Pacific Journal of Mathematics

HOMOLOGY 3-SPHERES WHICH ADMIT NO PL INVOLUTIONS

JOHN R. MYERS

Vol. 94, No. 2 June 1981

PACIFIC JOURNAL OF MATHEMATICS
Vol. 94, No. 2, 1981

HOMOLOGY 3-SPHERES WHICH ADMIT NO PL INVOLUTIONS

ROBERT MYERS

An infinite family of irreducible homology 3-spheres is constructed, each member of which admits no PL involutions.

1. Introduction. In Problem 3.24 of [6] H. Hilden and J. Montesinos ask whether every homology 3-sphere is the double branched covering of a knot in S^3. The interest in this question lies in the fact that there is an algorithm, due to J. Birman and H. Hilden [1], for deciding whether such a 3-manifold is homeomorphic to S^3. In addition, the Smith Conjecture for homotopy 3-spheres [4] implies that every homotopy 3-sphere of this type must be homeomorphic to S^3.

In this paper an infinite family of irreducible homology 3-spheres is exhibited which admit no PL involutions. This gives a negative answer to the above question since the nontrivial covering translation of a branched double cover is a PL involution.

2. Preliminaries. We shall work throughout in the PL category.

A *knot* K is an oriented simple closed curve in the oriented 3-sphere S^3 which does not bound a disk. The *exterior* $Q = Q(K)$ is the closure of the complement of a regular neighborhood of K. A *meridian* $\mu = \mu(K)$ of K is an oriented simple closed curve in ∂Q which bounds a disk in $S^2 - \operatorname{Int} Q$ and has linking number $+1$ with K. A *longitude* $\lambda = \lambda(K)$ of K is an oriented simple closed curve in ∂Q such that λ bounds a surface in Q and $\lambda \sim K$ in $S^3 - \operatorname{Int} Q$. ("$\sim$" means "is homologous to").

K is $\pm$ *amphicheiral* if there is an orientation reversing homeomorphism g of S^3 such that $g(K) = \pm K$. K is *invertible* if there is an orientation preserving homeomorphism g of S^3 such that $g(K) = -K$.

For the definitions of simple knot, torus knot, and fibered knot we refer to [8]. For the definitions of irreducible 3-manifold, incompressible surface, and of parallel surfaces in a 3-manifold we refer to [5]. Note that a knot K is simple if and only if every incompressible torus in $Q(K)$ is parallel to $\partial Q(K)$. If K is simple and $Q(K)$ contains an incompressible annulus which is not parallel to an annulus in $\partial Q(K)$, then K is a torus knot [3].

Suppose h is an involution on a homology 3-sphere M. Then by Smith theory [2] the fixed point set Fix $\langle h \rangle$ is homeomorphic to S^0

or S^2 if h reverses orientation and is empty or homeomorphic to S^1 if h preserves orientation.

3. The construction. Let K_0 and K_1 be knots. Let $Q_i = Q(K_i)$, $\mu_i = \mu(K_i)$, and $\lambda_i = \lambda(K_i)$, $i = 0, 1$. We construct $M = M(K_0, K_1)$ by identifying ∂Q_0 and ∂Q_1 so that $\mu_0 = \lambda_1$ and $\lambda_0 = -\mu_1$. We denote $Q_0 \cap Q_1$ by T and μ_0, λ_0 by α, β, respectively. Note that M is an irreducible homology 3-sphere and that T is incompressible in M.

LEMMA 3.1. *If K_0 and K_1 are simple knots, other than torus knots, then every incompressible torus in $M(K_0, K_1)$ is isotopic to T.*

Proof. Let T' be an incompressible torus in M. Isotop T' so that T and T' are in general position and meet in a minimal number of components.

Suppose some component J of $T \cap T'$ bounds a disk D' in T'. We may assume $D' \cap T = \partial D'$. By the incompressibility of T, $\partial D' = \partial D$ for some disk D in T. By the irreducibility of M, $D \cup D'$ bounds a 3-cell B in M. So T' can be isotoped by pushing D' across B and off D to remove at least J from $T \cap T'$. This contradicts minimality and so cannot happen. A similar argument shows that no component of $T \cap T'$ bounds a disk in T.

Thus if $T \cap T' \neq \emptyset$, $T' \cap Q_i$ consists of incompressible annuli. Let A' be such an annulus in Q_0. Since K_0 is simple and not a torus knot, A' is parallel in Q_0 to an annulus A in T. Therefore T' can be isotoped by pushing A' across the solid torus bounded by $A \cup A'$ and off A to remove at least ∂A from $T \cap T'$. By minimality this cannot occur.

Thus T' lies in some Q_i. Since K_i is simple, T' is parallel to T and we are done.

4. Involutions on $M(K_0, K_1)$. An involution h on $M(K_0, K_1)$ is *good* if $h(Q_i) = Q_i$, $i = 0, 1$, Fix $\langle h \rangle$ and T are in general position, $h(\alpha) \sim \pm\alpha$, and $h(\beta) \sim \pm\beta$.

LEMMA 4.1. *Let K_0 and K_1 be simple knots, other than torus knots, such that Q_0 and Q_1 are not homeomorphic. Then every involution of $M(K_0, K_1)$ is conjugate to a good involution.*

Proof. By Theorem 1 of Tollefson [1] and Lemma 3.1 there is an isotopy f_t of M such that $f_0 = id$, $f_1(T)$ and Fix $\langle h \rangle$ are in general position, and either $h(f_1(T)) = f_1(T)$ or $h(f_1(T)) \cap f_1(T) = \emptyset$. Let $h' = f_1^{-1} \circ h \circ f_1$. Then either $h'(T) = T$ or $h'(T) \cap T = \emptyset$.

Suppose $h'(T) \cap T = \emptyset$. We may assume $h'(T) \subset \operatorname{Int} Q_0$. If

$h(Q_0) \subset \operatorname{Int} Q_0$, then $Q_0 = h^2(Q_0) \subset \operatorname{Int} h(Q_0) \subset \operatorname{Int} h^2(Q_0) = \operatorname{Int} Q_0$, which is absurd. Thus $Q_1 \subset \operatorname{Int} h(Q_0)$. But since ∂Q_1 is parallel to $\partial h(Q_0)$ in $h(Q_0)$, Q_0 and Q_1 are homeomorphic, a contradiction. Therefore $h'(T) = T$ and so $h'(Q_i) = Q_i$.

Finally $h(\alpha) = h(\mu_0) = h(\lambda_1) \sim \pm\lambda_1 = \pm\alpha$ and similarly $h(\beta) \sim \pm\beta$.

LEMMA 4.2. *Suppose K_0 is non-amphicheiral. Then every good involution on $M(K_0, K_1)$ is orientation preserving.*

Proof. $h(\beta) \sim \pm\beta$ implies that $h(\mu_0) \sim \pm\mu_0$ and thus that the orientation reversing homeomorphism $h|Q_0$ can be extended to an orientation reversing homeomorphism g of S^3 such that $g(K_0) = \pm K_0$, a contradiction.

LEMMA 4.3. *Suppose K_1 is non-invertible. If h is a good, orientation preserving invoution on $M(K_0, K_1)$, then* Fix $\langle h \rangle \cap T = \emptyset$.

Proof. Suppose not. Then Fix $\langle h \rangle$ is a simple closed curve meeting T transversely in finitely many points $x_1, \cdots, x_n$. Let T^* be the orbit space of T under $h|T$. The projection $q: T \to T^*$ is a 2-fold covering branched over $x_1^*, \cdots, x_n^*$, where $x_i^* = q(x_i)$. An Euler characteristic argument shows that T^* is a 2-sphere and $n = 4$.

Let γ^* and δ^* be arcs in T^* such that γ^* joins x_1^* and x_2^*, δ^* joins x_2^* and x_3^*, and each misses the other two branch points. Then $\gamma = q^{-1}(\gamma^*)$ and $\delta = q^{-1}(\delta^*)$ are simple closed curves meeting transversely in the single point x_2. After choosing orientations, γ and δ form a basis for $H_1(T)$. Moreover $h(\gamma) \sim -\gamma$ and $h(\delta) \sim -\delta$. It follows that $h(\mu_1) \sim -\mu_1$ and $h(\lambda_1) \sim -\lambda_1$. Then $h|Q_1$ can be extended to an orientation preserving homeomorphism g of S^3 such that $g(K_1) = -K_1$, a contradiction.

LEMMA 4.4. *Let h be an orientation preserving free involution on a torus T. Let $\alpha \cup \beta$ be a pair of simple closed curves in T which meet transversely in a single point. Then $\alpha \cup \beta$ can be isotoped so that either*

(i) $h(\alpha) = \alpha$ *and* $h(\beta) \cap \beta = \emptyset$, *or*

(ii) $h(\beta) = \beta$ *and* $h(\alpha) \cap \alpha = \emptyset$, *or*

(iii) $h(\alpha) \cap \alpha = \emptyset = h(\beta) \cap \beta$.

Proof. Note that h induces the identity on $H_1(T)$. Isotop $\alpha \cup \beta$ so that $h(\alpha) \cap \alpha$ is minimal.

Suppose $h(\alpha) \cap \alpha \neq \emptyset$. Since $h(\alpha) \sim \alpha$ there is a disk D in T with $\partial D = \gamma \cup \delta$, where γ and δ are arcs in α and $h(\alpha)$, respectively, and $(\alpha \cup h(\alpha)) \cap \operatorname{Int} D = \emptyset$. Suppose $h(D) \cap D = \emptyset$. Then α can be isotoped by pushing γ across D and off δ to obtain a new curve having four fewer intersection points with its image. This contradicts minimality and so does not occur. Suppose $h(D) \cap D$ is a single point p. Then α can be isotoped by pushing γ across D and off $\delta - p$ to obtain a curve having two fewer intersections with its image. So this cannot happen. Therefore $h(D) \cap D$ consists of two points p and q. In fact $h(\alpha) \cap \alpha = \{p, q\}$. Isotop α by pushing γ across D to δ. Then $h(\alpha) = \alpha$.

Now isotop β, keeping α pointwise fixed, so that $h(\alpha) \cap \beta$ is a single point. (This is only necessary if $h(\alpha) \cap \alpha = \emptyset$.) Then isotop β, keeping α and $h(\alpha)$ setwise fixed, so that $h(\beta) \cap \beta$ is minimal. As in the case of α above, the result will be that either $h(\beta) \cap \beta = \emptyset$ or that β can be isotoped so that $h(\beta) = \beta$. This can be done keeping α and $h(\alpha)$ setwise fixed because the analogous disk D used in the isotopies meets each of α and $h(\alpha)$ in at most a point of $\gamma \cap \delta$ or an arc with one endpoint in each of $\operatorname{Int}(\gamma)$ and $\operatorname{Int}(\delta)$.

LEMMA 4.5. *Let h be a good orientation preserving involution on $M(K_0, K_1)$ such that* $\operatorname{Fix}\langle h\rangle \cap T = \emptyset$. *Then* $\operatorname{Fix}\langle h\rangle = \emptyset$ *and $\alpha \cup \beta$ can be isotoped so that $h(\alpha) \cap \alpha = \emptyset = h(\beta) \cap \beta$.*

Proof. We may assume that $\alpha \cup \beta$ satisfies one of the three possible outcomes of Lemma 4.4. Suppose (i) is true. Then $h|Q_0$ can be extended to an involution g on S^3 with $K_0 \subset \operatorname{Fix}\langle g\rangle$. By Smith theory $K_0 = \operatorname{Fix}\langle g\rangle$. By the period two Smith Conjecture [14] K_0 is unknotted, a contradiction. A similar argument rules out (ii). Thus (iii) holds. If $\operatorname{Fix}\langle h\rangle \neq \emptyset$, then $\operatorname{Fix}\langle h\rangle \subset \operatorname{Int} Q_i$ for some i. Then the homology 3-sphere $M(K_i, K_i)$ admits an involution g with $\operatorname{Fix}\langle g\rangle$ homeomorphic to $S^1 \cup S^1$. This contradicts Smith theory, so $\operatorname{Fix}\langle h\rangle = \emptyset$.

LEMMA 4.6. *Suppose K_0 has a unique isotopy class of incompressible spanning surface. If h is a good, orientation preserving free involution on $M(K_0, K_1)$, then K_0 is a fibered knot.*

Proof. Let Q_0^* be the orbit space of Q_0 under h. Let $q\colon Q_0 \to Q_0^*$ be the quotient map and set $\mu_0^* = q(\mu_0)$, $\lambda_0^* = q(\lambda_0)$, and $T^* = q(T)$. Let $i\colon T^* \to Q_0^*$ be the inclusion map. Choose an oriented simple closed curve ξ which meets λ_0^* transversely in a single point. It follows from Lemma 4.5 that μ_0^* and λ_0^* meet transversely in two points, so $\mu_0^* = 2\xi + k\lambda_0^*$. (We now confuse curves in T^* with their homology classes.)

Claim. $H_1(Q_0^*) \cong Z$ and is generated by ξ.

Since ∂Q_0^* is a torus, $H_1(Q_0^*)$ is infinite. This fact, together with the exact sequence

$$1 \longrightarrow \pi_1(Q_0) \xrightarrow{q_*} \pi_1(Q_0^*) \xrightarrow{\rho} Z_2 \longrightarrow 1$$

implies that

$$q_*[\pi_1(Q_0), \pi_1(Q_0)] = [\pi_1(Q_0^*), \pi_1(Q_0^*)] .$$

Hence we have the exact sequence $0 \to H_1(Q_0) \xrightarrow{q_*} H_1(Q_0^*) \xrightarrow{\rho} Z_2 \to 0$. So $H_1(Q_0^*)$ is either Z or $Z \oplus Z_2$. Suppose $H_1(Q_0^*) \cong Z \oplus Z_2$ with generators γ, δ for Z, Z_2, respectively. Then $i_*(\xi) = m\gamma + n\delta$. So $\gamma = i_*q_*(\mu_0) = i_*(\mu_0^*) = i_*(2\xi) = 2m\gamma + 2n\delta = 2m\gamma$, which is impossible. Thus $H_1(Q_0^*) \cong Z$ with generator γ. Then $i_*(\xi) = m\gamma$ and $2\gamma = i_*q_*(\mu_0) = i_*(\mu_0^*) = i_*(2\xi) = 2m\gamma$ implies $m = 1$. This establishes the claim.

Now choose a map $f\colon Q_0^* \to S^1$ which realizes the epimorphism $\pi_1(Q_0^*) \to Z$. Modify f on ∂Q_0^* so that $(f \mid T^*)^{-1}(p) = \lambda_0^*$ for some point p in S^1. Using standard surgery techniques (as in Lemma 6.5 of [5]) modify f on $\operatorname{Int} Q_0^*$ so that some component F^* of $f^{-1}(p)$ is an incompressible surface with $\partial F^* = \lambda_0^*$. Since $\pi_1(F^*) \leqq [\pi_1(Q_0^*), \pi_1(Q_0^*)] \leqq q_*\pi_1(Q_0)$, $f^{-1}(F^*)$ consists of two disjoint incompressible surfaces F_0 and F_1 which are interchanged by h. Since $\partial F_i \sim \lambda_0$ in T, the F_i are spanning surfaces for K_0 and so by assumption are isotopic. By Lemma 5.3 of [13] they cobound a product $F \times [0, 1]$ in Q_0. Since $Q_0 = (F \times [0, 1]) \cup h(F \times [0, 1])$ and $(F \times [0, 1]) \cap h(F \times [0, 1]) = F_0 \cup F_1$, K_0 is a fibered knot.

5. The examples.

THEOREM 5.1. *There is an infinite family of pairwise non-homeomorphic irreducible homology* 3*-spheres each of which admits no* PL *involutions.*

Proof. To construct one such example, it is sufficient, by the results of the previous section, to find simple knots K_0 and K_1, other than torus knots, having non-homeomorphic exteriors, such that K_0 is non-amphicheiral, has a unique isotopy class of incompressible spanning surface, and is not fibered, and K_1 is non-invertible.

Let K_0 be a twist knot [8, p. 112] with q twists, $q \leqq -2$. K_0 has bridge number 2 and so is simple [10]. K_0 has signature -2 and is therefore non-amphicheiral [8, p. 217]. K_0 has Alexander polynomial $qt^2 - (2q + 1)t + q$ and is therefore nonfibered [8, p. 326]; so K_0 is not a torus knot. By Lyon [7] K_0 has a unique isotopy type of incompressible spanning surface.

Let K_1 be the (3, 5, 7) pretzel knot [12]. K_1 has genus one and is therefore prime [9]. Since K_1 has bridge number 3 this implies [10] that K_1 is simple. Trotter [12] has shown that K_1 is non-invertible. K_1 has Alexander polynomial $18t^2 - 35t + 18$ and so is not a torus knot and has exterior not homeomorphic to that of K_0.

An infinite family of different examples is obtained by letting K_0 range over all twist knots with $q \leqq -2$ twists. No two of these are homeomorphic since, by Lemma 3.1, any homeomorphism between $M(K_0, K_1)$ and $M(K_0', K_1)$ could be deformed so that it carries Q_0 homeomorphically onto Q_0'. However, these are distinguished by the Alexander polynomials of K_0 and K_0'.

REFERENCES

1. J. Birman and H. Hilden, *Heegaard splittings of branched coverings of* S^3, Trans. Amer. Math. Soc., **213** (1975), 315-352.

2. G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

3. C. D. Feustel, *On the torus theorem and its applications*, Trans. Amer. Math. Soc., **217** (1976), 1-43.

4. C. McA. Gordon and R. Litherland, *The Smith conjecture for homotopy* 3-*spheres*, Notices Amer. Math. Soc., **26** (1978), A-252.

5. J. Hempel, 3-*Manifolds*, Princeton University Press, 1976.

6. R. Kirby, *Problems in low dimensional manifold theory*, Algebraic and Geometric Topology (Proceedings of Symposia in Pure Mathematics, Volume XXXII Part 2), Amer. Math. Soc., (1978), 273-312.

7. H. C. Lyon, *Simple knots with unique spanning surfaces*, Topology, **13** (1974), 275-279.

8. D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley, 1976.

9. H. Schubert, *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl., **3** (1949), 57-104.

10. ———, *Über eine numerische Knoteninvariante*, Math. Z., **61** (1954), 245-288.

11. J. Tollefson, *Periodic homeomorphisms of* 3-*manifolds fibered over* S^1, Trans. Amer. Math. Soc., **223** (1976), 223-234.

12. H. F. Trotter, *Non-invertible knots exist*, Topology, **2** (1964), 275-280.

13. F. Waldhausen, *On irreducible* 3-*manifolds which are sufficiently large*, Ann. of Math., (2), **87** (1968), 56-88.

14. ———, *Über Involutionen der* 3-*Sphäre*, Topology, **8** (1969), 81-91.

Received February 6, 1980.

OKLAHOMA STATE UNIVERSITY
STILLWATER, OK 74078

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $102.00 a year (6 Vols., 12 issues). Special rate: $51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 94, No. 2 June, 1981