

# Pacific Journal of Mathematics

**THE VOLUME CUT OFF A SIMPLEX BY A HALF-SPACE**

LEON GERBER

## THE VOLUME CUT OFF A SIMPLEX BY A HALF-SPACE

LEON GERBER

A formula for the volume cut off an  $n$ -dimensional simplex by a half-space has immediate application in probability theory. This note presents a derivation of such a formula in a short and completely elementary way and also yields the moments of this volume about the coordinate hyperplanes, and the volumes cut off the lower dimensional faces of the simplex and their moments.

Before stating our results we introduce some notation. The "minus function"  $(x)_-^k$  equals  $x^k$  if  $x < 0$  and is zero otherwise. The  $n$ th divided difference of a real-valued function  $f(x)$  is the symmetric function of  $n + 1$  arguments defined inductively by

$$D\{f(x): x_0, x_1\} = \frac{f(x_0) - f(x_1)}{x_0 - x_1},$$

$$D\{f(x): x_0, x_1, \dots, x_n\} = \frac{D\{f(x): x_0, x_1, \dots, x_{n-1}\} - D\{f(x): x_n, x_1, \dots, x_{n-1}\}}{x_0 - x_n}.$$

If  $x_0 = x_n$  we define

$$D\{f(x): x_0, x_1, \dots, x_{n-1}, x_0\} = \frac{\partial}{\partial x_0} D\{f(x): x_0, x_1, \dots, x_{n-1}\}$$

and similarly for all repeated arguments. The algorithm of Varsi [1, p 317] enables the calculation of such differences for  $f(x) = (x)_-^k$  without the use of limiting processes. A proof of the symmetry and other elementary properties of divided differences may be found in [4].

The unit  $n$ -simplex with vertices  $\mathbf{a}_0 = (0, \dots, 0)$ ,  $\mathbf{a}_1 = (1, 0, \dots, 0)$ ,  $\dots$ ,  $\mathbf{a}_n = (0, \dots, 0, 1)$  will be denoted by  $\mathcal{S}_n$ , and  $\mathcal{B}_n$  will denote an  $n$ -simplex with vertices  $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n$  where  $\mathbf{b}_i$  has cartesian coordinates  $(b_{i1}, \dots, b_{in})$ . We denote by  $\mathcal{B}_k$  the  $k$ -face of  $\mathcal{B}_n$  with vertices  $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_k$ . The  $n$ -volume ( $n$ -dimensional volume) of  $\mathcal{B}_n$  and the  $k$ -volume of  $\mathcal{B}_k$ ,  $k = 1, \dots, n$  are given by

$$v_n(\mathcal{B}_n) = \frac{1}{n!} \begin{vmatrix} 1 & b_{01} & \dots & b_{0n} \\ & & \dots & \\ & & & \\ 1 & b_{n1} & \dots & b_{nn} \end{vmatrix},$$

$$v_k(\mathcal{B}_k) = \frac{1}{k!} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & \mathbf{b}_0 \cdot \mathbf{b}_0 & \mathbf{b}_0 \cdot \mathbf{b}_1 & \cdots & \mathbf{b}_0 \cdot \mathbf{b}_k \\ & & \cdots & & \\ 1 & \mathbf{b}_k \cdot \mathbf{b}_0 & \mathbf{b}_k \cdot \mathbf{b}_1 & \cdots & \mathbf{b}_k \cdot \mathbf{b}_k \end{vmatrix}^{1/2}.$$

We are now ready for our main result.

**THEOREM.** Consider a linear function  $L(\mathbf{x}) = a_0 + a_1x_1 + \cdots + a_nx_n$  and the half-space  $\mathcal{H}$  defined by  $L(\mathbf{x}) \leq 0$ . Let  $p$  be any non-negative integer. Then for  $k = 1, 2, \dots, n$ :

$$1. \int_{\mathcal{B}_k} [L(\mathbf{x})]^p dv_k$$

$$= \frac{k! p!}{(k+p)!} v_k(\mathcal{B}_k) D\{x^{k+p}: L(\mathbf{b}_0), L(\mathbf{b}_1), \dots, L(\mathbf{b}_k)\}.$$

2. The  $k$ -volume cut off the  $k$ -simplex  $\mathcal{B}_k$  by  $\mathcal{H}$  is given by

$$v_k(\mathcal{B}_k \cap \mathcal{H}) = v_k(\mathcal{B}_k) D\{(x)^k: L(\mathbf{b}_0), L(\mathbf{b}_1), \dots, L(\mathbf{b}_k)\}.$$

3. If  $p = p_1 + \cdots + p_n$ , then the  $p$ th mixed moment of  $\mathcal{B}_k \cap \mathcal{H}$  about the coordinate hyperplanes is given by

$$\int_{\mathcal{B}_k \cap \mathcal{H}} x_1^{p_1} \cdots x_n^{p_n} dv_k$$

$$= \frac{k!}{(k+p)!} v_k(\mathcal{B}_k) \frac{\partial^p}{\partial a_1^{p_1} \cdots \partial a_n^{p_n}} D\{(x)^{k+p}: L(\mathbf{b}_0), L(\mathbf{b}_1), \dots, L(\mathbf{b}_k)\}.$$

*Proof.* Our first observation is that for all  $p$

$$\begin{aligned} & \int_{\mathcal{S}_n} [c_0 + (c_1 - c_0)y_1 + \cdots + (c_n - c_0)y_n]^p d\mathbf{y} \\ &= \frac{p!}{(n+p)!} \{x^{n+p}: c_0, c_1, \dots, c_n\} \end{aligned}$$

where  $\int_{\mathcal{S}_n} f(\mathbf{y}) d\mathbf{y}$  means  $\int_{y_1=0}^1 \int_{y_2=0}^{1-y_1} \cdots \int_{y_n=0}^{1-y_1-\cdots-y_{n-1}} f(\mathbf{y}) dy_n \cdots dy_1$ . The proof is by a straightforward induction on  $n$  when the  $c_i$  are distinct, and by continuity otherwise.

To prove part 1 for  $k = n$  consider the transformation defined by

$$x_1 = b_{01} + (b_{11} - b_{01})y_1 + \cdots + (b_{n1} - b_{01})y_n$$

...

$$x_n = b_{0n} + (b_{1n} - b_{0n})y_1 + \cdots + (b_{nn} - b_{0n})y_n,$$

so when  $\mathbf{y} = \mathbf{a}_i$  then  $\mathbf{x} = \mathbf{b}_i$ ,  $i = 0, 1, \dots, n$ . Note that the Jacobian of this transformation is

$$\begin{vmatrix} b_{11} - b_{01} & \cdots & b_{n1} - b_{01} \\ & \cdots & \\ b_{1n} - b_{0n} & \cdots & b_{nn} - b_{0n} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_{01} & b_{11} & \cdots & b_{n1} \\ & \cdots & & \\ b_{0n} & b_{1n} & \cdots & b_{nn} \end{vmatrix} = n! v_n(\mathcal{B}_n).$$

Then

$$\begin{aligned} & \int_{\mathcal{S}_n} [L(x_1, \dots, x_n)]^p dx_1 \cdots dx_n / n! v_n(\mathcal{B}_n) \\ &= \int_{\mathcal{S}_n} \{L(\mathbf{b}_0) + [[L(\mathbf{b}_1) - L(\mathbf{b}_0)]y_1 + \cdots + [L(\mathbf{b}_n) - L(\mathbf{b}_0)]y_n\}^p dy \\ &= \frac{p!}{(n+p)!} D\{x^{n+p}: L(\mathbf{b}_0), L(\mathbf{b}_1), \dots, L(\mathbf{b}_n)\}. \end{aligned}$$

For the case  $1 \leq k \leq n - 1$  we may assume  $\mathcal{B}_k$  is not orthogonal to the coordinate  $k$ -flat  $\mathcal{F}$  defined by  $x_{k+1} = \cdots = x_n = 0$  (since  $\mathcal{B}_k$  cannot be orthogonal to all  $\binom{n}{k}$  such  $k$ -flats) so  $\mathcal{B}_k$  can be defined by linear equations in the form

$$x_j = M_j(x_1, \dots, x_k), \quad j = k + 1, \dots, n.$$

Let  $\mathbf{x}^*$  denote the orthogonal projection of a point  $\mathbf{x}$  on  $\mathcal{F}$  and let

$$L^*(x_1, \dots, x_k) = L(x_1, \dots, x_k, M_{k+1}(x_1, \dots, x_k), \dots, M_n(x_1, \dots, x_k))$$

so for any point  $\mathbf{x}$  of  $\mathcal{B}_k$  we have  $L^*(\mathbf{x}^*) = L(\mathbf{x})$ , and in particular,  $L^*(\mathbf{b}_i^*) = L(\mathbf{b}_i)$ ,  $i = 0, 1, \dots, k$ . Since orthogonal projection shrinks  $dv_k$  and  $v_k(\mathcal{B}_k)$  by the same factor, we have

$$\begin{aligned} \int_{\mathcal{B}_k} [L(\mathbf{x})]^p dv_k / v_k(\mathcal{B}_k) &= \int_{\mathcal{B}_k^*} [L^*(x_1, \dots, x_k)]^p dx_1 \cdots dx_k / v_k(\mathcal{B}_k^*) \\ &= \frac{k! p!}{(k+p)!} D\{x^{k+p}: L^*(\mathbf{b}_0^*), L^*(\mathbf{b}_1^*), \dots, L^*(\mathbf{b}_k^*)\} \end{aligned}$$

which completes the proof of the first result.

Now it is easy to verify that if  $a \neq 0$  then

$$\int (ax + b)_-^p dx = \frac{ax + b)_-^{p+1}}{a(p+1)} + c.$$

This means that the minus function integrates formally like a polynomial and we may conclude that

$$\int_{\mathcal{B}_k} [L(\mathbf{x})]_+^p dv_k = \frac{k! p!}{(k+p)!} v_k(\mathcal{B}_k) D\{(x)_-^{p+k}: L(\mathbf{b}_0), L(\mathbf{b}_1), \dots, L(\mathbf{b}_k)\}.$$

In particular, since  $\mathbf{x} \in \mathcal{H}$  if and only if  $L(\mathbf{x}) \leq 0$  we have

$$v_k(\mathcal{B}_k \cap \mathcal{H}) = \int_{\mathcal{B}_k} [L(\mathbf{x})]_0^0 dv_k = v_k(\mathcal{B}_k) D\{(x)_-^k: L(\mathbf{b}_0), L(\mathbf{b}_1), \dots, L(\mathbf{b}_k)\}$$

which is our second result.

Our final result comes from the observation that

$$\int_{\mathcal{B}_k \cap \mathcal{H}} x_1^{p_1} \cdots x_n^{p_n} dv_k = \frac{1}{p!} \int \frac{\partial^p}{\partial a_1^{p_1} \cdots \partial a_n^{p_n}} [L(\mathbf{x})]^2 dv_k.$$

**COROLLARY.** *Let  $\mathcal{H}$  be the half-space defined by  $a_0 + a_1 x_1 + \cdots + a_n x_n \leq 0$ . Then*

$$\begin{aligned} v_n(\mathcal{A}_n \cap \mathcal{H})/v_n(\mathcal{A}_n) &= D\{(x)_-^n: a_0, a_0 + a_1, \dots, a_0 + a_n\} \\ &= D\{(x + a_0)_-^n: 0, a_1, \dots, a_n\} \end{aligned}$$

where  $v_n(\mathcal{A}_n) = 1/n!$ , and

$$v_{n-1}(\mathcal{A}_{n-1} \cap \mathcal{H})/v_{n-1}(\mathcal{A}_{n-1}) = D\{(x + a_0)_{n-1}^-: a_1, \dots, a_n\}$$

where  $\mathcal{A}_{n-1}$  is the  $(n-1)$ -face with vertices  $a_1, \dots, a_n$ , and  $v_{n-1}(\mathcal{A}_{n-1}) = \sqrt{n}/(n-1)!$

Varsi [5] discovered an algorithm equivalent to the first result of the corollary by dissecting  $\mathcal{A}_n$  into  $O(2^n)$  simplexes; his algorithm requires  $O(n^2)$  computations and is hence quite efficient. Ali [1] observed that Varsi's algorithm was equivalent to divided differences and proved this result using the Fourier-Stieltjes transform; he also proved the second result of the corollary. The technique of integrating a "plus function" was used in [2] to find the volume cut off a hypercube by a half-space, a result which was obtained previously in [3, pp. 48-50] using a more elementary technique which, curiously enough, involved a density function defined on  $2^n$  overlapping regions of space.

*Added in proof.* The third part of the theorem can be put in a more useful form. For example, using  $L_i$  for  $L(\mathbf{b}_i)$ , the centroid  $\mathcal{B}_k \cap \mathcal{H}$  is given by

$$\frac{D\{(x)_-^{k+1}: L_0, L_0, L_1, L_2, \dots, L_k\} \mathbf{b}_0 + 0\{(x)_-^{k+1}: L_0, L_1, L_1, L_2, \dots, L_k\} \mathbf{b}_1 + \cdots}{(k+1)D\{(x)_-^k: L_0, L_1, L_2, \dots, L_k\}}.$$

## REFERENCES

1. M. M. Ali, *Content of the frustrum of a simplex*, Pacific J. Math., **48** (1973), 313-322.
2. D. L. Barrow and P. W. Smith, *Spline notation applied to a volume problem*, Amer. Math. Monthly, **86** (1979), 50-51.
3. M. G. Kendall, *A Course in the Geometry of  $n$  Dimensions*, Griffin, London, 1961.
4. J. F. Steffenson, *Interpolation*, Williams and Wilkins, 1927.
5. G. Varsi, *The Multidimensional content of the frustrum of the simplex*, Pacific J. Math., **46** (1973), 303-314.

Received October 19, 1979.

ST. JOHN'S UNIVERSITY  
JAMAICA, NY 11439

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DONALD BABBITT (Managing Editor)

University of California  
Los Angeles, CA 90024

HUGO ROSSI

University of Utah  
Salt Lake City, UT 84112

C. C. MOORE and ANDREW OGG

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University  
Stanford, CA 94305

## ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF HAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1981 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

Thomas E. Armstrong and William David Sudderth, Nearly strategic measures .....	251
John J. Buoni, Artatrana Dash and Bhushan L. Wadhwa, Joint Browder spectrum .....	259
Jack Paul Diamond, Hypergeometric series with a $p$ -adic variable .....	265
Raymond Frank Dickman, Jack Ray Porter and Leonard Rubin, Completely regular absolutes and projective objects .....	277
James Kenneth Finch, On the local spectrum and the adjoint .....	297
Benno Fuchssteiner, An abstract disintegration theorem .....	303
Leon Gerber, The volume cut off a simplex by a half-space .....	311
Irving Leonard Glicksberg, An application of Wermer's subharmonicity theorem .....	315
William Goldman, Two examples of affine manifolds .....	327
Yukio Hirashita, On the Weierstrass points on open Riemann surfaces .....	331
Darrell Conley Kent, A note on regular Cauchy spaces .....	333
Abel Klein and Lawrence J. Landau, Periodic Gaussian Osterwalder-Schrader positive processes and the two-sided Markov property on the circle .....	341
Brenda MacGibbon, $\mathcal{H}$ -Borelian embeddings and images of Hausdorff spaces .....	369
John R. Myers, Homology 3-spheres which admit no PL involutions .....	379
Boon-Hua Ong, Invariant subspace lattices for a class of operators .....	385
Chull Park, Representations of Gaussian processes by Wiener processes .....	407
Lesley Millman Sibner and Robert Jules Sibner, A sub-elliptic estimate for a class of invariantly defined elliptic systems .....	417
Justin R. Smith, Complements of codimension-two submanifolds. III. Cobordism theory .....	423
William Albert Roderick Weiss, Small Dowker spaces .....	485
David J. Winter, Cartan subalgebras of a Lie algebra and its ideals. II .....	493