

Pacific Journal of Mathematics

**CARTAN SUBALGEBRAS OF A LIE ALGEBRA AND ITS
IDEALS. II**

DAVID J. WINTER

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Cartan subalgebras H of a Lie algebra L and Cartan subalgebras \hat{H} of its p -closure \bar{L} are related. This is used to prove that $I_0(\text{ad } H \cup I)$ is a Cartan subalgebra of I if $p=0$ or $(\text{ad}_I I)^p \subset \text{ad}_I I$, by reduction to the known case $(\text{ad } L)^p \subset \text{ad } L$.

In Winter [3], the following theorem is proved about a Lie algebra L with Cartan subalgebra H over a field of characteristic $p \geq 0$.

THEOREM 1. *Let I be an ideal of L . Then $I_0(\text{ad}(H \cap I))$ is a Cartan subalgebra of I if either $p = 0$ or $(\text{ad } L)^p \subset \text{ad } L$ and $(\text{ad}_I I)^p \subset \text{ad}_I I$.*

The purpose of this note is to relate Cartan subalgebras of L and those of the p -closure \bar{L} of L , in Theorem 2 below, and use this to show in Theorem 3 that the hypothesis $(\text{ad } L)^p \subset \text{ad } L$ in Theorem 1 can be dropped. This result is used in Winter [5].

We refer the reader to Jacobson [1] for preliminaries on Lie p -algebras (restricted Lie algebras) for $p > 0$.

THEOREM 2. *Let L be a subalgebra of a Lie p -algebra M and let H be a Cartan subalgebra of L . Let \bar{L} be the p -closure $\bar{L} = \sum_{p=0}^{\infty} L^{p^e}$ of L in M where L^{p^e} is the span of $\{x^{p^e} | x \in L\}$. Then*

- (1) *every ideal I of L is an ideal of \bar{L} and $[\bar{L}, \bar{I}] \subset I$;*
- (2) *$\bar{L} = \hat{H} + L$ for any Cartan subalgebra \hat{H} of \bar{L} ;*
- (3) *for any Cartan subalgebra H of L , \bar{L} has a Cartan subalgebra \hat{H} such that $[\hat{H}, H] \subset H$ and $\hat{H} \cap L \subset H$.*

Proof. Statements (1) and (2) are proved in Winter [2], §7.1. For (3), note that $T = \bar{H}^{p^\infty} = \bigcap_{e=0}^{\infty} \bar{H}^{p^e}$ is a torus and $L_0(\text{ad } T) = H$, as proved in Winter [4]. Letting \hat{T} be a maximal torus of \bar{L} containing T , and letting $\hat{H} = \bar{L}_0(\text{ad } \hat{T})$, \hat{H} is a Cartan subalgebra of \bar{L} , by Winter [4]. Since $T \subset \hat{T}$, we have $\hat{H} = \bar{L}_0(\text{ad } \hat{T}) \subset \bar{L}_0(\text{ad } T)$. Thus, \hat{H} normalizes $\bar{L}_0(\text{ad } T) \cap L = H$ in the sense that $[\hat{H}, H] \subset H$; and $\hat{H} \cap L \subset \bar{L}_0(\text{ad } T) \cap L = H$. \square

THEOREM 3. *Let I be an ideal of L and suppose that either $p=0$ or $(\text{ad}_I I)^p \subset \text{ad}_I I$. Then $H_I = I_0(\text{ad}(H \cap I))$ is a Cartan subalgebra of I .*

Proof. If, furthermore, $(\text{ad } L)^p \subset \text{ad } L$, this is Theorem 1. In order to bypass this additional assumption, let M be a Lie p -algebra containing L as subalgebra. Then, by Theorem 2, there is a Cartan subalgebra \hat{H} of \bar{L} such that $[\hat{H}, H] \subset H$ and $\hat{H} \cap L \subset H$, and $\bar{L} = \hat{H} + L$. By the Theorem 1, $\hat{H}_I = I_0(\text{ad}(\hat{H} \cap I))$ is a Cartan subalgebra of I . But $\hat{H}_I = I_0(\text{ad}(\hat{H} \cap L \cap I)) \supset I_0(\text{ad}(H \cap I)) = H_I$ since $\hat{H} \cap L \subset H$. Thus, $H_I \subset \hat{H}_I$ and H_I is nilpotent. Since $H_I = I_0(\text{ad}(H \cap I))$, H_I is also selfnormalizing in I and is therefore a Cartan subalgebra of I ; e.g., $x \in I$ and $[x, H_I] \subset H_I$ implies $x \in I_0(\text{ad}(H \cap I)) = H_I$. \square

Note that H_I in the above proof is also maximal nilpotent in I , so that $H_I = \hat{H}_I = I_0(\text{ad}(\hat{H} \cap I))$.

We can now consolidate and supplement some of our conclusions as follows.

THEOREM 4. *Let L be a subalgebra of a Lie p -algebra M , let H be a Cartan subalgebra of L and choose (using Theorem 2) a Cartan subalgebra \hat{H} of \bar{L} such that $[\hat{H}, H] \subset H$. Then*

- (1) $\bar{L} = \hat{H} + L$ and $\hat{H} \cap L \subset H$;
- (2) if I is an ideal of L and $p = 0$ or $(\text{ad}_I I)^p \subset \text{ad}_I I$, then $I_0(\text{ad } H \cap I)$, $I_0(\text{ad } \hat{H} \cap I)$ are equal and are Cartan subalgebras of I ;
- (3) if $p = 0$ or $(\text{ad}_L L)^p \subset \text{ad}_L L$, then $H = L_0(\text{ad } \hat{H} \cap L)$.

Proof. For (1), note that $\bar{L} = \hat{H} + L$ by Theorem 2 and $\hat{H} \subset \bar{L}_0(\text{ad } H)$ since $[\hat{H}, H] \subset H$, so that $\hat{H} \cap L \subset \bar{L}_0(\text{ad } H) \cap L = L_0(\text{ad } H) = H$. And (2) follows from the observation following Theorem 3. Finally, (3) follows from (2), taking $L = I$. \square

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