LOCALLY SMOOTH TORUS GROUP ACTIONS ON INTEGRAL COHOMOLOGY COMPLEX PROJECTIVE SPACES

Kai Wang
Let $X$ be a $HCP^n$ which admits a nontrivial smooth $S^1$ action. Petrie defined and studied a set functions $\delta_i(m)$ which are important in the study of local representations. In this paper, we extended Petrie's result to locally smooth torus group actions on integral cohomology complex projective spaces.

Introduction. Let $X$ be a closed smooth manifold homotopically equivalent to $CP^n$ which admits a nontrivial smooth $S^1$ action. An interesting problem is to study the structure of the representations of $S^1$ on the normal fibers to the various components of the fixed point set. Let the fixed point set $X_{S^1}$ consist of $k$ connected components $X_i$, $i = 1, 2, \ldots, k$. Let $\eta$ be the equivariant Hopf bundle [3]. Choose $x_i \in X_i$ and define $k$ integers $a_i$ by $\eta|_{x_i}(t) = t^{a_i}$. Then Petrie [3] proved the following.

**THEOREM 0.1.** The $k$ integers $a_i$ are distinct.

Suppose further that $X_i = x_i$, isolated points. Let $TX|_{x_i}(t) = \sum_{j=1}^n t^{a_{ij}}$. For each integer $m$ and each $i = 1, \ldots, k$, set

$$n_i(m) = \text{number of } j \neq i \text{ such that } m \text{ divides } a_j - a_i;$$
$$d_i(m) = \text{number of } j = 1, \ldots, n \text{ such that } m \text{ divides } x_{i,j},$$
$$\delta_i(m) = n_i(m) - d_i(m).$$

Then Petrie [3] proved the following.

**THEOREM 0.2.** $\delta_i(m) \geq 0$ and $\delta_i(p^r) = 0$ if $p$ is a prime.

Although so far the actions are smooth, it is not difficult to see that the numbers $a_i$ can be defined in the same way for an $S^1$ action on an integral cohomology complex projective space and the numbers $x_{i,j}$ are defined if the action is locally smooth [1]. It turns out that these results are also true for locally smooth $S^1$ actions on integral cohomology complex projective spaces. The main purpose of this paper is to extend these results to the category of locally smooth torus actions on integral cohomology complex projective spaces such that the fixed point sets do not necessarily consist of isolated points.
Our approach is different from [3] and is more elementary in the sense that we do not need equivariant K-theory and the Atiyah Singer Index Theorem. This paper is organized as follows. In §1, we study the torus actions on principal $S^1$-bunles in a very general setting and prove an extension of Theorem 0.1. In §2, we study the upper bound of the dimensions of certain invariant subspace. §3, we study the functions $\delta(H)$ and prove an extension of Theorem 0.2.

1. Torus actions on principal $S^1$ bundles. Let $X$ be a paracompact space which supports a left $T^*$ action. Let $\pi: P \to X$ be a principal $S^1$ bundle. The following result is due to Stewart and Su [3, p. 126].

**Theorem 1.1.** If $H^1(X; \mathbb{Z}) = 0$, then a $T^*$ action on $X$ lifts to a $T^*$ action on $P$ which commutes with the principal $S^1$ action on $P$. If $(t, p) \to t \cdot p$ and $(t, p) \to t \circ p$ denote two liftings of $T^*$ to $P$, then there is a homomorphism $\theta: T^* \to S^1$ such that

$$t \circ p = t \cdot p \cdot \theta(t).$$

Let $(t, p) \to tp$ be a fixed lifting. We define an equivariant complex line bundle $\eta$ over $X$ by letting $E(\eta) = P \times_{S^1} C$ and $t[p, z] = [tp, z]$ where $t \in T^*$, $z \in C$, $p \in P$. Suppose that the fixed point set $X^*_i$ is the disjoint union of $k + 1$ components $X_i$, $i = 0, i, \ldots, k$. For each $i$, choose $x_i \in X_i$ and define a character $\chi_i$ of $T^*$ by $\eta|_{x_i}(t) = \chi_i(t)$. In this section we will study the general properties of these characters under the assumption that the fixed point set of any $T^*$ action on $P$ is connected (including the empty set as usual).

**Lemma 1.2.** For each $i$, there is a lifting of $T^*$ to $P$ such that the associated equivariant complex line bundle $\eta_i$ satisfies

$$\eta_i|_{x_i}(t) = \chi_i(t)\chi_i(t)^{-1}.$$  

We will say that $\eta_i$ is the normalization of $\eta$ at $x_i$ and the lifting is normalized at $x_i$.

**Proof.** Define a new lifting $(t, p) \to t \cdot p$ by $t \circ p = t p \chi_i(t)$ and let $\eta_i$ be the associated equivariant complex line bundle. Let $p \in \pi^{-1}(X_j)$ and $t \in T^*$, $z \in C$,

$$[p, \eta_i|_{x_i}(t)z] = t [p, z] = [t \circ p, z] = [t p \chi_i(t), z] = t[p \chi_i(t), z]$$

$$= [p \chi_i(t), \chi_i(t)z] = [p, \chi_i(t) \chi_i(t)^{-1}z].$$

Hence $\eta_i|_{x_i}(t) = \chi_i(t)\chi_i(t)^{-1}$. 

\[\square\]
For the rest of this section, we will always assume that the lifting is normalized at $x_0$ and $\chi_0(t) = 1$.

**Lemma 1.3.** Let $H \subset T^s$ be a subgroup. Then $\pi^{-1}(X_j) \subset P''$ if and only if $\chi_i(g) = 1$ for all $g \in H$.

**Proof.** Suppose that $\chi_i(g) = 1$ for all $g \in H$. Let $p \in \pi^{-1}(X_j)$.

$$[gp, z] = g[p, z] = [p, \chi_i(g)z] = [p, z].$$

Now it is clear that $gp = p$ for all $g \in H$ and consequently, $p \in P''$.

Conversely, suppose that $\pi^{-1}(X_j) \subset P''$. Let $p \in \pi^{-1}(X_j)$.

$$[p, z] = [gp, z] = g[p, z] = [p, \chi_i(g)z].$$

Hence $\chi_i(g) = 1$ for all $g \in H$. \hfill \square

**Theorem 1.4.** The $k + 1$ characters $\chi_i$ are distinct.

**Proof.** If the $k + 1$ characters are not distinct, we may assume without loss of generality that $\chi_i = \chi_0$. By Lemma 1.3, we have $X_0 \cup X_i \subset \pi(P^{rs})$. Since we have assumed that $P^{rs}$ is connected, $\pi(P^{rs})$ is a connected subset of $X^{rs}$. It follows that $X_i$ and $X_0$ are in the same component of $X^{rs}$. This is a contradiction. Hence $\chi_i \neq \chi_j$ for $i \neq j$. \hfill \square

**Corollary 1.5.** $\pi^{-1}(X_0) = P^{rs}$. \hfill \square

**Proposition 1.6.** Let $H \subset T^s$ be a subgroup. If $X_i$ and $X_0$ are contained in the same component of $X''$, then $\chi_i(g) = 1$ for all $g \in H$.

Conversely, if $P''$ is connected and $\chi_i(g) = 1$ for all $g \in H$, then $X_i$ and $X_0$ are contained in the same component of $X''$.

**Proof.** If $X_i$ and $X_0$ are contained in the same component of $X''$, then, for $g \in H$,

$$\chi_i(g) = \eta|_{x_j}(g) = \eta|_{x_0}(g) = \chi_0(g) = 1.$$  

Conversely, if $\chi_i(g) = 1$ for all $g \in H$, then $\pi^{-1}(X_0 \cup X_i) \subset P''$ by Lemma 1.3. Since by assumption, $P''$ is connected, $X_0$ and $X_i$ are contained in the same component, namely $\pi(P'')$, of $X''$. \hfill \square

2. The dimensions of invariant subspaces. For convenience, let $G = T^s$. Let $E_g \to B_g$ be the universal principal $G$-bundle. It is well-known that $H^*(B_g, Q) = Q[t_1, \ldots, t_s]$ where $deg t_i = 2$. For the sake of simplicity, we will write $t = (t_1, \ldots, t_s)$. For a graded
$H^*(B_G, Q)$-module $M$, we denote by $M[t^{-1}]$ for the localization of $M$ with respect to the multiplicative system generated by $t$. For a $G$-space $A$, let $A_G = A \times_G B_G$ which the associated bundle over $B_G$ with fiber $A$. We will need the following result in [2].

**PROPOSITION 2.1.** Let $A$ be a $G$-space. Then

(i) The inclusion $j: A_G \to A$ induces an isomorphism

$$j^*: H^*(A_G, Q)[t^{-1}] \to H^*(A, Q)[t^{-1}].$$

(ii) the top class in $H^*(A, Q)$ does not die in $H^*(A_G, Q)[t^{-1}]$ if $A$ is orientable and $A_G \neq \emptyset$.

Let $X$ be a closed topological manifold of dimension $2n$ which is also a rational cohomology complex projective space and which admits a $G$ action. Write the fixed point set $X_G = X_0 \cup \cdots \cup X_k$ as a disjoint union of its components. By a theorem of Bredon [1, p. 378], each $X_i$ is a rational cohomology complex projective space and the inclusion $X_i \to X$ induces isomorphisms $j^*: H^i(X, Q) \to H^i(X_G, Q)$ for $i \leq \dim X_i$. Let $Y \subset X$ be an oriented invariant submanifold and let $C = \{i \mid X_i \subset Y\}$. In this section, we will study the dimension of $Y$ under the assumption that $Y_G = \bigcup_{i \in C} X_i$, where $C \neq \emptyset$. We would like to point out that the most interesting case is that $Y$ is a component of the fixed point set of a subgroup of $G$.

**THEOREM 2.2.** With the above assumptions.

$$\dim Y \leq 2 \left( \sum_{i \in C} \frac{1}{2} \dim X_i + 1 \right).$$

**Proof.** By Proposition 2.1, $H^*(Y_G, Q)[t^{-1}](H^*(X_G, Q)[t^{-1}])$, respectively) is isomorphic to $H^*(Y_G, Q)[t^{-1}](H^*(X_G, Q)[t^{-1}])$, respectively). Since $Y_G = Y_G \times B_G$ and $X_G = X_G \times B_G$.

$$H^*(X_G, Q)[t^{-1}] = \bigoplus_{i=0}^k H^*(X_i) \otimes H^*(B_G)[t^{-1}]$$

and

$$H^*(Y_G, Q)[t^{-1}] = \bigoplus_{i \in C} H^*(X_i) \otimes H^*(B_G)[t^{-1}].$$

This implies the rank of $H^*(Y_G, G)[t^{-1}]$ over $Q[t, t^{-1}]$ is equal to $\sum_{i \in C} (1/2 \dim X_i + 1)$. Let $\alpha \in H^2(H_G, Q)$ represent the generator of $H^2(X, Q)$ and let $\alpha_Y$ be its image in $H^2(Y_G, Q)$. Since $H^*(X_G, Q)[t^{-1}]$ is a free algebra over $Q[t, t^{-1}]$ with basis $1, \alpha, \cdots, \alpha^*$. $H^*(Y_G, Q)[t^{-1}]$ is also a free algebra over $Q[t, t^{-1}]$ with basis $1, \alpha_1, \cdots, \alpha_k$, where $k$
is easily seen to be $\sum_{i \in C} (\frac{1}{2} \dim X_i + 1)$. Since the top classes in $H^*(Y, Q)$ does not die in $H^*(Y_\sigma, Q)[t^{-1}]$, we have

$$\dim Y \leq 2\left(\sum_{i \in C} \frac{1}{2} \dim X_i + 1\right).$$

**Corollary 2.3.** With the above assumptions,

$$\dim Y \leq 2(rkH^*(Y, Q) - 1)$$

and the equality holds if and only if $Y$ is a rational cohomology complex projective space.

**Proof.** Since $rkH^*(Y_\sigma, Q) \leq rkH^*(Y, Q)$ [see [1, p. 163]], and $rkH^*(Y_\sigma, Q) = 1/2 \sum_{i \in C} (\dim X_i + 1)$, the inequality follows immediately. If the equality holds then $rkH^*(Y, Q) = rkH^*(Y_\sigma, Q)$ and $H_{odd}(Y, Q) = 0$. This implies that $Y$ is totally nonhomologous to zero in $Y_\sigma$. Let $a = j^*\alpha_\nu$ where $j: Y \to Y_\sigma$ is the inclusion. It follows that $H^*(Y, Q)$ is generated by $a$. This implies that $Y$ is a rational cohomology complex projective space. The converse is obvious.

**Remark.** Petrie’s examples in [4] provide exotic $S^1$ actions on $\mathbb{C}P^n$ such that the fixed point set component $Y$ of $Z_{pq} \subset S^1$ has property that $\dim Y < 2(rkH^*(Y, Q) - 1)$.

3. The functions $\delta_t(H)$. Let $G$ be a compact Lie Groups and let $V$ be an euclidian space on which $H$ operates orthogonally. We say that a $G$ action on a paracompact space $X$ is locally smooth if for each orbit $Y$ of type $G/H$ there is a linear tube

$$\varphi: G \times_H V \longrightarrow X$$

about $Y$ in $X$, i.e., $\varphi$ is an equivariant embedding onto an open neighborhood of $Y$ in $X$. Note that $X$ must be a topological manifold and the components of the fixed point set are topological submanifolds. If $x$ is a fixed point, then the action in a neighborhood of $x$ is equivalent to an orthogonal action. See [1] for details.

Let $X$ be a closed integral cohomology complex projective space of dimension $2n - 2$ and let $X$ admit a locally smooth $T^s$ action. Then $X$ is the orbit space of a free $S^1$ action on a closed integral cohomology sphere $\Sigma$ and the $T^s$ action on $X$ lifts to a $T^s$ action on $\Sigma$ which commutes with the free $S^1$ action. Let $\eta$ be the equivariant complex line bundle over $X$ associated to a fixed lifting. Then $\eta$ is the equivariant Hopf bundle defined in [4]. Let the fixed point set

$$X^{T^s} = X_0 \cup X_1 \cup \cdots \cup X_k$$
be the disjoint union its components $X_i$. Each $X_i$ is a closed integral cohomology complex projective space of dimension $2n_i - 2$ and $\sum_{i=0}^{k} n_i = n$. Choose $x_i \in X_i$. Define $k + 1$ characters $\chi_i$ as in §1. Since $\Sigma$ is an integral cohomology sphere, $\Sigma^{T^s}$ is connected. By Theorem 1.4, the $k + 1$ characters $\chi_i$ are distinct. For each $i$, let $\rho_i(t) = \sum_{j=0}^{\infty} \chi_{i,j}(t)$ be the representation in the normal fiber at $x_i$ to $X_i$.

The following definition is essentially due to Petrie [3].

**Definition 3.1.** For each subgroup $H \subset T^s$ and each $i = 0, 1, \cdots, k$, set $\alpha_i(H) = \sum_{j \in A_i} n_j$ where $A_i = \{ j \neq i | \chi_j(g) = \chi_i(g) \text{ for all } g \in H\}$, $\beta_i(H) = \text{card } B_i$ where $B_i = \{ j | \chi_{i,j}(g) = 1 \text{ for all } g \in H\}$. Let $\delta_i(H) = \alpha_i(H) - \beta_i(H)$. 

It is easy to see that the function $\delta_i(H)$ are independent of the choice of the lifting. In this section, we will study the properties of $\delta_i(H)$.

**Lemma 3.2.** Let $p$ be a prime and let $H \subset T^s$ be a subgroup of order $p^a$. Then $X^H$ has no $p$-torsions.

**Proof.** Choose a prime $q \neq p$ such that $X^H$ has no $q$-torsions. Let $K \subset T^s$ be a cyclic subgroup of order $q^b$ where $b$ is so large that $X^K = X^{T^s}$. It follows from the results of [1],

$$rk H^*(X; Z) = rk_p H^*(X; Z_p) = rk_p H^*(X^H; Z_p) \geq rk H^*(X^H; Z)$$

$$= rk_q H^*(X^H; Z_q) \geq rk_q H^*(X^H)^{K}; Z_q) = rk_q H^*(X^{T^s}; Z_q)$$

$$= rk H^*(X^{T^s}; Z) = rk H^*(X; Z).$$

It follows that $rk_p H^*(X^H; Z_p) = rk H^*(X^H; Z)$ and $X^H$ has no $p$-torsions.

**Proposition 3.3.** Let $H \subset T^s$ be a subgroup. Then $X^H$ is orientable.

**Proof.** Let $K \subset H$ be a subgroup of order $2^s$ such that $H/K$ has no element of even order. By Lemma 3.2, $X^K$ has no 2-torsions. Hence $X^K$ is orientable. It has been shown in [1] that the fixed point set of a locally smooth action of a finite group of odd order or a torus group on an orientable manifold is always orientable. It follows that $X^H$ is orientable.

**Theorem 3.4.** $\delta_i(H) \geq 0$ for any subgroup $H \subset T^s$ and $\delta_i(H) = 0$ if $H$ is a toral subgroup or a subgroup of order $p^a$ where $p$ is a prime.
Proof. We may assume without loss of generality that $i = 0$. Let $Y$ be a component of $X''$ which contains $X_0$. Let $C = \{j | X_j \subseteq Y\}$. It is obvious that $Y'' = \bigcup_{j \in C} X_j$. By Proposition 1.6, $C \subseteq A_0 \cup \{0\}$.

It is clear that

$$\frac{1}{2} \dim Y = (n_0 - 1) + \beta_0(H).$$

By Theorem 2.2, we have

$$\frac{1}{2} \dim Y \leq \sum_{j \in C} n_j - 1 \leq \sum_{j \in A_0} n_j + n_0 - 1 = \alpha_0(H) + n_0 - 1.$$

Hence $\delta_0(H) = \alpha_0(H) - \beta_0(H) = \alpha_0(H) + n_0 - 1 - 1/2 \dim Y \geq 0$.

If $H$ is a subgroup of order $p^s$, it follows from P. A. Smith Theorem that $\sum''$ is connected. Hence $C = A_0 \cup \{0\}$ by Proposition 1.6. By Lemma 3.2, $Y$ is a mod $p$ cohomology complex projective space without $p$-torsions. Hence $Y$ is a rational cohomology complex projective spaces. It follows easily from Corollary 2.3 $\delta_0(H) = 0$. Similarly, we can prove the case that $H$ is a toral subgroup. □

COROLLARY 3.5. If $\delta_i(H) = 0$ for $i = 0, \cdots, k$, then $X''$ is a disjoint union of rational cohomology complex projective spaces for any subgroup $H \subset T^s$ and $\sum''$ is connected for any normalized lifting.

We also include the following properties for future application.

PROPOSITION 3. Let $H \subset T^s$ be a subgroup. If $X_i$ and $X_j$ are contained in the same component of $X''$, then $\delta_i(H) = \delta_j(H)$.

Proof. Let $Y$ be the component of $X''$ which contains $X_i$ and $X_j$. It is clear that $\alpha_i(H) + n_i = \alpha_j(H) + n_j$ and $1/2 \dim Y = \beta_i(H) + n_i - 1 = \beta_j(H) + n_j - 1$. It follows that $\delta_i(H) = \delta_j(H)$. □

PROPOSITION 3. Let $H \subset T^s$ be a subgroup. If $\chi_i(H) = \chi_j(H) = 1$ and $\delta_j(H) = 0$, then $\delta_i(H) = 0$.

Proof. It is clear from the proof of Theorem 3.4 that $\delta_j(H) = 0$ implies $\sum''$ is connected where lifting is normalized at $x_j$. By Proposition 1.6, $X_i$ and $X_j$ are contained in the same component of $X''$. It follows from Proposition 3.5 that $\delta_i(H) = \delta_j(H) = 0$. □

REFERENCES


4. ———, *A setting for smooth $S^1$ actions with applications to real algebraic actions on $P(C^n)$*, Topology, **13** (1974), 363-374.

Received October 30, 1978 and in revised form October 17, 1979.

State University of New York at Buffalo
Buffalo, NY 14214
Willy Brandal, Conditions for being an FGC domain ........................................ 1
Allan Calder and Frank Williams, Incompressibility of maps and the homotopy invariance of Čech cohomology .......................... 13
Jacques Chaumat, Quelques propriétés du couple d’espaces vectoriels $(L^1 m / H ∞ ⊥ , H ∞ )$ .......................................................... 21
Manfred Droste and Rüdiger Göbel, Products of conjugate permutations ... 47
Jean Esterle, Rates of decrease of sequences of powers in commutative radical Banach algebras ........................................ 61
Allan Fryant, Ultraspherical expansions and pseudo analytic functions ...... 83
John Hannah, Homogenization of regular rings of bounded index. II .... 107
Shigeru Haruki, On the theorem of S. Kakutani-M. Nagumo and J. L. Walsh for the mean value property of harmonic and complex polynomials ..... 113
Hugh M. Hilden, Representations of homology 3-spheres ..................... 125
Craig Huneke, A characterization of locally Macaulay completions ........ 131
Takesi Isiwata, Closed ultrafilters and realcompactness ....................... 139
Joseph Weston Kitchen, Jr. and David A. Robbins, Tensor products of Banach bundles ...................................................... 151
Allan J. Kroopnick, Note on bounded $L^p$-solutions of a generalized Liénard equation ................................................................. 171
Ajay Kumar and Ajit Kaur Chilana, Spectral synthesis in products and quotients of hypergroups .................................................. 177
Charles Livingston, Homology cobordisms of 3-manifolds, knot concordances, and prime knots ............................................. 193
Hans Opolka, Projective representations of finite groups in cyclotomic fields ................................................................. 207
V. D. Pathak, Isometries of $C^{(n)} [0, 1]$ .................................................. 211
Mark Allan Pinsky, On the spectrum of Cartan-Hadamard manifolds ...... 223
Judith Roitman, The number of automorphisms of an atomic Boolean algebra ................................................................. 231
Kai Wang, Locally smooth torus group actions on integral cohomology complex projective spaces ........................................ 243