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ON THE LOCAL SPECTRUM AND THE ADJOINT

JAMES KENNETH FINCH

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Let X be a Banach space, X^* the dual space, and suppose that T is a closed linear operator on X . Assume that the domain of T is dense in X , so that the adjoint operator T^* is a closed linear operator on X^* . The local spectrum $\sigma(T, x)$ is defined below. In this paper we investigate some of the relations between $\sigma(T, x)$ and $\sigma(T^*, x^*)$. In particular we show that if $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are both empty, then $x^*x = 0$.

The resolvent function of T is $R_T(\lambda) = (\lambda I - T)^{-1}$; it is an operator valued function, and is defined and analytic for λ not in $\sigma(T)$, the spectrum of T . Setting $f_x(\lambda) = R_T(\lambda)x$, then f_x is analytic and satisfies $(\lambda I - T)f_x(\lambda) = x$ for all λ not in $\sigma(T)$. However, it may be possible to find analytic solutions to $(\lambda I - T)f(\lambda) = x$ for some (or all) values of λ that are in the spectrum of T . So we are led to define a *local resolvent function* of T at x as a vector valued analytic function f which satisfies $(\lambda I - T)f(\lambda) = x$. It is easily shown that for λ not in $\sigma(T)$, the only local resolvent is $f(\lambda) = R_T(\lambda)x$. But for λ in $\sigma(T)$, there may be more than one local resolvent function. The *local resolvent set* is the union of the domains of all the local resolvent functions. The point at infinity is included if there is a local resolvent function which is defined and bounded for $|\lambda| > r$. The *local spectrum* $\sigma(T, x)$ is the complement of the local resolvent set. Clearly $\sigma(T, x)$ is a closed subset of the spectrum $\sigma(T)$; it may be equal to $\sigma(T)$, properly contained in it, or even empty.

The operator T has the *single valued extension property* if there is at most one local resolvent function defined near any λ in \mathbb{C} ; that is, whenever both f and g are local resolvent functions defined near λ , then $f = g$ there. In this case, there is a unique local resolvent with maximal domain. It can be shown that T has the single valued extension property iff $\sigma(T, x)$ is not empty for any $x \neq 0$. (Note that $\sigma(T, 0)$ is always empty; the local resolvent $f(\lambda) = 0$ is defined for all λ .)

If both T and T^* have the single valued extension property, then $\sigma(T, x) \cap \sigma(T^*, x^*) = \emptyset$ implies $x^*x = 0$ (see Lemma 1). This is not true if $\sigma(T, x) = \emptyset$ for some $x \neq 0$. For then $x^*x = 0$ for all x^* , from which it would follow that $x = 0$. Similarly, it is not true if $\sigma(T^*, x^*)$ is empty. But in Theorem 3 we show that if both $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are empty, then $x^*x = 0$.

Suppose that T does not have the single valued extension

property. Then for some open set $D \subseteq \mathbb{C}$ there are distinct local resolvent functions for some x . Subtracting them, we find that there is a nonzero analytic function $f: D \rightarrow X$ which satisfies $(\lambda I - T)f(\lambda) = 0$ for λ in D . For a fixed λ_0 in D , f has a series expansion $f(\lambda) = \sum_{n=0}^{\infty} x_n(\lambda - \lambda_0)^n$. It is not hard to show that $(\lambda_0 I - T)x_{n+1} = -x_n$, $(\lambda_0 I - T)x_0 = 0$, and $(\lambda_0 I - T)^{n+1}x_n = 0$. Also, $\sigma(T, x_n)$ is empty for $n = 0, 1, 2, \dots$. To show this, we give two resolvent functions for x_n , whose domains together cover the extended complex plane. Set

$$\begin{aligned} g_n(\lambda) &= \sum_{k=0}^n (-1)^k (\lambda_0 I - T)^k x_n / (\lambda - \lambda_0)^{k+1} \\ &= \sum_{k=0}^n x_{n-k} / (\lambda - \lambda_0)^{k+1} \quad \text{for } \lambda \neq \lambda_0, \end{aligned}$$

and

$$h_n(\lambda) = \sum_{k=0}^{\infty} x_{n+1+k}(\lambda - \lambda_0)^k \quad \text{for } |\lambda - \lambda_0| < r.$$

The series for h converges in the same disk as the series for f . Straightforward calculation shows that $(\lambda I - T)g_n(\lambda) = x_n$ for $\lambda \neq \lambda_0$. To show that $(\lambda I - T)h_n(\lambda) = x_n$ for $|\lambda - \lambda_0| < r$, first consider the finite sum $S_m(\lambda) = \sum_{k=0}^{m-1} x_{n+1+k}(\lambda - \lambda_0)^k$. Then $(\lambda I - T)S_m(\lambda) = x_n - x_{n+m}(\lambda - \lambda_0)^m$, which converges to x_n as $m \rightarrow \infty$. But $S_m \rightarrow h_n$ as $m \rightarrow \infty$. Since T is a closed operator, we have $(\lambda I - T)h_n(\lambda) = x_n$.

From the above remarks we see that when an operator T does not have the single valued extension property, it acts something like a left shift operator. For a specific example, consider an orthonormal basis $\{e_n\}$ for a separable Hilbert space H . Let T be the left shift: $Te_{n+1} = e_n$ and $Te_0 = 0$. Then we have $(\lambda I - T)f(\lambda) = 0$ on $|\lambda| < 1$ for $f(\lambda) = \sum_{n=0}^{\infty} e_n \lambda^n$. And so $\sigma(T, e_n)$ is empty for all n . Actually $\sigma(T, x)$ is contained in the unit circle $\{\lambda: |\lambda| = 1\}$ for all x in H , although $\sigma(T) = \{\lambda: |\lambda| \leq 1\}$. The function $g_x(\lambda) = -\sum \lambda^n S^{n+1}x$ is a local resolvent defined on $|\lambda| < 1$, where S is the right shift: $Se_n = e_{n+1}$. Further, it is not hard to see that $\sigma(T^*, x^*)$ is the entire unit disk $\{\lambda: |\lambda| \leq 1\}$ for all x^* in H^* . In Theorem 2 below we show that this behavior is typical. This example was first given by Kakutani; see Dunford and Schwartz, page 1932. Kakutani's formulation is on the Hilbert space of functions analytic on the unit disk.

LEMMA 1 (Bishop 1959). *Suppose f and g are analytic functions with range in X and X^* respectively, and which satisfy $(\lambda I - T)f(\lambda) = x$ and $(\lambda I^* - T^*)g(\lambda) = x^*$. If the union of the domains of f and g is the entire complex plane, and if f or g is bounded in a neighborhood of infinity, then $x^*x = 0$.*

The proof is an application of Liouville's Theorem to the function $h(\lambda) = x^*f(\lambda)$ for λ in the domain of f , and $h(\lambda) = g(\lambda)x$ otherwise.

THEOREM 2. *Let T be a closed linear operator with domain dense in a Banach space X . Suppose that T does not have the single valued extension property, so that there is an analytic function $f: D \subseteq \mathbb{C} \rightarrow X$ with $(\lambda I - T)f(\lambda) = 0$ for λ in D . Assume that D is a connected open set. Then for each x^* in X^* , either*

- (i) x^* is identically zero on the subspace spanned by $\{f(\lambda): \lambda$ in $D\}$, or
- (ii) $\sigma(T^*, x^*)$ contains all of D .

Proof. Let Y be the closed linear span of $\{f(\lambda): \lambda$ in $D\}$. For any λ_0 in D , expand f in a Taylor series around λ_0 , $f(\lambda) = \sum x_n(\lambda - \lambda_0)^n$. Each x_n is in Y , since

$$x_n = (2\pi i)^{-1} \int_{\Gamma} f(\lambda)(\lambda - \lambda_0)^{-n-1} d\lambda,$$

where Γ is a circle around λ_0 contained in D . Moreover, Y is actually spanned by $\{x_n\}$. For if $x^*x_n = 0$ for all n , then $x^*f(\lambda) = 0$ wherever the series converges. But $x^*f(\lambda)$ is analytic on the connected set D , so $x^*f(\lambda) = 0$ for λ in D . It follows that $x^*y = 0$ for all y in Y , and so $\{x_n\}$ spans Y .

Now suppose that λ_0 is in D , but is not in the local spectrum $\sigma(T^*, x^*)$. Then, by definition, there is an analytic function h defined near λ_0 with $(\lambda I^* - T^*)h(\lambda) = x^*$. For each n , $g_n(\lambda) = \sum_{k=0}^n (-1)^k (\lambda_0 I - T)^k x_n / (\lambda - \lambda_0)^{k+1}$ satisfies $(\lambda I - T)g_n(\lambda) = x_n$ for $\lambda \neq \lambda_0$. Then from Lemma 1, $x^*x_n = 0$ for all n . Hence x^* is identically zero on Y . \square

Suppose that the local spectrums $\sigma^*(T, x)$ the $\sigma(T^*, x^*)$ are both empty. Can we conclude that $x^*x = 0$? The answer is yes, but not directly from Theorem 2, for we do not know that x is in a subspace Y as described there. Instead, we show that x is in the sum of a finite number of such subspaces. Then since x^* is identically zero on each subspace, it is zero on the sum. And then $x^*x = 0$.

Suppose that $\sigma(T, x)$ is empty. Then in particular the point at infinity is not in $\sigma(T, x)$; by definition this means that there is a bounded function $g(\lambda)$ defined on $D = \{\lambda: |\lambda| > r\}$ which satisfies $(\lambda I - T)g(\lambda) = x$. This function g has a special form.

LEMMA 3. *The function $g(\lambda)$ described above has the series representation $g(\lambda) = \sum_{n=0}^{\infty} T^n x / \lambda^{n+1}$.*

Proof. Since g is bounded for λ in D , it must have a series

representation $\sum_{n=0}^{\infty} x_n/\lambda^n$. Then $g(\lambda) \rightarrow x_0$ as $\lambda \rightarrow \infty$, and so $g(\lambda)/\lambda \rightarrow 0$. But $T(g(\lambda)/\lambda) = T(g(\lambda))/\lambda = (\lambda g(\lambda) - x)/\lambda \rightarrow x_0$. T is a closed operator, hence $x_0 = T(0) = 0$. Now, $\lambda g(\lambda) \rightarrow x_1$ as $\lambda \rightarrow \infty$, and so $T(g(\lambda)) = \lambda g(\lambda) - x$ converges to $x_1 - x$. But $g(\lambda) \rightarrow 0$; thus $0 = T(0) = x_1 - x$, or $x_1 = x$.

We show that $Tx_n = x_{n+1}$ for $n \geq 1$, and therefore that $x_n = T^n x$, using induction. Let

$$\begin{aligned} h(\lambda) &= x_n + x_{n+1}/\lambda + x_{n+2}/\lambda^2 + \cdots \\ &= \lambda^n g(\lambda) - \lambda^{n-1} x_1 - \cdots - \lambda x_{n-1}. \end{aligned}$$

Then

$$\begin{aligned} Th(\lambda) &= \lambda^n (\lambda g(\lambda) - x) - \lambda^{n-1} x_2 - \cdots - \lambda x_n \\ &= \lambda^{n+1} g(\lambda) - \lambda^n x_1 - \cdots - \lambda x_n \\ &= x_{n+1} + x_{n+2}/\lambda + \cdots. \end{aligned}$$

Thus as $\lambda \rightarrow \infty$, $h(\lambda) \rightarrow x_n$ and $Th(\lambda) \rightarrow x_{n+1}$. Since T is closed, we have $Tx_n = x_{n+1}$. \square

THEOREM 4. *Suppose T is a closed linear operator on a Banach space X with domain dense in X . If $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are both empty, then $x^*x = 0$.*

Proof. The idea is to show that if $\sigma(T, x)$ is empty, then x is in a finite sum of subspaces spanned by $f_k(\lambda)$, where $(\lambda I - T)f_k(\lambda) = 0$ for λ in the domain of f_k . It then follows from Theorem 2 that $x^*x = 0$.

If $\sigma(T, x)$ is empty, then for every complex number z , there is an analytic function $g_z(\lambda)$ which satisfies $(\lambda I - T)g_z(\lambda) = x$ and which is defined (at least) on some disk D_z around z . In particular, there is such a function $g_\infty(\lambda)$ defined on $D_\infty = \{\lambda: |\lambda| \geq r_0\}$. Take B_z an open disk around z contained in D_z ; and B_∞ a neighborhood of ∞ contained in D_∞ . The B 's form an open cover of the extended complex plane, which is compact. Hence a finite number of them, $B_\infty, B_1, \dots, B_n$ also form an open cover. Let g_0, g_1, \dots, g_n be the corresponding functions. Now the idea is that where the domains of g_i and g_j overlap, the function $f_{jk} = g_j - g_k$ satisfies $(\lambda I - T)f_{jk}(\lambda) = 0$. The problem is to show that x is in the sum of the subspaces generated by all the f_{jk} 's.

We may assume that no B_j is in the union of the previous B 's; drop any that is. We may also assume that no B_j is in the union of the closure of the others, by taking the B_j slightly smaller if necessary. Now define closed sets by $C_0 = \text{cl}(B_\infty)$ and $C_k = \text{cl}(B_k - C_0 - \cdots - C_{k-1})$. We see that $B_\infty \cup B_1 \cup \cdots \cup B_k$ is contained in $C_0 \cup \cdots \cup C_k$ and that C_k is disjoint from $B_\infty, B_1, \dots, B_{k-1}$. Thus the

family C_k is a closed cover of the extended complex plane. The sets C_k may have more than one component, but each component is bounded by a finite number of circular arcs. Let Γ_j be the boundary of C_j . Given a point on Γ_j , any neighborhood of that point contains interior points of C_j and interior points of some other C_k ; by the construction, the point must then be on Γ_k . That is, every point on Γ_j is on at least one other Γ_k . Also, a point where three or more of the Γ_k meet must be a point where two or more of the original circular boundaries of the B_k meet. Only a finite number of such points exist, since distinct circles intersect in at most two points.

Note that $2\pi ix = \sum_j \int_{\Gamma_j} g_j(\lambda) d\lambda$, where the integral around Γ_0 is in the counterclockwise direction and the others in the clockwise direction. This is true since $\int_{\Gamma_j} g_j = 0$ for $j = 1, \dots, n$, and $\int_{\Gamma_0} g_0 = 2\pi ix$ from Lemma 3. Now consider an arc γ on Γ_j that extends from one intersection of three or more Γ_k to another, without passing through a third (if there are not two such intersections on Γ_j , let $\gamma = \Gamma_j$). Each interior point of γ is on exactly one other Γ_k ; in fact, all interior points are on the same Γ_k since otherwise an intersection of three Γ_k would be an interior point of γ . The paths of integration of $\int_{\Gamma_j} g_j$ and $\int_{\Gamma_k} g_k$ both contain γ , and the integrals "go over" γ in opposite directions. Thus these portions of those integrals may be combined to $\pm \int_{\gamma} (g_j - g_k)$. Doing this for each arc on each Γ_j , we see that the expression for $2\pi ix$ becomes a linear combination of integrals of the form $\int_{\gamma} (g_j - g_k)$. Since $(\lambda I - T)(g_j - g_k) = 0$, setting $f_{jk} = g_j - g_k$, we have x in the sum of the spaces spanned by the f_{jk} . \square

THEOREM 5. *Suppose that T is as above, and in addition the range of T is closed. If for every $x \neq 0$ the local spectrum $\sigma(T, x)$ contains the disk $D = \{\lambda: |\lambda| \leq r\}$, then T^* does not have the single valued extension property. In particular, there are elements $x^* \neq 0$ in X^* for which $\sigma(T^*, x^*)$ is empty.*

Proof. First T is one-one. For if $Tx = 0$, then $f(\lambda) = x/\lambda$ is a local resolvent function defined for all $\lambda \neq 0$. And then $\sigma(T, x) \subseteq \{0\}$, contrary to our assumption. However, 0 is in the spectrum of T . Thus T cannot be onto. Since the range of T is closed, T^* is onto but not one-one. This is sufficient to show that T^* does not have the single valued extension property. For take any x_0^* with $x_0^* = 1$ and $T^*x_0^* = 0$, and form a sequence $x_0^*, x_1^*, x_2^*, \dots$ with $T^*x_{n+1}^* = x_n^*$.

Because T^* is an open mapping, this can be done so that $\|x_{n+1}^*\| < k\|x_n^*\|$. It is then fairly easy to show that $f^*(\lambda) = \sum x_n^* \lambda^n$ converges for $|\lambda| < 1/k$ and satisfies $(\lambda I^* - T^*)f^*(\lambda) = 0$. \square

REFERENCES

1. Errett Bishop, *A duality theory for arbitrary operators*, Pacific J. Math., **9** (1959), 379-397.
2. Nelson Dunford and Jacob Schwartz, *Linear Operators*, New York: Interscience, 1957.
3. James Finch, *The single valued extension property on a Banach space*, Pacific J. Math., **58** (1975), 61-69.
4. F. H. Vasilescu, *Residual properties for closed operators on Frèchet spaces*, Illinois J. Math., **15** (1971), 377-386.

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