-1)^2k^2/4$ with equality in case $K \rightarrow -k^2$ sufficiently fast at infinity. This general result is obtained by analyzing a system of ordinary differential equations. If either d=2 or the manifold possesses appropriate symmetry, the result is obtained under weaker conditions by analyzing a Riccati equation. Finally the case k=0 is treated separately.

1. Description of results. The infimum of the L^2 spectrum is defined by

(1.0)
$$\lambda_1 = \inf_{\phi \neq 0} \frac{\int_M |d\phi|^2}{\int_M \phi^2}$$

when the infinum is taken over H_0^1 , the closure of $C_0^{\infty}(M)$ in the norm $\int_M (\phi^2 + |d\phi|^2)$. Let $K_x(P)$ be the sectional curvature of the two-plane $P \subseteq M_x$, the tangent space at x. Let $\gamma(t) = \gamma(t; 0, \xi)$ be the unit-speed geodesic emanating from $0 \in M$ and having initial velocity $\xi \in M_0$. Let

$$arepsilon(t) = \sup_{|arepsilon| = 1} \sup_{P \subseteq M_{arepsilon(t)}} |K_{arepsilon(t)}(P) + k^2|$$

where k is a positive constant. Our main result is the following upper bound.

THEOREM. Suppose that

(1.1)
$$\int_{1}^{\infty} \varepsilon(t) dt < \infty .$$

Then

$$0<\lambda_{1}\leq (d-1)^{2}k^{2}/4$$
 .

This immediately implies

COROLLARY 1. Suppose that outside of some compact set M has constant sectional curvature $K = -k^2 < 0$. Then $0 < \lambda_1 \leq (d-1)^2 k^2/4$.

Finally, we have the result stated in the first paragraph.

COROLLARY 2. Suppose that (1.1) holds and that $K \leq -k^2 < 0$ everywhere on M. Then $\lambda_1 = (d-1)^2 k^2/4$.

2. Proofs. We will study Jacobi fields J(t) along a geodesic $\{\gamma(t), t \ge 0\}$ where J(0) = 0, $J(t) \ne 0$, $(J(t), \gamma') = 0$. For this purpose, let $\{E_i(t), 2 \le i \le d\}$ be a parallel field of orthonormal vectors along γ with $(E_i, \gamma') = 0$. Write

(2.0)
$$J(t) = \sum_{i=2}^{d} f_i(t) E_i(t)$$

From the Jacobi equation we have the following system of equations [2]

$$(2.1) f_i''(t) + \sum_{j=2}^a (R(E_i, \gamma')\gamma', E_j)f_j(t) = 0 (2 \le i \le d) .$$

By the representation of R in terms of sectional curvature, we have

$$(R(E_i,\,\gamma')\gamma',\,E_j)=\,-\,k^2\delta_{ij}+arepsilon_{ij}$$

where $|\varepsilon_{ij}| \leq \varepsilon(t)$.

We use the following result from ordinary differential equations.

PROPOSITION. Consider the system

(2.2)
$$f_i''(t) - k^2 f_i(t) = \sum_{j=2}^d \varepsilon_{ij}(t) f_j(t) \quad (2 \le i \le d)$$

where $\int_{1}^{\infty}|arepsilon_{ij}(t)|\,dt<\infty.$ Then (2.2) has solutions $f_{i}^{(1)},\,f_{i}^{(2)}$ with

$$\begin{array}{ll} f_i^{(1)} \sim e^{kt}, & f_i^{(1)'} \sim k e^{kt} & (t \longrightarrow \infty) \\ f_i^{(2)} \sim e^{-kt}, & f_i^{(2)'} \sim -k e^{-kt} & (t \longrightarrow \infty) \end{array}$$

For the proof see Hartman [5, p. 381] for the case d = 2. To apply this to (2.1) we recall that from the Rauch comparison theorem [2] $|J(t)| \rightarrow \infty$ when $t \rightarrow \infty$. Now let

(2.3)
$$f_i(t) = \sum_{j=2}^d \left[c_{ij} f_j^{(1)}(t) + d_{ij} f_j^{(2)}(t) \right] \, .$$

We claim that $c_{ij} \neq 0$ for at least one value of (i, j). Indeed, if $c_{ij} \equiv 0$, then $f_i(t) = \mathcal{O}(e^{-kt})$, $t \to \infty$ which implies that $|J(t)| \to 0$, a contradiction. Now

$$(2.4) \qquad \frac{(J'(t), J(t))}{(J(t), J(t))} = \frac{\sum_{i=2}^{d} f_i(t) f'_i(t)}{\sum_{i=2}^{d} f_i(t)^2} \\ = \frac{\sum_{i,j} c_{ij}^2 f_i^{(1)} f_i^{(1)'}}{\sum_{i,j} c_{ij}^2 f_i^{(1)2}} (1 + o(1)) \quad (t \longrightarrow \infty) \\ = k(1 + o(1)) \qquad (t \longrightarrow \infty)$$

Thus we have proved the following proposition.

LEMMA 1. Let J(t) be a Jacobi field along γ with J(0) = 0, $(J(t), \gamma') = 0$, $J(t) \neq 0$. If (1.1) is satisfied, then

(2.5)
$$\frac{(J'(t), J(t))}{(J(t), J(t))} \longrightarrow k , \qquad t \longrightarrow \infty .$$

LEMMA 2. Let r be the geodesic distance from $0 \in M$. Then (1.1) implies that

$$(2.6) \qquad \qquad \varDelta r(\gamma(t)) \longrightarrow (d-1)k \qquad (t \longrightarrow \infty)$$

where the convergence is uniform over S^{d-1} .

Proof. Let $\gamma(t; 0, \xi)$ be the geodesic emanating from $0 \in M$ with initial velocity ξ . Let $\{J_i(t), 2 \leq i \leq d\}$ be Jacobi fields along γ with $J_i(0) = 0, J'_i(0) = E_i$ where $(\gamma'(0), E_2, \dots, E_d)$ is an orthonormal basis of M_0 . Then from the second variation of arclength [1], we have

(2.7)
$$\Delta r(\gamma(t)) = \sum_{k=2}^{d} \frac{(J'_{k}(t), J_{k}(t))}{(J_{k}(t), J_{k}(t))}$$

Using Lemma 1 the result follows.

LEMMA 3. Let m = (d - 1)k, $0 < R_0 < R_1 < \infty$, (2.8) $\phi(r) = \begin{cases} e^{-mr/2} \sin \pi \frac{(r - R_0)}{R_1 - R_0} R_0 \leq r \leq R_1 \\ 0 & otherwise. \end{cases}$

Then

(2.9)
$$\Delta \phi + \left| \frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2} \right| \phi = (\Delta r - m) \phi'(r) .$$

Proof. Calculus and the formula $\Delta \phi = \phi'' + (\Delta r)\phi'$ [1, p. 134]. Now let B be the annular domain $R_0 \leq r \leq R_1$. LEMMA 4.

$$(2.10) \qquad -\int_{B} (\phi')^{2} + \left| \frac{m^{2}}{4} + \frac{\pi^{2}}{(R_{1} - R_{0})^{2}} \right| \int_{B} \phi^{2} = \int_{B} (\varDelta r - m) \phi \phi' \, .$$

Proof. Multiply equation (2.9) by ϕ , integrate by parts and use the boundary condition $\phi = 0$.

Proof of the theorem. Let $X = \|\phi'\|_{L^2(B)}^2$, $I = \|\phi\|_{L^2(E)}^2$, $c = m^2/4 + \pi^2/(R_1 - R_0)^2$. Applying Schwarz's inequality we have

$$\left|\int_{\scriptscriptstyle B} (arDelta r \, - \, m) \phi \phi'
ight| \, \leq \, arepsilon_{\scriptscriptstyle 1}(R_{\scriptscriptstyle 0}) \, \| \, \phi \, \|_{_{L_2(B)}} \, \| \, \phi' \, \|_{_{L_2(B)}}$$

where $\varepsilon_{\scriptscriptstyle 1}(R_{\scriptscriptstyle 0}) \to 0$ when $R_{\scriptscriptstyle 0} \to \infty$.

Applying this to (2.10), we have

$$(2.11) |X - cI| \leq \varepsilon_{\scriptscriptstyle 1}(R_{\scriptscriptstyle 0}) \sqrt{IX} \, .$$

But this implies that X is smaller than the largest root of the corresponding equation, i.e.,

$$\sqrt{X} \leq \sqrt{T} \left\{ rac{arepsilon_{ ext{i}}(R_{ ext{o}})}{2} + \sqrt{c + rac{arepsilon_{ ext{i}}(R_{ ext{o}})^2}{4}}
ight\}$$

A glance at the definition (1.0) shows that $\lambda_1 \leq X/I$. This holds for all $R_1 > R_0$; letting $R_1 \rightarrow \infty$, we have

$$\sqrt{\lambda_{\scriptscriptstyle 1}} \leq rac{arepsilon_{\scriptscriptstyle 1}(R_{\scriptscriptstyle 0})}{2} + \sqrt{rac{m^2}{4} + rac{arepsilon_{\scriptscriptstyle 1}^2(R_{\scriptscriptstyle 0})}{4}}$$

Finally letting $R_{\scriptscriptstyle 0}
ightarrow \infty$, we have the result $\lambda_{\scriptscriptstyle 1} \leq m^2/4$.

To prove the lower bound, we first note that for some δ

$$(2.12) \qquad \qquad \varDelta r \geqq \delta < 0 \; .$$

Indeed, outside of some sufficiently large compact set we can use Lemma 2. On the other hand, the proof of the Rauch comparison theorem implies that for any Jacobi field along γ with J(0) = 0, $(J(t), \gamma') = 0, J(t) \neq 0$, we have $(J'(t), J(t))/(J(t), J(t)) \geq 1/r$. Hence

Having proved (2.12), we can use the method of McKean. For this purpose let $G(t, \xi) = |J_2(t) \wedge \cdots \wedge J_d(t)|$. From (2.12) we see that $G_t/G \geq \delta$. Now M is the image of R^d under \exp_0 . Integrals over

M can be computed over R^d according to the following: For any $\phi \in H^1_0$, $f \in L^1$

(2.13)
$$\int_{\mathcal{M}} f = \int_{S^{d-1}} d\omega \int_{0}^{\infty} f(\exp_{0} t\omega) G(t, \omega) dt$$

(2.14)
$$\int_{M} |d\phi|^{2} \geq \int_{M} |d\phi(\partial/\partial r)|^{2} .$$

But

$$egin{aligned} &\int_0 \phi^2 G(t,\,\omega) dt &\leq rac{1}{\delta} \int_0^\infty \phi^2 G_t dt \ &= -rac{2}{\delta} \int_0^\infty \phi \phi_t G dt \ &\leq rac{2}{\delta} \Bigl(\int_0^\infty \phi^2 G dt \Bigr)^{1/2} \left(\int_0^\infty \phi_t^2 G dt
ight)^{1/2}. \end{aligned}$$

Thus

$$\int_{_0}^{^\infty}\!\phi_t^2 Gdt \ge rac{\partial^2}{4}\!\int_{_0}^{^\infty}\!\phi^2 Gdt \;.$$

Integrating this inequality on S^{d-1} and referring to (2.13)-(2.14), it is clear that we have proved

$$\int_{\scriptscriptstyle M} \lvert d\phi
vert^2 \geqq rac{\delta^2}{4} \! \int_{\scriptscriptstyle M} \! \phi^2 \quad (\phi \in H^{\scriptscriptstyle 1}_{\scriptscriptstyle 0}) \; .$$

Thus $\lambda_1 \geq \delta^2/4 > 0$, as required.

3. On condition (1.1). In certain cases one may relax the technical condition (1.1). These are the following

$$(3.1)$$
 $d = 2$

$$(3.2) M ext{ is a model [4]}.$$

The latter means that for every orthogonal transformation ϕ in M_0 , there exists an isometry $\Phi: M \to M$ such that $\Phi(0) = 0$, $\Phi^*(0) = \phi$.

PROPOSITION. Suppose that the CH manifold M satisfies either (3.1) or (3.2) and in addition

$$(3.3) \qquad \qquad \varepsilon(t) \longrightarrow 0 \qquad (t \longrightarrow \infty) \ .$$

Then

$$0<\lambda_{\scriptscriptstyle 1} \leqq (d-1)^{\scriptscriptstyle 2} k^{\scriptscriptstyle 2} / 4$$
 .

Proof. Following the proof of the theorem, the result will follow once we prove Lemma 1. In case (3.1), the Jacobi equation is a single scalar equation

(3.4)
$$J''(t) + K(t)J(t) = 0$$

where K(t) is the Gaussian curvature. Let h(t) = J'(t)/J(t). Then

(3.5)
$$h'(t) + h(t)^2 = -K(t)$$
.

Recall the following asymptotic result [7] concerning solutions of (3.5).

(3.5a)
$$\liminf_{t\to\infty} \sqrt{-K(t)} \leq \liminf_{t\to\infty} h(t) \leq \limsup_{t\to\infty} h(t) \leq \limsup_{t\to\infty} \sqrt{-K(t)} .$$

Thus (3.3) implies that $h(t) \rightarrow k$, which proves Lemma 1 in this case.

To treat the case (3.2), we use the following result of Greene-Wu [4, p. 25]: every proper Jacobi field J(t) along a geodesic γ which is orthogonal to γ' and vanishes at 0 has the form

$$J(t) = f(t)E(t)$$

when E(t) is a parallel vector field along γ and f(t) is a real-valued function. The Jacobi equation then takes the form

(3.6)
$$f''(t) + K(t)f(t) = 0$$

where K(t) is the sectional curvature of the 2-plane spanned by $(\gamma'(t), E(t))$. Observing that (3.6) is of the same form as (3.4), we can copy the above proof for d = 2 to conclude Lemma 1 in this case also, thus completing the proof of the proposition.

Finally, using the method of Gage [3], we can obtain results using only Ricci curvature. Indeed, Gage has proved that

(3.7)
$$G_{rr} + \frac{R_{11}}{d-1}G = -\frac{G}{2(d-1)^2}\Sigma(\mu_i - \mu_j)^2$$

where $G = |J_2 \wedge \cdots \wedge J_d|^{1/(d-1)}$, R_{11} is the Ricci curvature in the direction $\gamma(t)$ and (μ_2, \dots, μ_d) are the eigenvalues of the second fundamental form relative to the geodesic sphere. Ignoring the right hand member of (3.7) gives an inequality. Letting h = G'/G, we have the Riccati inequality

$$h'(t) + h(t)^2 \leq -rac{R_{11}}{d-1} \ .$$

Let $h_1(t)$ be the solution of the corresponding equation, with the same initial behavior. Then standard comparison methods yield

$$h(t) \leq h_1(t)$$
.

But the asymptotic result (3.5a) now applies to the $h_1(t)$. Combining all of the above, we have the following

PROPOSITION. Suppose that for the CH manifold M

$$R_{_{11}}(\gamma(t)) \longrightarrow -(d-1)k^2 \qquad (t \longrightarrow \infty)$$

then

$$\lambda_1 \leq rac{(d-1)^2 k^2}{4}$$

4. Asymptotic flatness. The previous results are all formulated under the hypothesis $k \neq 0$, which we now remove.

DEFINITION. The CH manifold M is asymptotically flat if k=0 and either (1.1) holds or (3.3) holds with d=2 or (3.3) holds where M is a model.

PROPOSITION 4.1. Suppose that the CH manifold M is asymptotically flat. Then $\lambda_1 = 0$.

Proof. In this case $\Delta r \to 0$ when $r \to \infty$. Using the trial function $f = \sin \pi (r - R_0)/(R_1 - R_0)$ in the definition of λ_1 , the previous proof remains unchanged, with the conclusion $\lambda_1 = 0$.

Conversely, we have the following negative result.

PROPOSITION 4.2. There exists a CH manifold with $\lambda_1 = 0$ and curvature function K which satisfies $\liminf_{r\to\infty} K < 0$.

For the proof we will construct a 2-dimensional CH manifold M with metric

$$ds^{\scriptscriptstyle 2}=dr^{\scriptscriptstyle 2}+G(r)^{\scriptscriptstyle 2}d heta^{\scriptscriptstyle 2}$$

where G'' + KG = 0, G(0) = 0, G'(0) = 1. The curvature function K(r) is

$$K(r) = egin{cases} \mathbf{0} & r
otin (a_k, a_k + arepsilon_k) \ -\mathbf{1} & r
otin (a_k, a_k + arepsilon_k) \end{cases}$$

where a_k, ε_k are to be specified below.

Let h = G'/G. Then h satisfies the Riccati equation $h' + h^2 = -K$, with h(r) = 1/r for $0 < r < a_1$. Note the following facts:

(i) On any interval $(a_k, a_k + \varepsilon_k)$, $h' = 1 - h^2 \leq 1$ and thus $h(a_k + \varepsilon_k) \leq h(a_k) + \varepsilon_k$.

(ii) On any interval $(a_k + \varepsilon_k, a_{k+1})$, the Riccati equation has the explicit solution $h(r) = (a_k + \varepsilon_k)h(a_k + \varepsilon_k)/r$.

Now let $\varepsilon_k = 1/2^{k+1} (k \ge 1)$, $a_1 = 4$, $a_{k+1} > 4(a_k + \varepsilon_k)$. Such a choice is clearly possible, we will show that $h(r) \to 0$. First we show inductively that $h(a_k + \varepsilon_k) < 1/2^k$.

On the interval $0 < r < a_1$, h(r) = 1/r and thus $h(a_1) < 1/4$. Using (i) above, we have $h(a_1 + \varepsilon_1) \leq h(a_1) + \varepsilon_1 < 1/4 + \varepsilon_1 = 1/2$. Now if $h(a_k + \varepsilon_k) < 1/2^k$, then on the interval $a_k + \varepsilon_k < r < a_{k+1}$, $h(r) = h(a_k + \varepsilon_k)(a_k + \varepsilon_k)/r$ and thus $h(a_{k+1}) < (1/4)h(a_k + \varepsilon_k) < 1/2^{k+2}$. Using (i) again, $h(a_{k+1} + \varepsilon_{k+1}) \leq h(a_{k+1}) + \varepsilon_{k+1} < 1/2^{k+1}$.

Finally, we check that $h(r) \to 0$ as $r \to \infty$. Indeed, on the interval $(a_k + \varepsilon_k, a_{k+1})h$ is decreasing, and thus $h(r) \leq h(a_k + \varepsilon_k) < 1/2^k$. On the interval $(a_{k+1}, a_{k+1} + \varepsilon_{k+1})$ we have $h' \leq 1$ and thus $h(r) \leq h(a_{k+1}) + (r - a_{k+1}) \leq 1/2^k + 1/2^{\kappa+2}$.

We can now prove that $\lambda_1 = 0$. Indeed, from earlier work [8] we know that $(4\lambda_1)^{1/2} \leq \lim_{r \to \infty} G_r/G$. Thus $\lambda_1 = 0$, as required.

REMARKS 1. By modifying the above example, it is possible to find a metric for which $\lambda_1 = 0$ and $\liminf_{r\to\infty} K(r) = -\infty$. Indeed, it suffices to replace -1 by a sequence going to $-\infty$ and choose $\varepsilon_k \to 0$ sufficiently fast.

2. It would be interesting to find a necessary condition for $\lambda_1 = 0$, expressed in terms of the curvature function. From our previous paper [8] we know that $\lambda_1 = 0$ implies $\liminf_{r\to\infty} h(r) = 0$. But we do not know what this says about K(r).

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