HOMOLOGY 3-SPHERES WHICH ADMIT NO PL INVOLUTIONS

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An infinite family of irreducible homology 3-spheres is constructed, each member of which admits no PL involutions.

1. Introduction. In Problem 3.24 of [6] H. Hilden and J. Montesinos ask whether every homology 3-sphere is the double branched covering of a knot in \( S^3 \). The interest in this question lies in the fact that there is an algorithm, due to J. Birman and H. Hilden [1], for deciding whether such a 3-manifold is homeomorphic to \( S^3 \). In addition, the Smith Conjecture for homotopy 3-spheres [4] implies that every homotopy 3-sphere of this type must be homeomorphic to \( S^3 \).

In this paper an infinite family of irreducible homology 3-spheres is exhibited which admit no PL involutions. This gives a negative answer to the above question since the nontrivial covering translation of a branched double cover is a PL involution.

2. Preliminaries. We shall work throughout in the PL category.

A knot \( K \) is an oriented simple closed curve in the oriented 3-sphere \( S^3 \) which does not bound a disk. The exterior \( Q = Q(K) \) is the closure of the complement of a regular neighborhood of \( K \). A meridian \( \mu = \mu(K) \) of \( K \) is an oriented simple closed curve in \( \partial Q \) which bounds a disk in \( S^3 - \text{Int} \, Q \) and has linking number +1 with \( K \). A longitude \( \lambda = \lambda(K) \) of \( K \) is an oriented simple closed curve in \( \partial Q \) such that \( \lambda \) bounds a surface in \( Q \) and \( \lambda \sim K \) in \( S^3 - \text{Int} \, Q \). ("\( \sim \)" means "is homologous to").

\( K \) is \( \pm \) amphicheiral if there is an orientation reversing homeomorphism \( g \) of \( S^3 \) such that \( g(K) = \pm K \). \( K \) is invertible if there is an orientation preserving homeomorphism \( g \) of \( S^3 \) such that \( g(K) = -K \).

For the definitions of simple knot, torus knot, and fibered knot we refer to [8]. For the definitions of irreducible 3-manifold, incompressible surface, and of parallel surfaces in a 3-manifold we refer to [5]. Note that a knot \( K \) is simple if and only if every incompressible torus in \( Q(K) \) is parallel to \( \partial Q(K) \). If \( K \) is simple and \( Q(K) \) contains an incompressible annulus which is not parallel to an annulus in \( \partial Q(K) \), then \( K \) is a torus knot [3].

Suppose \( h \) is an involution on a homology 3-sphere \( M \). Then by Smith theory [2] the fixed point set \( \text{Fix} \, \langle h \rangle \) is homeomorphic to \( S^0 \).
or $S^2$ if $h$ reverses orientation and is empty or homeomorphic to $S^1$ if $h$ preserves orientation.

3. The construction. Let $K_0$ and $K_1$ be knots. Let $Q_i = Q(K_i)$, $\mu_i = \mu(K_i)$, and $\lambda_i = \lambda(K_i)$, $i = 0, 1$. We construct $M = M(K_0, K_1)$ by identifying $\partial Q_0$ and $\partial Q_1$ so that $\mu_0 = \lambda_1$ and $\lambda_0 = -\mu_1$. We denote $Q_0 \cap Q_1$ by $T$ and $\mu_0, \lambda_0$ by $\alpha, \beta$, respectively. Note that $M$ is an irreducible homology 3-sphere and that $T$ is incompressible in $M$.

**Lemma 3.1.** If $K_0$ and $K_1$ are simple knots, other than torus knots, then every incompressible torus in $M(K_0, K_1)$ is isotopic to $T$.

Proof. Let $T'$ be an incompressible torus in $M$. Isotop $T'$ so that $T$ and $T'$ are in general position and meet in a minimal number of components.

Suppose some component $J$ of $T \cap T'$ bounds a disk $D'$ in $T'$. We may assume $D' \cap T = \partial D'$. By the incompressibility of $T$, $\partial D' = \partial D$ for some disk $D$ in $T$. By the irreducibility of $M$, $D \cup D'$ bounds a 3-cell $B$ in $M$. So $T'$ can be isotoped by pushing $D'$ across $B$ and off $D$ to remove at least $J$ from $T \cap T'$. This contradicts minimality and so cannot happen. A similar argument shows that no component of $T \cap T'$ bounds a disk in $T$.

Thus if $T \cap T' \neq \emptyset$, $T' \cap Q_i$ consists of incompressible annuli. Let $A'$ be such an annulus in $Q_0$. Since $K_0$ is simple and not a torus knot, $A'$ is parallel in $Q_0$ to an annulus $A$ in $T$. Therefore $T'$ can be isotoped by pushing $A'$ across the solid torus bounded by $A \cup A'$ and off $A$ to remove at least $\partial A$ from $T \cap T'$. By minimality this cannot occur.

Thus $T'$ lies in some $Q_i$. Since $K_i$ is simple, $T'$ is parallel to $T$ and we are done.

4. Involutions on $M(K_0, K_1)$. An involution $h$ on $M(K_0, K_1)$ is good if $h(Q_i) = Q_i$, $i = 0, 1$, Fix $\langle h \rangle$ and $T$ are in general position, $h(\alpha) \sim \pm \alpha$, and $h(\beta) \sim \pm \beta$.

**Lemma 4.1.** Let $K_0$ and $K_1$ be simple knots, other than torus knots, such that $Q_0$ and $Q_1$ are not homeomorphic. Then every involution of $M(K_0, K_1)$ is conjugate to a good involution.

Proof. By Theorem 1 of Tollefson [1] and Lemma 3.1 there is an isotopy $f_i$ of $M$ such that $f_0 = id$, $f_i(T)$ and Fix $\langle h \rangle$ are in general position, and either $h(f_i(T)) = f_i(T)$ or $h(f_i(T)) \cap f_i(T) = \emptyset$. Let $h' = f_1^{-1} \circ h \circ f_i$. Then either $h'(T) = T$ or $h'(T) \cap T = \emptyset$.

Suppose $h'(T) \cap T = \emptyset$. We may assume $h'(T) \subset \text{Int} Q_0$. If
\[ h(Q_0) \subset \text{Int } Q_0, \text{ then } Q_0 = h^3(Q_0) \subset \text{Int } h(Q_0) \subset \text{Int } h^3(Q_0) = \text{Int } Q_0, \text{ which is absurd. Thus } Q_1 \subset \text{Int } h(Q_0). \] But since \( \partial Q_1 \) is parallel to \( \partial h(Q_0) \) in \( h(Q_0) \), \( Q_0 \) and \( Q_1 \) are homeomorphic, a contradiction. Therefore \( h'(T) = T \) and so \( h'(Q_1) = Q_1 \).

Finally \( h(\alpha) = h(\mu_0) = h(\lambda_1) \sim \pm \lambda_1 = \pm \alpha \) and similarly \( h(\beta) \sim \pm \beta \).

**Lemma 4.2.** Suppose \( K_0 \) is non-amphichiral. Then every good involution on \( M(K_0, K_i) \) is orientation preserving.

**Proof.** \( h(\beta) \sim \pm \beta \) implies that \( h(\mu_0) \sim \pm \mu_0 \) and thus that the orientation reversing homeomorphism \( h \mid Q_0 \) can be extended to an orientation reversing homeomorphism \( g \) of \( S^3 \) such that \( g(K_0) = \pm K_0 \), a contradiction.

**Lemma 4.3.** Suppose \( K_i \) is non-invertible. If \( h \) is a good, orientation preserving involution on \( M(K_0, K_i) \), then \( \text{Fix } \langle h \rangle \cap T = \emptyset \).

**Proof.** Suppose not. Then \( \text{Fix } \langle h \rangle \) is a simple closed curve meeting \( T \) transversely in finitely many points \( x_1, \ldots, x_n \). Let \( T^* \) be the orbit space of \( T \) under \( h \mid T \). The projection \( g: T \to T^* \) is a 2-fold covering branched over \( x_1^*, \ldots, x_n^* \), where \( x_i^* = g(x_i) \). An Euler characteristic argument shows that \( T^* \) is a 2-sphere and \( n = 4 \).

Let \( \gamma^* \) and \( \delta^* \) be arcs in \( T^* \) such that \( \gamma^* \) joins \( x_1^* \) and \( x_2^* \), \( \delta^* \) joins \( x_3^* \) and \( x_4^* \), and each misses the other two branch points. Then \( \gamma = g^{-1}(\gamma^*) \) and \( \delta = g^{-1}(\delta^*) \) are simple closed curves meeting transversely in the single point \( x_2 \). After choosing orientations, \( \gamma \) and \( \delta \) form a basis for \( H_1(T) \). Moreover \( h(\gamma) \sim -\gamma \) and \( h(\delta) \sim -\delta \). It follows that \( h(\mu_i) \sim -\mu_i \) and \( h(\lambda_i) \sim -\lambda_i \). Then \( h \mid Q_1 \) can be extended to an orientation preserving homeomorphism \( g \) of \( S^3 \) such that \( g(K_i) = -K_i \), a contradiction.

**Lemma 4.4.** Let \( h \) be an orientation preserving free involution on a torus \( T \). Let \( \alpha \cup \beta \) be a pair of simple closed curves in \( T \) which meet transversely in a single point. Then \( \alpha \cup \beta \) can be isotoped so that either

(i) \( h(\alpha) = \alpha \) and \( h(\beta) \cap \beta = \emptyset \), or

(ii) \( h(\beta) = \beta \) and \( h(\alpha) \cap \alpha = \emptyset \), or

(iii) \( h(\alpha) \cap \alpha = \emptyset = h(\beta) \cap \beta \).

**Proof.** Note that \( h \) induces the identity on \( H_1(T) \). Isotop \( \alpha \cup \beta \) so that \( h(\alpha) \cap \alpha \) is minimal.

Suppose \( h(\alpha) \cap \alpha \neq \emptyset \). Since \( h(\alpha) \sim \alpha \) there is a disk \( D \) in \( T \) with \( \partial D = \gamma \cup \delta \), where \( \gamma \) and \( \delta \) are arcs in \( \alpha \) and \( h(\alpha) \), respectively,
and \((\alpha \cup h(\alpha)) \cap \text{Int } D = \emptyset\). Suppose \(h(D) \cap D = \emptyset\). Then \(\alpha\) can be isotoped by pushing \(\gamma\) across \(D\) and off \(\delta\) to obtain a new curve having four fewer intersection points with its image. This contradicts minimality and so does not occur. Suppose \(h(D) \cap D\) is a single point \(p\). Then \(\alpha\) can be isotoped by pushing \(\gamma\) across \(D\) and off \(\delta - p\) to obtain a curve having two fewer intersections with its image. So this cannot happen. Therefore \(h(D) \cap D\) consists of two points \(p\) and \(q\). In fact \(h(\alpha) \cap \alpha = \{p, q\}\). Isotop \(\alpha\) by pushing \(\gamma\) across \(D\) to \(\delta\). Then \(h(\alpha) = \alpha\).

Now isotop \(\beta\), keeping \(\alpha\) pointwise fixed, so that \(h(\alpha) \cap \beta\) is a single point. (This is only necessary if \(h(\alpha) \cap \alpha = \emptyset\).) Then isotop \(\beta\), keeping \(\alpha\) and \(h(\alpha)\) setwise fixed, so that \(h(\beta) \cap \beta\) is minimal. As in the case of \(\alpha\) above, the result will be that either \(h(\beta) \cap \beta = \emptyset\) or that \(\beta\) can be isotoped so that \(h(\beta) = \beta\). This can be done keeping \(\alpha\) and \(h(\alpha)\) setwise fixed because the analogous disk \(D\) used in the isotopies meets each of \(\alpha\) and \(h(\alpha)\) in at most a point of \(\gamma \cap \delta\) or an arc with one endpoint in each of \(\text{Int } (\gamma)\) and \(\text{Int } (\delta)\).

**Lemma 4.5.** Let \(h\) be a good orientation preserving involution on \(M(K_0, K_1)\) such that \(\text{Fix } h \cap T = \emptyset\). Then \(\text{Fix } h = \emptyset\) and \(\alpha \cup \beta\) can be isotoped so that \(h(\alpha) \cap \alpha = \emptyset = h(\beta) \cap \beta\).

**Proof.** We may assume that \(\alpha \cup \beta\) satisfies one of the three possible outcomes of Lemma 4.4. Suppose (i) is true. Then \(h|_{Q_0}\) can be extended to an involution \(g\) on \(S^3\) with \(K_0 \subset \text{Fix } g\). By Smith theory \(K_0 = \text{Fix } g\). By the period two Smith Conjecture [14] \(K_0\) is unknotted, a contradiction. A similar argument rules out (ii). Thus (iii) holds. If \(\text{Fix } h \neq \emptyset\), then \(\text{Fix } h \subset \text{Int } Q_i\) for some \(i\). Then the homology 3-sphere \(M(K_0, K_i)\) admits an involution \(g\) with \(\text{Fix } g\) homeomorphic to \(S^1 \cup S^2\). This contradicts Smith theory, so \(\text{Fix } h = \emptyset\).

**Lemma 4.6.** Suppose \(K_0\) has a unique isotopy class of incompressible spanning surface. If \(h\) is a good, orientation preserving free involution on \(M(K_0, K_1)\), then \(K_0\) is a fibered knot.

**Proof.** Let \(Q_0^*\) be the orbit space of \(Q_0\) under \(h\). Let \(q: Q_0 \to Q_0^*\) be the quotient map and set \(\mu_0^* = q(\mu_0), \lambda_0^* = q(\lambda_0),\) and \(T^* = q(T)\). Let \(i: T^* \to Q_0^*\) be the inclusion map. Choose an oriented simple closed curve \(\xi\) which meets \(\lambda_0^*\) transversely in a single point. It follows from Lemma 4.5 that \(\mu_0^*\) and \(\lambda_0^*\) meet transversely in two points, so \(\mu_0^* = 2\xi + k\lambda_0^*\). (We now confuse curves in \(T^*\) with their homology classes.)
Claim. \( H_1(Q^*_0) \cong \mathbb{Z} \) and is generated by \( \xi \).

Since \( \partial Q^*_0 \) is a torus, \( H_1(Q^*_0) \) is infinite. This fact, together with the exact sequence

\[
1 \longrightarrow \pi_1(Q_0) \overset{q_{*}}{\longrightarrow} \pi_1(Q^*_0) \overset{\rho}{\longrightarrow} \mathbb{Z}_2 \longrightarrow 1
\]

implies that

\[
q_{*}[\pi_1(Q_0), \pi_1(Q_0)] = [\pi_1(Q^*_0), \pi_1(Q^*_0)] .
\]

Hence we have the exact sequence \( 0 \rightarrow H_1(Q_0) \overset{q_{*}}{\longrightarrow} H_1(Q^*_0) \overset{\rho}{\longrightarrow} \mathbb{Z}_2 \rightarrow 0 \). So \( H_1(Q^*_0) \) is either \( \mathbb{Z} \) or \( \mathbb{Z} \oplus \mathbb{Z}_2 \). Suppose \( H_1(Q^*_0) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \) with generators \( \gamma, \delta \) for \( \mathbb{Z}, \mathbb{Z}_2 \), respectively. Then \( i_{*}(\xi) = m\gamma + n\delta \). So \( \gamma = i_{*}q_{*}(\mu^*_0) = i_{*}(\mu^*_0) = i_{*}(2\xi) = 2m\gamma + 2n\delta = 2m\gamma \), which is impossible. Thus \( H_1(Q^*_0) \cong \mathbb{Z} \) with generator \( \gamma \). Then \( i_{*}(\xi) = m\gamma \) and \( 2\gamma = i_{*}q_{*}(\mu^*_0) = i_{*}(\mu^*_0) = i_{*}(2\xi) = 2m\gamma \) implies \( m = 1 \). This establishes the claim.

Now choose a map \( f: Q^*_0 \rightarrow S^1 \) which realizes the epimorphism \( \pi_1(Q^*_0) \rightarrow \mathbb{Z} \). Modify \( f \) on \( \partial Q^*_0 \) so that \( (f|T^*)_\sim(p) = \lambda^*_0 \) for some point \( p \) in \( S^1 \). Using standard surgery techniques (as in Lemma 6.5 of [5]) modify \( f \) on \( \text{Int} Q^*_0 \) so that some component \( F^*_0 \) of \( f^{-1}(p) \) is an incompressible surface with \( \partial F^* = \lambda^*_0 \). Since \( \pi_1(F^*) \cong [\pi_1(Q^*_0), \pi_1(Q^*_0)] \cong q_{*} \pi_1(Q_0) \), \( f^{-1}(F^*) \) consists of two disjoint incompressible surfaces \( F^*_0 \) and \( F^*_1 \) which are interchanged by \( h \). Since \( \partial F^*_i \sim \lambda_0 \) in \( T \), the \( F^*_i \) are spanning surfaces for \( K_0 \) and so by assumption are isotopic. By Lemma 5.3 of [13] they cobound a product \( F \times [0, 1] \) in \( Q_0 \). Since \( Q_0 = (F \times [0, 1])\cup h(F \times [0, 1]) \) and \( (F \times [0, 1])\cap h(F \times [0, 1]) = F_0 \cup F'_1 \), \( K_0 \) is a fibered knot.

5. The examples.

**Theorem 5.1.** There is an infinite family of pairwise non-homeomorphic irreducible homology 3-spheres each of which admits no PL involutions.

**Proof.** To construct one such example, it is sufficient, by the results of the previous section, to find simple knots \( K_0 \) and \( K_1 \), other than torus knots, having non-homeomorphic exteriors, such that \( K_0 \) is non-amphicheiral, has a unique isotopy class of incompressible spanning surface, and is not fibered, and \( K_1 \) is non-invertible.

Let \( K_0 \) be a twist knot [8, p. 112] with \( q \) twists, \( q \leq -2 \). \( K_0 \) has bridge number 2 and so is simple [10]. \( K_0 \) has signature \(-2 \) and is therefore non-amphicheiral [8, p. 217]. \( K_0 \) has Alexander polynomial \( qt^2 - (2q + 1)t + q \) and is therefore nonfibered [8, p. 326]; so \( K_0 \) is not a torus knot. By Lyon [7] \( K_0 \) has a unique isotopy
Let $K_i$ be the $(3, 5, 7)$ pretzel knot [12]. $K_i$ has genus one and is therefore prime [9]. Since $K_i$ has bridge number 3 this implies [10] that $K_i$ is simple. Trotter [12] has shown that $K_i$ is non-invertible. $K_i$ has Alexander polynomial $18t^5 - 35t + 18$ and so is not a torus knot and has exterior not homeomorphic to that of $K_0$.

An infinite family of different examples is obtained by letting $K_0$ range over all twist knots with $q \leq -2$ twists. No two of these are homeomorphic since, by Lemma 3.1, any homeomorphism between $M(K_0, K_i)$ and $M(K'_0, K_i)$ could be deformed so that it carries $Q_0$ homeomorphically onto $Q'_0$. However, these are distinguished by the Alexander polynomials of $K_0$ and $K'_0$.

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