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Hypergeometric series with a p-adic variable and ratios of such series, as originally considered by B. Dwork, are evaluated at $x=1$. Koblitz's conjecture on the limit of ratios of partial sums of hypergeometric series in the supersingular case is examined and a sufficient condition for the validity of this conjecture is given.

Introduction. In studying the zeta function of a hypersurface, B. Dwork was led to a study of ratios of p-adic hypergeometric series. In [4] and [5] he showed that under certain conditions these ratios had an analytic continuation beyond their disc of convergence. N. Koblitz has recently shown, [6], that the value of the continuation of $F(a, b; 1; x)/F(a', b'; 1; x^p)$ at $x = 1$ is $\Gamma_p(a)\Gamma_p(b)/\Gamma_p(a + b)$, where Γ_p is Morita's p-adic gamma function. Koblitz then conjectured that the ratio of the partial sums

$$F_{s+1}(a, b; 1; 1)/F_s(a', b'; 1; 1) ,$$

where

$$F_s(a, b; c; x) = \sum_{0 \leq n < p^s} \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

has a limit as s approaches infinity for all a and b except for a special case in which the ratio is 0/0. In addition, he gave an expected formula in terms of Γ_p.

In § 1 we will calculate the value of the continuation of $F(a, b; c; x)/F(a', b'; c'; x^p)$ at $x = 1$ for any appropriate a, b and c. In § 2 we will consider the value at $x = 1$ of hypergeometric series in which $c \in \Omega_p - \boldsymbol{Z}_p$ and of certain cases of generalized hypergeometric series. It will be seen, in particular, that Dixon's theorem and Saalschütz's formula hold for p-adic variables. In the last section we consider Koblitz's conjecture, generalized to allow for other c and x. While we give some examples where the conjecture is not quite true, the basic result is a condition on the size of $F_s(a', b'; c'; 1)$ which is sufficient to prove Koblitz's conjecture and its generalization to $c \neq 1$. The proof this theorem connects some of the results of § 2, where c was not in $\boldsymbol{Z}_p$, with Dwork's work, in which $c \in \boldsymbol{Z}_p$.

1. Ratios of hypergeometric functions. If a, b and c are in $\boldsymbol{Z}_p$, then the hypergeometric series

$$F(a, b; c; x) = \sum_{n \geq 0} \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

does not converge at $x = 1$ unless the series terminates. While the series does not usually have an analytic continuation to $x = 1$, Dwork, in [4] and [5], has shown that the ratio

$$\mathscr{F}(a, b; c; x) = \frac{F(a, b; c; x)}{F(a', b'; c'; x^p)}$$

has an analytic continuation to $x = 1$ if certain conditions on a, b and c are satisfied. u' is defined as $(u + \bar{u})/p$, where $\bar{u}$ is the least nonnegative integer $\equiv -u(\operatorname{mod} p)$. When $c = 1$ and the conditions on a and b for Dwork's theory are met, Koblitz, in [6], has shown that

$$\mathscr{F}(a, b; 1; 1) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a + b)}.$$

Koblitz's result can be generalized to allow for c in $\boldsymbol{Z}_p$ other than 1. The result being the classical formula of Gauss, but with the p-adic gamma function.

$u^{(i)}$ is defined as $(u^{(i-1)})'$, with $u^{(0)} = u$. $\bar{u}^{(i)}$ is $\overline{u^{(i)}}$.

THEOREM 1.1. *If $a, b, c \in \boldsymbol{Z}_p$ and the following conditions are satisfied for $i = 0, 1, \cdots$*

(i) $|c^{(i)}| = 1$

(ii) *if $c \neq 1$, then $\bar{a}^{(i)}, \bar{b}^{(i)} < \bar{c}^{(i)}$*

(iii) $|F_1(a^{(i)}, b^{(i)}; c^{(i)}; 1)| = 1$

then

$$\mathscr{F}(a, b; c; x) = \frac{F(a, b; c; x)}{F(a', b'; c'; x^p)}$$

has an analytic continuation to $x = 1$ with the value

$$\mathscr{F}(a, b; c; 1) = \frac{\Gamma_p(c)\Gamma_p(c - a - b)}{\Gamma_p(c - a)\Gamma_p(c - b)}. \tag{1}$$

Conditions (i), (ii) and (iii) are the assumptions Dwork showed to be sufficient for the analytic continuation to $x = 1$. In order to evaluate $\mathscr{F}(a, b; c; 1)$, we need to replace (ii) and (iii) by a nearly equivalent set of conditions.

THEOREM 1.2. *Given that $a, b, c \in \boldsymbol{Z}_p$ and $|c^{(i)}| = 1$, the assumptions $\bar{a}^{(i)}, \bar{b}^{(i)} \leq \bar{c}^{(i)}$ and $|F_1(a^{(i)}, b^{(i)}; c^{(i)}; 1)| = 1$ are equivalent to*

$$a_i + b_i \leqq c_i$$

where a_i, b_i *and* c_i *are the* ith *digits in the* p*-adic expansions of* $-a$, $-b$ *and* $-c$.

Proof. It is an immediate consequence of the definitions that $\bar{a}^{(i)} = a_i$. Hence the conditions $a_i + b_i \leqq c_i$ are the same as $\bar{a}^{(i)} + \bar{b}^{(i)} \leqq \bar{c}^{(i)}$.

Suppose that $\bar{a}^{(i)}, \bar{b}^{(i)} \leqq \bar{c}^{(i)}$ and $|F_1(a^{(i)}, b^{(i)}; c^{(i)}; 1)| = 1$ for all i. To prove Theorem 1.2 it is sufficient to work with $i = 0$. The given condition that $|c'| = 1$ implies that $c + \bar{c} \neq 0 (\mathrm{mod}\, p^2)$. Suppose that $\bar{b} \leqq \bar{a}$. Then,

$$F_1(a, b; c; 1) \equiv \sum_{j=0}^{\bar{b}} \frac{(-\bar{b})_j (a)_j}{(c)_j j!} \pmod p .$$

If $\bar{a} = \bar{c}$, then $\bar{b} \neq 0$ leads to the contradiction $F_1(a, b; c; 1) \equiv 0 \pmod p$. If $\bar{a} < \bar{c}$, let $M = \bar{c} - \bar{a}$. If $\bar{a} + \bar{b} > \bar{c}$, then $1 \leqq M < \bar{b}$. This leads to a contradiction as follows.

$$F_1(a, b; c; 1) \equiv \frac{1}{(c)_M} D_x^M (x^{c+M-1}(1-x)^{\bar{b}})|_{x=1} \equiv 0 \pmod p .$$

Hence, $\bar{a} + \bar{b} \leqq \bar{c}$.

Conversely, suppose $a_i + b_i \leqq c_i$. Again, it sufficient to work with $i = 0$. Obviously, $\bar{a}, \bar{b} \leqq \bar{c}$. Let $M = \bar{c} - \bar{a}$. Then $\bar{b} \leqq M \leqq \bar{c}$ and $a \equiv c + M \pmod p$. As before,

$$F_1(a, b; c; 1) \equiv \frac{1}{(c)_M} D_x^M (x^{c+M-1}(1-x)^{\bar{b}})|_{x=1} \pmod p .$$

Application of Leibnitz's formula for the Mth derivative shows that $F_1(a, b; c; 1) \not\equiv 0 \pmod p$.

Proof of Theorem 1.1. Suppose $a, c \in \boldsymbol{Z}_p$ and c satisfies (i) of Theorem 1.1. Let

$$S(a, c) = \{b \colon b \in \boldsymbol{Z}_p \text{ and } a, b, c \text{ satisfy (ii) and (iii)}\}$$

and suppose $S(a, c)$ is not empty.

The right side of (1) is continuous in b on $S(a, c)$ and the negative integers in $S(a, c)$ are dense in $S(a, c)$. Koblitz observed that in Dwork's construction of $\mathscr{F}(a, b; 1; x)$ the mapping $b \to \mathscr{F}(a, b; 1; 1)$ is the uniform limit of a sequence of continuous functions and hence continuous. This is equally true for $b \to \mathscr{F}(a, b; c; 1)$, so it is sufficient to prove (1) for the negative integers in $S(a, c)$.

When b is a negative integer Gauss' classical formula for $F(a, b; c; 1)$ reduces to

$$F(a, b; c; 1) = \frac{(c-a)\cdots(c-a-b-1)}{c\cdots(c-b-1)}.$$

This is an identity for polynomials and is therefore valid for p-adic hypergeometric series.

If one uses the functional equation of the p-adic gamma function and the result of Theorem 1.2 that $\bar{a} + \bar{b} \leqq \bar{c}$, the expression

$$\frac{\Gamma_p(c)\Gamma_p(c-a-b)}{\Gamma_p(c-a)\Gamma_p(c-b)}$$

is seen to equal $\mathscr{F}(a, b; c; 1)$.

2. Hypergeometric series at $x = 1$.

In this section we will look at hypergeometric series and certain generalized hypergeometric series which have some of their parameters in $\Omega_p - \boldsymbol{Z}_p$. The following elementary lemma will provide the convergence at $x = 1$ for the series which will be considered.

LEMMA 2.1. *Suppose* $u, v \in \Omega_p$.

(i) *If* $\operatorname{dist}(u, \boldsymbol{Z}_p) < \operatorname{dist}(v, \boldsymbol{Z}_p)$, *then* $\lim_{n\to\infty} (u)_n/(v)_n = 0$.

(ii) *If* $\operatorname{dist}(u, \boldsymbol{Z}_p) = \operatorname{dist}(v, \boldsymbol{Z}_p) \neq 0$, *then* $(u)_n/(v)_n$ *is bounded as* n *runs through the positive integers.*

Furthermore, the convergence in (i) is uniform over all u, v at fixed distances from $\boldsymbol{Z}_p$ and the bound in (ii) depends only on the distance of u from $\boldsymbol{Z}_p$.

First we will consider a ${}_2F_1$. If suitable conditions are placed on a, b, c then $\log F(a, b; c; 1)$ can be expressed in terms of the p-adic log gamma function G_p in the same form as Gauss' formula. This result has also been demonstrated by Koblitz in a different manner.

THEOREM 2.2. *If* $b \in \boldsymbol{Z}_p$ *and* $\operatorname{dist}(a, \boldsymbol{Z}_p) < \operatorname{dist}(c, \boldsymbol{Z}_p)$ *then* $\log F(a, b; c; 1) = G_p(c) + G_p(c-a-b) - G_p(c-a) - G_p(c-b)$.

Proof. When b is a negative integer we can apply Gauss' formula and the identity $G_p(x+1) = G_p(x) + \log x$ to obtain the theorem.

When a and c are fixed, the series for $F(a, b; c; 1)$ converges uniformly with respect to b in $\boldsymbol{Z}_p$. Hence the mapping $b \to F(a, b; c; 1)$ is continuous on $\boldsymbol{Z}_p$.

Before considering $\log F(a, b; c; 1)$ it is necessary to be sure that $F(a, b; c; 1)$ is never zero.

Suppose that $F(a, b; c; 1) = 0$. Then there a sequence of negative integers, b_i, approaching b, with

$$\lim_{i\to\infty} F(a, b_i; c; 1) = 0 .$$

It follows from the formula for $F(a, b_i; c; 1)$ that

$$F(-a, b_i; c - a; 1) = 1/F(a, b_i; c; 1) .$$

Hence $\lim_{i\to\infty} F(-a, b_i; c - a; 1) = \infty$. This contradicts the fact that $F(-a, b; c - a; 1)$ is finite.

Now we know both sides of the equation in Theorem 2.2 are continuous functions of b in $\boldsymbol{Z}_p$, so the theorem follows.

If b is a positive integer, $F(a, b; c; 1)$ can be evaluated in closed form. This is just an application of one of Gauss' formulas between contiguous functions.

LEMMA 2.3. *If b is a positive integer and* $\operatorname{dist}(a, \boldsymbol{Z}_p) < \operatorname{dist}(c, \boldsymbol{Z}_p)$, *then*

$$F(a, b; c; 1) = \frac{(c-1)\cdots(c-b)}{(c-a-1)\cdots(c-a-b)} .$$

There are classical formulas for the values of certain generalized hypergeometric functions at $x = 1$. We will consider two such results for a ${}_3F_2$, Dixon's theorem and Saalschütz's formula. The function ${}_3F_2(x) = {}_3F_2(a, b, c; d, e; x)$ is called well-poised if

$$1 + a = b + d = c + e .$$

THEOREM 2.4. *If $b \in \boldsymbol{Z}_p$,* $\operatorname{dist}(c, \boldsymbol{Z}_p) < \operatorname{dist}(e, \boldsymbol{Z}_p)$ *and ${}_3F_2(x)$ is well-poised, then ${}_3F_2(x)$ converges for all x with $|x| \leqq 1$ and*

$$(2) \qquad {}_3F_2(1) = \frac{{}_2F_1(b, c; 1/2(c+e+1); 1)}{{}_2F_1(b, c; c+e; 1)} .$$

Proof. If b is a negative integer, (2) is a polynomial identity which is a special case of Dixon's theorem for complex variables, see [1].

In general, the p-adic convergence in (2) is a consequence of Lemma 2.1. Lemma 2.1 also shows that if c and e are held constant, ${}_3F_2(1)$ is a continuous function of b on $\boldsymbol{Z}_p$. Since we have already shown the denominator of the right side of (2) does not vanish, both sides are continuous and the theorem follows.

A ${}_3F_2$ is said to be Saalschützian if $a + b + c = d + e - 1$. In this case we have

THEOREM 2.5. *If* ${}_3F_2(x)$ *is Saalschützian,* $b \in \boldsymbol{Z}_p$ *and* $\operatorname{dist}(a, \boldsymbol{Z}_p)$, $\operatorname{dist}(c, \boldsymbol{Z}_p) < \operatorname{dist}(e, \boldsymbol{Z}_p)$, *then*

$$ {}_3F_2(a, b; c; 1) = \frac{{}_2F_1(a, b; e; 1)}{{}_2F_1(a, b; e - c; 1)} . $$

Unlike Dixon's theorem, this result is not valid in general with complex parameters. However, Saalschütz's theorem says that if b is a negative integer the above formula holds. The usual continuity argument establishes the result in general for p-adic parameters.

In the first section we considered ratios of hypergeometric series with $a, b, c \in \boldsymbol{Z}_p$. In the next section we will return to these ratios and want to approximate c by a number not in $\boldsymbol{Z}_p$. The following theorem generalizes Theorem 1.1 to the case in which $c \notin \boldsymbol{Z}_p$.

First we need to extend some definitions to allow for numbers not in $\boldsymbol{Z}_p$. $\Gamma_p(x)$ is defined for $x \in \boldsymbol{Z}_p$, but Morita showed that there is a power series for $\Gamma_p(x)$ when $\operatorname{ord}(x) \geqq 1$. This series, together with the equation for $\Gamma_p(x+1)$ shows that $\Gamma_p(x)$ has a natural extension to

$$ \mathscr{D} = \{x\colon x \in \Omega_p \text{ and } \operatorname{dist}(x, \boldsymbol{Z}_p) \leqq 1/p\} . $$

Let $\mathscr{D}_i = \{x\colon |x + i| \leqq 1/p\}$. Γ_p is holomorphic on each $\mathscr{D}_i$. If $x \in \mathscr{D}_i$, $0 \leqq i \leqq p - 1$, we will let $\bar{x} = i$ and $x' = (x + \bar{x})/p$. Let

$$ \mathscr{F}(a, b; c; x) = \frac{F(a, b; c; x)}{F(a', b'; c'; x^p)} . $$

THEOREM 2.6. *If* $a, b, c \in \mathscr{D}$, $b \in \boldsymbol{Z}_p$ *and* $\operatorname{dist}(a, \boldsymbol{Z}_p) < \operatorname{dist}(c, \boldsymbol{Z}_p)$, *then*

$$ \mathscr{F}(a, b; c; 1) = \varepsilon(a, b, c)\frac{\Gamma_p(c)\Gamma_p(c - a - b)}{\Gamma_p(c - a)\Gamma_p(c - b)} , $$

where

$$ \varepsilon(a, b, c) = \begin{cases} 1 & \text{if } \bar{c} \geqq \bar{a} + \bar{b} \\ p(c' - a' - b') & \text{if } \bar{a} + \bar{b} > \bar{c} \geqq \bar{a}, \bar{b} \\ (c' - a' - b')/(c' - a') & \text{if } \bar{a} > \bar{c} \geqq \bar{b} \\ (c' - a' - b')/p(c' - a')(c' - b') & \text{if } \bar{a}, \bar{b} > \bar{c} . \end{cases} $$

Proof. The conditions on a, b, c guarantee the convergence of both $F(a, b; c; 1)$ and $F(a', b'; c'; 1)$. If a and c are held constant, each side of the above equation is continuous in b on $\boldsymbol{Z}_p$. Hence it is sufficient to verify the theorem when b is a negative integer.

This is the same type of calculation made in Theorem 1.1 except that there is no restriction that $\bar{c} \geqq \bar{a} + \bar{b}$.

3. Koblitz's conjecture. When Dwork showed that $\mathscr{F}(a, b; c; x)$ had an analytic continuation for certain a, b, c, he wrote

$$\mathscr{F}(a, b; c; x) = \lim_{s\to\infty} \frac{F_{s+1}(a, b; c; x)}{F_s(a', b'; c'; x^p)}$$

with

$$F_s(a, b; c; x) = \sum_{n=0}^{p^s-1} \frac{(a)_n (b)_n}{(c)_n n!} x^n ,$$

and then showed the limit on the right gave the continuation for x not near 0.

When Koblitz calculated $\mathscr{F}(a, b; 1; 1)$, see [6], he conjectured that $\lim_{s\to\infty} F_{s+1}(a, b; 1; 1)/F_s(a', b'; 1; 1)$ exists for all $a, b \in \boldsymbol{Z}_p$, provided $F_s(a', b'; 1; 1)$ does not vanish, and that its value is

$$\varepsilon(a, b)\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)} \qquad \varepsilon(a, b) = \begin{cases} 1 \text{ if } \bar{a} + \bar{b} < p \\ -p(a+b)' \text{ if } \bar{a} + \bar{b} \geqq p . \end{cases}$$

While we will show that in certain cases this value for the limit is not quite correct, it seems likely that in most cases the conjecture is valid.

We will look at Koblitz's conjecture in a more general setting. Namely, under what circumstances does

$$\bar{F}(x) = \bar{F}(a, b; c; x) = \lim_{s\to\infty} \frac{F_{s+1}(a, b; c; x)}{F_s(a', b'; c'; x^p)}$$

exist, what is its value and what relation is there between the values of this limit for different values of x? We will not answer these questions in general here, but will consider a special case with x in a neighborhood of 1 and then focus on $x = 1$.

THEOREM 3.1. *If $a, c \in \boldsymbol{Z}_p$, c is not 0 or a negative integer and b and $a - c$ are nonnegative integers, then $\bar{F}(a, b; c; x)$ is a quotient of holomorphic functions on each of the discs $D(0, 1^-)$ and $D(1, 1^-)$.*

Proof. If $x \in D(0, 1^-)$, then clearly

$$\bar{F}(a, b; c; x) = F(a, b; c; x)/F(a', b'; c'; x^p) .$$

In order to consider $x \in D(1, 1^-)$ we refer to a result of Cassou-Nogues, [2]. She showed that if $n \to a_n$ can be extended to a uniformly differentiable mapping of $\boldsymbol{Z}_p \to \boldsymbol{Z}_p$, then

$$\lim_{s\to\infty} p^{-s} \sum_{n=0}^{p^s-1} a_n x^n$$

exists and defines a holomorphic function on $D(1, 1^-)$. The condition that b and $a - c$ are nonnegative integers allows us to apply Cassou-Nogues' result separately to the numerator and denominator after inserting a factor of p^{-s}/p^{-s} into the expression for $\bar{F}(a, b; c; x)$.

As an example, let's consider $a = b = c = 1$. Then, $\bar{F}(x) = 1 + x + \cdots x^{p-1}$ for x in both $D(0, 1^-)$ and $D(1, 1^-)$.

The situation is not so simple when $a = 2$ and $b = c = 1$. Then, if $x \in D(0, 1^-)$, $\bar{F}(x) = (1 - x^p)/(1 - x)^2$, and if $x \in D(1, 1^-)$,

$$\bar{F}(x) = (1 - x^p)/(1 - x)^2 + (x^p - 1)/(x - 1) \log x .$$

When $x = 1$, the natural generalization of Koblitz's conjecture is that if $F_s(a', b'; c', 1)$ does not become 0 for large s, $\bar{F}(1)$ exists and its value is given by the formulas in Theorem 2.6. In fact, this conjecture is false in certain cases, but when we look deeper into this problem we will see there are good reasons to expect it to be true in many cases.

First, let's look at some examples. If $a \in \boldsymbol{Z}_p$ and $b = c = 1$, then

$$F_s(a, 1; 1; 1) = (a + 1)\cdots(a + p^s - 1)/(p^s - 1)! ,$$

and a short calculation shows $\bar{F}(a, 1; 1; 1) = pa'/a$. Though $\bar{F}(1)$ exists for all $a \in \boldsymbol{Z}_p$ if $b = c = 1$, the value when $\bar{a} \neq 0$ is the negative of the result conjectured by Koblitz.

A direct calculation can also be made when $b = 2$ and $c = 1$. In this case $\bar{F}(1)$ exists for all $a \in \boldsymbol{Z}_p$ and agrees with Koblitz's value if $\bar{a} = 0$ or 1, but is again off by a minus sign if $\bar{a} > 1$.

A different type of behavior occurs when $b = 1$, $c = 2$ and a is a positive integer greater than 2. Here we can use the idea in Cassou-Nogues' work and find

$$\bar{F}(a, 1; 2; 1) = \frac{pa'}{a-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{a-1}\right) .$$

This expression is discontinuous at each value of a.

We believe the cause of this complicated value of $\bar{F}(1)$ is that $\bar{b} > \bar{c}$. If c is 1, $\bar{c} = p - 1$, so $\bar{a}$ and $\bar{b}$ never exceed $\bar{c}$ and difficulties of this type probably do not occur.

For the remainder of this section we will take $x = 1$. One way to look at Koblitz's conjecture and its generalization to $c \neq 1$ is that we want to be able to reverse the limits in the expression

$$\lim_{\gamma\to c} \lim_{s\to\infty} \frac{F_{s+1}(a, b; \gamma; 1)}{F_s(a', b'; \gamma'; 1)} ,$$

where $\gamma \notin \boldsymbol{Z}_p$. While the examples already given show this cannot always be done, we will show that if $F_s(a', b'; c'; 1)$ does not decrease too rapidly as $s \to \infty$ and the digits in the p-adic expansion of $-c$ are not too small, then the limits may be reversed. In proving this result we will need to know where $\lim_{\gamma \to c} F(a, b; \gamma; 1) = 0$. The following theorem answers this question.

Let $\boldsymbol{Z}^+$ denote the positive integers, $\boldsymbol{Z}^-$ the negative integers, $\boldsymbol{Z}_+ = \boldsymbol{Z}^+ \cup \{0\}$ and $\boldsymbol{Z}_- = \boldsymbol{Z}^- \cup \{0\}$.

THEOREM 3.2. *If $a, b, c \in \boldsymbol{Z}_p$, $c \notin \boldsymbol{Z}_-$ and $\gamma \notin \boldsymbol{Z}_p$, then*

$$\lim_{\gamma \to c} F(a, b; \gamma; 1) = 0$$

if and only if either

(i) *$b, c \in \boldsymbol{Z}^+$ with $b \geqq c$ and if also $a \in \boldsymbol{Z}^+$, then $a \geqq c$ or*

(ii) *$b \in \boldsymbol{Z}^-$, $a - c \in \boldsymbol{Z}_+$ and $b < c - a$ or*

(iii) (i) *or* (ii) *holds with a and b reversed.*

Proof. If (i) holds then Lemma 2.3 shows that the limit is 0. If (ii) holds the same result follows from the formula that applies when $b \in \boldsymbol{Z}^-$.

Now suppose that none of (i), (ii) or (iii) hold. Gauss' relations between contiguous hypergeometric series are formal power series identities, so we can use them with p-adic numbers. By letting $x = 1$ in one of these identities the equation

$$(\gamma - a)(\gamma - b)F(a, b; \gamma + 1; 1) = \gamma(\gamma - a - b)F(a, b; \gamma; 1)$$

is obtained. This leads to

$$F(a, b; \gamma + m; 1) = \frac{(\gamma)_m(\gamma - a - b)_m}{(\gamma - a)_m(\gamma - b)_m} F(a, b; \gamma; 1)$$

and also

$$F(a, b; \gamma - m; 1) = \frac{(\gamma - a - m)_m(\gamma - b - m)_m}{(\gamma - m)_m(\gamma - a - b - m)_m} F(a, b; \gamma; 1) .$$

If $a - c$ and $b - c \notin \boldsymbol{Z}_+$, there is no problem letting $\gamma \to c$ in the formula for $F(a, b; \gamma + m; 1)$. If, however, say, $a - c \in \boldsymbol{Z}_+$, the formula for $F(a, b; \gamma - m; 1)$ will be used unless $c \in \boldsymbol{Z}^+$.

If $c \in \boldsymbol{Z}^+$ and $a - c \in \boldsymbol{Z}_+$, then Lemma 2.3 (with a and b interchanged and c replaced by γ) shows that unless (i) or (ii) holds, $\lim_{\gamma \to c} F(a, b; \gamma; 1) \neq 0$.

Now assume that $\lim_{\gamma \to c} F(a, b; \gamma; 1) = 0$ and $a - c, b - c \notin \boldsymbol{Z}_+$. Then,

$$\lim_{\gamma \to c} F(a, b; \gamma + m; 1) = 0 \qquad \text{for } m = 0, 1, 2, \cdots .$$

Let B_m be a disc around $c+m$ with $|F(a, b; \gamma; 1)| < 1$ when $\gamma \in B_m$. Let $\mathscr{B}$ be a finite covering of Z_p chosen from the B_m. Inspection of the series for $F(a, b; \gamma; 1)$ as a function of γ shows that $F(a, b; \gamma; 1)$ is an analytic element on each set of the form $\operatorname{dist}(\gamma, Z_p) \geqq \delta$ (δ real), and, hence, by Krasner's Mittag-Leffler theorem, must attain its maximum value at some γ where $\operatorname{dist}(\gamma, Z_p) = \delta$. When γ is large, $|F(a, b; \gamma; 1)| = 1$. This leads to a contradiction if δ is chosen sufficiently small.

The above approach also works when $a-c$ or $b-c$ is in Z_+, so Theorem 3.2 is proved.

Note that if (ii) of Theorem 3.2 holds then an argument similar to the proof of Theorem 1.2 shows that $F_s(a', b'; c'; 1) = 0$ when $p^s > b'$. Hence, as Koblitz did for $c=1$, we must exclude this possibility in order to be able to define $\bar{F}(a, b; c; 1)$.

We can now give a sufficient condition for a generalization of Koblitz's conjecture.

THEOREM 3.3. *Suppose* $a, b, c \in Z_p$, $c \notin Z_-$, $\gamma \in \Omega_p - Z_p$, $F_s(a', b'; c'; 1) \neq 0$ *when* s *is large,* $\lim_{\gamma \to c} F(a', b'; \gamma'; 1) \neq 0$, $|c^{(i)}| = 1$ *and if* $c \neq 1$, *then* $\bar{c}^{(i)} > \bar{a}^{(i)}, \bar{b}^{(i)}$ *for* $i = 0, 1, 2, \cdots$. *Then,* $\lim_{s\to\infty} p^{-s} F_s(a', b'; c'; 1) = \infty$ *implies*

$$\lim_{s\to\infty} \frac{F_{s+1}(a, b; c; 1)}{F_s(a', b'; c'; 1)} = \varepsilon(a, b, c) \frac{\Gamma_p(c)\Gamma_p(c-a-b)}{\Gamma_p(c-a)\Gamma_p(c-b)}$$

with

$$\varepsilon(a, b, c) = \begin{cases} 1 \text{ if } \bar{c} \geqq \bar{a} + \bar{b} \\ p(c' - a' - b') \text{ if } \bar{c} < \bar{a} + \bar{b}\,. \end{cases}$$

Proof. For simplicity, let

$$g(z) = \varepsilon(a, b, z) \frac{\Gamma_p(z)\Gamma_p(z-a-b)}{\Gamma_p(z-a)\Gamma_p(z-b)} \quad z \in \mathscr{D}$$

and

$$\bar{F}_s(z) = \frac{F_{s+1}(a, b; z; 1)}{F_s(a', b'; z'; 1)} \quad z \in \mathscr{D}$$

with $\mathscr{D} = \{x;\ \operatorname{dist}(x, Z_p) \leqq 1/p\}$.

For $\gamma \in \mathscr{D}$ and each positive integer s there is the identity

$$\bar{F}_s(c) - g(c) = \bar{F}_s(c) - \bar{F}_s(\gamma) + \bar{F}_s(\gamma) - g(\gamma) + g(\gamma) - g(c)\,.$$

Since g and $\bar{F}_s$ are continuous at c, the terms $F_s(c) - F_s(\gamma)$ and $g(\gamma) - g(c)$ are small if γ is close enough to c. The problem now is to choose γ near c so that $F_s(\gamma) - g(\gamma)$ is small.

By Theorem 2.6, $g(\gamma) = F(a, b; \gamma; 1)/F(a', b'; \gamma'; 1)$, so

$$\bar{F}_s(\gamma) - g(\gamma) = \frac{F(a', b'; \gamma'; 1)F_{s+1}(a, b; \gamma; 1) - F(a, b; \gamma; 1)F_s(a'; b'; \gamma'; 1)}{F_s(a', b'; \gamma'; 1)F(a', b'; \gamma'; 1)} . \tag{3}$$

At this point we will deal just with $c \neq 1$. The case $c = 1$ will be treated afterwards. The basic result of Dwork's article, [5], shows that if $a, b, \gamma \in Z_p$, $|\gamma^{(i)}| = 1$ and $\bar{\gamma}^{(i)} > \bar{a}^{(i)}, \bar{b}^{(i)}$ then

$$F(a', b', c'; x^p)F_{s+1}(a, b; c; x) \equiv F(a, b; c; x)F_s(a', b'; c'; x^p) \pmod{p^{s+1}[[x]]} .$$

This is a formal power series congruence.

An examination of Dwork's proof shows that if we take γ in $\mathscr{D}$, rather than just in Z_p, the congruence is still valid provided that $\gamma^{(s)} \in \mathscr{D}$. (Only the proof of (1.3) of Lemma 1 of [4] uses $\gamma \in Z_p$. The result, however, can be proved with just $\gamma \in \mathscr{D}$.) If $\gamma = c + \theta$, $|\theta| \leqq 1/p$, then $\gamma' = c' + \theta/p$. Thus if γ is sufficiently close to c, $\gamma^{(s)} \in \mathscr{D}$. Furthermore, the conditions $|c^{(i)}| = 1$, $\bar{c}^{(i)} > \bar{a}^{(i)}$, $\bar{b}^{(i)}$ carry over to $\gamma^{(i)}$ when γ is close to c.

Since the series $F(a, b; \gamma; 1)$ and $F(a', b'; \gamma'; 1)$ converge when $\gamma \notin Z_p$, Dwork's formal congruence, with $x = 1$ becomes a numerical congruence. That is, if γ is sufficiently close to c,

$$F(a', b'; \gamma'; 1)F_{s+1}(a, b; \gamma; 1) \equiv F(a, b; \gamma; 1)F_s(a', b'; \gamma'; 1) \pmod{p^{s+1}} .$$

This gives sufficient control over the numerator in (3).

Since it is assumed that $\lim_{\gamma \to c} F(a', b'; \gamma'; 1) \neq 0$, there is some positive M depending only on a, b, c so that every neighborhood of c contains a $\gamma \notin Z_p$ such that $|F(a', b'; \gamma'; 1)| > M$.

Now for $F_s(a', b'; \gamma'; 1)$. A necessary condition for the formulation of Theorem 3.3 is that $F_s(a', b'; c'; 1) \neq 0$ when s is large. Hence, given s, when γ is close enough to c,

$$|F_s(a', b'; \gamma'; 1)| = |F_s(a', b'; c'; 1)| .$$

We are finally ready to show (3) is small when s is sufficiently large. First, by the hypothesis of the theorem, if we are given ε we can choose S such that $|p^{-s}F_s(a', b'; c'; 1)| > 1/M\varepsilon$ for all $s > S$. Then, for each $s > S$, if γ is chosen close enough to c to satisfy all of the conditions mentioned above and also chosen so $|F(a', b'; \gamma'; 1)| > M$, the above estimates can be put together to show

$$|\bar{F}_s(\gamma) - g(\gamma)| < \varepsilon .$$

We can now conclude that

$$\lim_{s\to\infty} \bar{F}_s(c) = g(c) .$$

When $c = 1$, the conditions $\bar{c}^{(i)} > \bar{a}^{(i)}, \bar{b}^{(i)}$ are unnecessarily restrictive. In order to handle this case we refer to Dwork's first paper, [4] involving the basic power series congruence used above. In it he considered hypergeometric series of the form $F(a, b; 1; x)$. We claim that Theorem 2 of [4], which establishes the congruence, can be used if γ is taken close to c. The point here is that $n!/(\gamma)_n$ behaves as well as $(a)_n/n!$ in Dwork's theory. If one looks at Corollary 1 on page 36 of [4] it is clear that the only possible difficulty in having things upside down is that (i) may fail. The following lemma shows there is no problem when γ is close to 1.

LEMMA 3.4. *Let* $z \in \Omega_p$, $\operatorname{ord}(z-1) \geqq 1$. *Then*

$$|A_z(n)/A_{z'}([n/p])| = 1 ,$$

where $A_z(n) = (z)_n/n!$.

The proof is a simple induction.

The proof of Theorem 3.3 when $c = 1$ continues in the same manner as when $c \neq 1$.

Theorem 1.2 and the case $c = 1$ of Theorem 3.3 suggest that the conditions $\bar{c}^{(i)} > \bar{a}^{(i)}, \bar{b}^{(i)}$ in Theorem 3.3 can be weakened to $\bar{c}^{(i)} \geqq \bar{a}^{(i)}, \bar{b}^{(i)}$.

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