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Benno Fuchssteiner

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BENNO FUCHSSTEINER

A Strassen-type disintegration theorem for convex cones with localized order structure is proved. As an example a flow theorem for infinite networks is given.

Introduction. It has been observed by several authors (e.g., [9], [10], [13] and [14]) that the essential part of the celebrated Strassen disintegration theorem ([12] or [7]) consists of a rather sophisticated Hahn-Banach argument combined with a measure theoretic argument like the Radon Nikodym theorem. For this we refer especially to a theorem given by M. Valadier [13] and also by M. Neumann [10]. We first extend this result (in the formulation of [10]) to convex cones. This extension is nontrivial because in the situation of convex cones one looses in general the required measure theoretic convergence properties if a linear functional is decomposed with respect to an integral over sublinear functionals. For avoiding these difficulties we have to combine a maximality-argument with, what we call, a localized order structure. On the first view this order structure looks somewhat artificial, but it certainly has some interest in its own since it turns out that the disintegration is compatible with this order structure. The usefulness of these order theoretic arguments is illustrated by giving as an example a generalization of Gale's [4] celebrated theorem on flows in networks (see also Ryser [11]).

1. A Disintegration theorem. Let (Ω, Σ, m) be a measure space with σ-algebra Σ and positive σ-finite measure m. By $L^1_*(m)$ we denote the convex cone of $\boldsymbol{R}_*$-valued ($\boldsymbol{R}_* = \boldsymbol{R} \cup \{-\infty\}$) measurable functions on Ω such that their positive part (but not necessarily the negative part) is integrable with respect to m. Of course, two elements of $L^1_*(m)$ are considered to be equal if they coincide almost everywhere. Note, that for every $f \in L^1_*(m)$ the integral $\int_\Omega f dm$ exists in $\boldsymbol{R}_*$, and that the function $-\infty$ is an element of $L^1_*(m)$.

Throughout the paper we assume $0 \cdot (-\infty) = 0$. We are interested in operators $p: F \to L^1_*(m)$, where F is some convex cone. As usual, such an operator is said to be sublinear if it is positively homogeneous (i.e., $P(\lambda x) = \lambda P(x) \forall x \in F$, $\forall \lambda \geqq 0$) and subadditive. If it is superadditive instead of subadditive then it is called superlinear. A linear operator is one which is both sublinear and superlinear.

For the study of operators $F \to L^1_*$ we introduce in F an order structure $\leqq_{\omega, \omega \in \Omega}$ which is *localized* on Ω. This means that for every

$\omega \in \Omega$ we have given an preorder $\leqq_\omega$ on F (a reflexive and transitive relation on F) which is compatible with the cone structure of F (see [2] or [3]).

An operator $p: F \to L^1_*(m)$ is said to be monotone with respect to this localized order structure (*Ω-monotone* for short) if for $x, y \in F$ we always have:

$$P(x) \leqq P(y) \ m\text{-almost everywhere on } \{\omega \in \Omega \mid x \leqq {}_\omega y\} .$$

DISINTEGRATION THEOREM 1. *Let $\mu: F \to \boldsymbol{R}_*$ be linear and let $P: F \to L^1_*(m)$ be an Ω-monotone sublinear operator with*

$$(1) \qquad \mu(x) \leqq \int_\Omega P(x)\, dm \ \textit{for all } x \in F .$$

Then there is an Ω-monotone linear operator $T: F \to L^1_*(m)$ with $T \leqq P$ (i.e., $T(x) \leqq P(x) \forall x \in F$) *such that*

$$(2) \qquad \mu(x) \leqq \int_\Omega T(x) dm \ \textit{for all } x \in F .$$

Proof. Let Φ be the convex cone consisting of all simple Σ-measurable functions $\phi: \Omega \to F$. Here, a function ϕ is called simple if $\phi(\Omega)$ is a finite set and if, for every $x \in F$, the set $\{\omega \in \Omega \mid \phi(\omega) = x\}$ belongs to Σ. In Φ we consider the preorder given by:

$$\phi_1 \leqq \phi_2 \Longleftrightarrow \phi_1(\omega) \leqq {}_\omega \phi_2(\omega) \text{ for } m\text{-almost all } \omega \in \Omega .$$

Then

$$p(\phi) = \int_\Omega P(\phi(\omega))(\omega) dm(\omega)$$

defines a monotone sublinear functional on Φ. And

$$\delta(\phi) = \begin{cases} \mu(x) \text{ if } \phi \text{ is constant (with value } x\text{) on } \Omega \\ -\infty \text{ otherwise} \end{cases}$$

gives us a superlinear functional on Φ, with $\delta \leqq p$. According to the sandwich theorem ([2] or [3]) there is a monotone linear ν with $\delta \leqq \nu \leqq p$. And by using Zorn's lemma we can further assume that ν is maximal among the linear functionals $\leqq p$. Now, for $A \in \Sigma$ and $x \in F$, we define

$$(3) \qquad d(A, x) = \nu(1_A x) ,$$

where 1_A is the characteristic function of A, i.e., $1_A(\omega) = \{1 \text{ if } \omega \in A, 0 \text{ otherwise}\}$. We claim that for $x, y \in F$ and $A \in \Sigma$ the following are true:

(4) $$d(A, \cdot) \text{ is linear on } F$$

(5) $$d(\cdot, x) \text{ is an additive set function on } \Sigma$$

(6) $$\mu(x) \leqq d(\Omega, x)$$

(7) $$d(A, x) \leqq \int_A P(x)dm$$

(8) when $x \leqq_\omega y$ for m-almost all $\omega \in A$ then $d(A, x) \leqq d(A, y)$

(9) if A_n is a sequence of pairwise disjoint sets in Σ then

$$d(\bigcup_{n \in N} A_n, x) = \liminf_{m \to \infty} \sum_{n=1}^{m} d(A_n, x) .$$

The assertions (4)-(9) prove the theorem in the following way: (5) and (9) show clearly that $d(\cdot, x)$ is a signed measure on Ω. Assertion (8) implies that this measure is absolutely continuous with respect to m. This is so, because from $m(A) = 0$ we obviously get $x \leqq_\omega 0$ and $0 \leqq_\omega x$ for almost all $\omega \in A$, and hence $d(A, x) = d(A, 0) = 0$.

Now, we apply the Radon-Nikodym theorem to find a measurable function $T(x)$ such that

(10) $$d(A, x) = \int_A T(x)dm .$$

Then, because of (7), the positive part of $T(x)$ is absolutely integrable with respect to m, so $T(x)$ must belong to $L^1_*(m)$. Assertion (4) gives that $x \to T(x)$ is linear, and from (6) and (7) we obtain (2) and $T(x) \leqq P(x)$. Finally, we show that $x \to T(x)$ is in fact Ω-monotone. Consider $x, y \in F$, put $B = \{\omega \in \Omega \mid x \leqq_\omega y\}$ and assume $T(x)(\omega) > T(y)(\omega)$ for $\omega \in A \subset B$ with $m(A) > 0$. Then, without loss of generality, we may further assume that $\int_A T(x)dm > -\infty$ (otherwise we replace A by a suitable subset). And we have in contradiction to (8)

$$d(A, x) = \int_A T(x)dm > \int_A T(y)dm = d(A, y) .$$

So we are left with:

Proof of (4)-(9): (4) and (5) are easy consequences of the linearity of ν, and (6) and (7) follow immediately from $\delta \leqq \nu \leqq p$. Let $x \leqq_\omega y$ for m-almost all $\omega \in A$. Then $1_A \cdot x \leqq 1_A \cdot y$ and by monotony of ν we get:

$$d(A, x) = \nu(1_A \cdot x) \leqq \nu(1_A \cdot y) = d(A, y) .$$

So we have also proved (8).

The proof of (9) is a little bit more complicated and depends essentially on the maximality of ν. So, let the A_n be as in (9) and define for arbitary $\phi \in \Phi$:

$$\rho(\phi) = \nu(1_Y \cdot \phi) + \liminf_{m \to \infty} \sum_{n=1}^{m} \nu(1_{A_n} \cdot \phi) , \tag{11}$$

where $\gamma = \Omega \backslash \bigcup_{n \in N} A_n$. Then ρ is superlinear (because of the inf in the lim inf). From $\nu \leqq p$ we get

$$\rho(\phi) \leqq \nu(1_Y \cdot \phi) + \sum_{n=1}^{\infty} \int_{A_n} P(\phi(\omega))(\omega) dm(\omega) = \int_{\Omega} P(\phi(\omega))(\omega) dm = p(\phi) .$$

Hence $\rho \leqq p$. The σ-additivity of m implies $\rho \geqq \nu$: To see this we use the following obvious inequality:

$$\rho(\phi) + \limsup_{m \to \infty} \nu(1_{Z_m} \cdot \phi) \geqq \nu(\phi) , \tag{12}$$

where $Z_m = \Omega \backslash (Y \cup \bigcup_{n=1}^{m} A_n)$. Since the Z_m are decreasing to $\varnothing$ we get:

$$\begin{aligned} \limsup_{m \to \infty} \nu(1_{Z_m} \cdot \phi) &\leqq \limsup_{m \to \infty} p(1_{Z_m} \cdot \phi) \\ &\leqq \limsup_{m \to \infty} \int_{Z_m} P(\phi(\omega))(\omega) dm(\omega) \leqq 0 . \end{aligned}$$

This inserted in (12) leads in fact to $\rho \geqq \nu$. Now, we apply the sandwich theorem to obtain a monotone linear $\bar{\nu}$ with $\rho \leqq \bar{\nu} \leqq \pi$. Then, because of $\rho \geqq \nu$ and the fact that ν was already maximal, this yields $\nu = \bar{\nu}$. Hence $\rho = \nu$. Inserting this in (11) and putting $\phi = 1_{\cup_{n \in N} A_n} \cdot x$ we get the desired result. □

REMARK 2. Without loss of generality one can assume the μ in Theorem 1 to be superlinear instead of linear, since the sandwich theorem applied to $\mu \leqq \int_{\Omega} P(\cdot) dm$ yields a linear $\bar{\mu}$ fulfilling the same inequality as μ. Then application of Theorem 1 to $\bar{\mu}$ gives the desired result.

REMARK 3. A similar disintegration result can be obtained for linear functionals attaining values in a Dedekind complete Riesz space. This can be done by replacing the use of the sandwich theorem by the vector valued sandwich theorem of [3]. Of course, in this case the arguments depending on the Radon-Nikodym theorem do not work, and therefore the disintegration theorem for this situation has to be stated in a less elegant form.

2. An Example. We consider a signed measure $\tilde{\mu}$ on some measurable space $(\tilde{\Omega}, \Sigma)$ and a positive finite measure τ on $\Omega = \tilde{\Omega} \times \tilde{\Omega}$. We recall that a bimeasure (see |6| or |8|) on Ω is a function $\nu: \Sigma \times \Sigma \to \boldsymbol{R}$ being separately in each variable a signed measure. By F we denote the convex cone of positive simple measurable functions on $\tilde{\Omega}$, i.e., functions of the form $x = \sum_{n=1}^{N} \alpha_n 1_{A_n}$, where $\alpha_n \geqq 0$, $A_n \in \Sigma$. To every $x \in F$ we assign a function $\hat{x}: \Omega \to \boldsymbol{R}$ by

$$\hat{x}(\omega_1, \omega_2) = \max(x(\omega_1) - x(\omega_2), 0), \ \omega_1, \omega_2 \in \tilde{\Omega} .$$

The map $x \to \hat{x}$ is sublinear and a simple calculation shows that

$$(*) \qquad \tilde{\mu}(A) \leqq \tau(A \times \complement A) \text{ for all } A \in \Sigma$$

is equivalent to

$$(**) \qquad \int_{\Omega} x d\tilde{\mu} \leqq \int_{\Omega} \hat{x} d\tau \text{ for all } x \in F .$$

Using this and the disintegration theorem we get

FLOW THEOREM. *The following are equivalent*:

(i) $\tilde{\mu}(A) \leqq \tau(A \times \complement A)$ *for all* $A \in \Sigma$

(ii) *There is a bimeasure* ν *on* $\Omega = \tilde{\Omega} \times \tilde{\Omega}$ *having the following properties*:

(a) $\tilde{\mu}(A) \leqq \nu(A, \tilde{\Omega})$ *for all* $A \in \Sigma$,

(b) $\nu(A, B) \leqq \tau(A \times B \cap \complement A)$ *for all* $A, B \in \Sigma$,

(c) $\nu(A, B) \geqq 0$ *whenever* $A, B \in \Sigma$ *are disjoint.*

Proof. (ii) $\Rightarrow$ (i) is quite trivial.

(i) $\Rightarrow$ (ii): We introduce in F an order structure localized on Ω by defining for $\omega = (\omega_1, \omega_2) \in \tilde{\Omega} \times \tilde{\Omega}$

$$x \leqq_{\omega} y \Longleftrightarrow x(\omega_1) \leqq y(\omega_1) \text{ and } x(\omega_2) \geqq y(\omega_2) .$$

Then the map $x \to P(x) \doteqdot \hat{x}$ is Ω-monotone. Now, consider the linear function $\mu: F \to \boldsymbol{R}$ given by

$$\mu(x) = \int_{\tilde{\Omega}} x d\tilde{\mu} .$$

According to $(*) \Rightarrow (**)$ the inequality (i) is equivalent to:

$$\mu(x) \leqq \int_{\Omega} P(x) d\tau \text{ for all } x \in F .$$

From our disintegration theorem we then obtain a Ω-monotone linear map $T: F \to L^1_*(\tau)$ such that

$$(13) \qquad \int_{\tilde{\Omega}} x d\tilde{\mu} \leqq \int_{\Omega} T(x) d\tau$$

$$T(x) \leqq \hat{x} \tag{14}$$

for all $x \in F$. We define

$$\nu(A, B) = \int_{\tilde{\Omega} \times B} T(1_A) d\tau \text{ for } A, B \in \Sigma ,$$

then ν has the required properties. The assertion (a) is a consequence of (13) and (b) comes directly out of (14). The Ω-monotony implies (c). All what remains to prove is the σ-additivity of ν in the first variable. Take an arbitrary sequence $A_n \downarrow \varnothing$, $A_n \in \Sigma$. Then (b) and the positivity of τ give $\nu(A_n, \tilde{\Omega}) \leqq \tau(A_n \times \tilde{\Omega})$. Hence

$$\lim_{n \to \infty} \nu(A_n, \tilde{\Omega}) = 0 . \tag{15}$$

If B_n is such that $B_n \cap A_n = \varnothing$ then we get from (c) and (b) that $0 \leqq \nu(A_n, B_n) \leqq \tau(A_n \times \tilde{\Omega})$. Hence

$$\lim_{n \to \infty} \nu(A_n, B_n) = 0 . \tag{16}$$

Now, using the additivity of ν in both variables one can express the sequence $\nu(\tilde{A}_n, \tilde{B})$ $(\tilde{A}_n \downarrow \varnothing,\ \tilde{A}_n, \tilde{B} \in \Sigma)$ in terms of sequences like (15) and (16). This gives the σ-additivity in the first variable. □

Specializing the Flow Theorem to the case of finite discrete sets $\tilde{\Omega}$ one immediately obtains Gale's theorem ([4], [11] or [1, page 38]). It is well known that this theorem is closely related to the Ford-Fulkerson theorem. But whereas the Ford-Fulkerson theorem for infinite networks can be obtained from the finite case via Tychonoff's theorem (see [5]) the situation is slightly more complicated in case of Gale's theorem (although not too different in principle).

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CALIFORNIA INSTITUTE OF TECHNOLOGY
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AND
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