SPACES OF REPRESENTATIONS AND ENVELOPING L.M.C. *-ALGEBRAS

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Given a l.m.c. *-algebra $E$ with a b.a.i., the space of representations $\mathcal{R}(E)$ and the enveloping algebra $\mathcal{E}(E)$ of $E$ are defined. Under a suitable condition for the extreme points of $E$, $\mathcal{R}(E)$, $\mathcal{R}(\mathcal{E}(E))$ coincide topologically, a fact contributing to the openness of the map defining the topology of $\mathcal{R}(E)$. Furthermore, one gets $\mathcal{E}(E) = \lim_{\alpha \to \infty} \mathcal{E}(E_{\alpha})$, within a topological algebraic isomorphism, where $(E_{\alpha})$ is the inverse system of Banach algebras corresponding to $E$.

1. Introduction. There is a vast literature concerning representation theory of abstract Banach *-algebras (resp. $C^*$-algebras). On the other hand, due to recent considerations, it would be interesting and useful to have these results extended within the frame of (non-normed) topological *-algebras, a fact arising not only from the part of pure mathematics (e.g., function algebras), but also from that of applications in theoretical physics (quantum mechanics).

The present paper provides within the context of l.m.c. *-algebras, extensions of various results referred to Banach *-algebras (resp. $C^*$-algebras) representation theory. More specifically, if $E$ is a l.m.c. *-algebra with a b.a.i., $\mathcal{R}(E)$ will denote the non-zero extreme points of $\mathcal{P}(E)$ (continuous positive linear forms on $E$), and $\mathcal{R}(E)$ the equivalence classes of all continuous topologically irreducible representations of $E$. The set $\mathcal{R}(E)$ endowed with the final topology $\tau_{\mathcal{E}}$ induced on it by the map $\delta_E: \mathcal{B}(E) \to \mathcal{R}(E)$ (an extension of the classical "Gel' fand-Naimark-Segal map"; Th. 3.4) is called the space of representations of $E$. Thus, the paper is mainly concerned with the study of $\mathcal{R}(E)$ and the openness of the map $\delta_E$. To this study, the notion of the enveloping algebra $\mathcal{E}(E)$ of $E$ having by its definition the crucial $C^*$-property (Def. 4.1), plays an important role. Now, the openness of $\delta_{\mathcal{E}(E)}$, with $E$ a bQ l.m.c. *-algebra with a b.a.i. (Def. 4.2) is obtained, leading thus to the required openness of $\delta_E$ (Th. 4.2), based besides on the fact that the spaces $\mathcal{B}(E)$, $\mathcal{R}(E)$ coincide topologically with the corresponding ones of $\mathcal{E}(E)$, when $\mathcal{B}(\mathcal{E}(E))$ is locally equicontinuous (Th. 4.1).

Furthermore, $\mathcal{E}(E/N(p_{\alpha}))$, $\mathcal{E}(E_{\alpha})$ are isomorphic as topological algebras (Lemma 4.3) where $(E/N(p_{\alpha}))$, $(E_{\alpha})$ are the inverse systems...
of normed respectively Banach algebras corresponding to $E$ [1], a fact further applied to get an inverse limit decomposition of $\mathcal{A}(E)$ in terms of $(\mathcal{A}(E_a))$ (Th. 4.3).

2. Preliminaries. We introduce in this section the notation and terminology applied throughout.

A representation $\phi$ (or a *-representation) of a *-algebra $E$ is an involution preserving homomorphism of $E$ into the $C^*$-algebra $\mathcal{L}(H_{\phi})$ of all bounded linear operators on some Hilbert space $H_{\phi}$ (representation space of $E$).

A representation $\phi$ on a Hilbert space $H_{\phi}$ is topologically irreducible if $H_{\phi}$, $\{0\}$ are the only closed linear subspaces of $H_{\phi}$ left invariant by $\phi(E)$. Moreover, $\phi$ is called non-degenerate if $\{\phi(x)(\xi): x \in E, \xi \in H_{\phi}\}^\perp = H_{\phi}$; in that case $\phi$ is called cyclic. Now, the representations $\phi, \psi$ of $E$ are equivalent, we write $\phi \sim \psi$ (cf. [7]), if there exists a Hilbert space isomorphism $U: H_{\psi} \to H_{\phi}$ such that $\psi(x) \circ U = U \circ \phi(x)$, $x \in E$.

A positive linear form on a *-algebra $E$ is a complex linear form $f$ on $E$ with $f(x^*x) \geq 0$, $x \in E$. If $E$ has an identity $e$, then we also suppose that $f(e) = 1$. The set of positive linear forms on $E$ is denoted by $P(E)$.

A topological algebra $E$ (topological vector space with a separately continuous multiplication) is called locally $m$-convex (l.m.c.) if it has a local basis $\mathcal{U}$ consisting of $m$-barrels, (cf. [11] and [9; Chapt. 1, Th. 1.1]), where by an $m$-barrel we mean a subset of $E$ which is closed, convex, balanced, absorbing and idempotent. We may always suppose that such a local basis is directed.

Given a l.m.c. algebra $E$ with a directed local basis $\mathcal{U} = \{U_a, \alpha \in A\}$, $\{p_a, \alpha \in A\}$ denote the family of submultiplicative semi-norms (gauges) corresponding to $\mathcal{U}$. Then, $U_a = \{x \in E: p_a(x) \leq 1\}$, $\alpha \in A$, [9; Chapt. 1, Lemma 2.3].

Now, by a l.m.c. *-algebra we mean a l.m.c. algebra $E$ with an involution * such that $p_a(x^*) = p_a(x)$, $\alpha \in A, x \in E$ (cf. also [5; p.p. 6, 7]). If moreover, $p_a(x^*x) = p_a(x)^2$, $\alpha \in A, x \in E$, $E$ is called l.m.c. $C^*$-algebra. Note that if $E$ is a l.m.c. algebra with an involution * such that $p_a(x)^2 \leq p_a(x^*x)$, $\alpha \in A, x \in E$, $E$ is a l.m.c. $C^*$-algebra.

By a Fréchet l.m.c. *-algebra, we mean a l.m.c. *-algebra whose underlying locally convex space is Fréchet.

Furthermore, if $N(p_a) = \ker(p_a)$, $\alpha \in A$, $(E/N(p_a))$, $(E_a)$ denote the projective systems of normed and Banach *-algebras correspond-
ing to \( E \), where \( E_\alpha \) is the completion of \( E/N(p_\alpha) \), \( \alpha \in A \) (cf. \[1\], [11]). The topology of \( E_\alpha \) is defined by the norm \( \hat{p}_\alpha \), with \( \hat{p}_\alpha(x_\alpha) = p_\alpha(x) \), \( x_\alpha = \pi_\alpha(x) = x + N(p_\alpha) \in E/N(p_\alpha) \), \( \alpha \in A \), where \( \pi_\alpha \) is the quotient map of \( E \) onto \( E/N(p_\alpha) \). If \( E \) is a l.m.c. \( C^* \)-algebra, each \( E_\alpha, \alpha \in A \), is a \( C^* \)-algebra.

Now, \( E_\lambda \) will denote the respective unital l.m.c. \( * \)-algebra of \( E \), with corresponding family of semi-norms \( (p^\lambda_\alpha) \) and involution \( * \) defined respectively by \( p^\lambda_\alpha(x, \lambda) = p_\alpha(x) + |\lambda| \), \( (x, \lambda)^* = (x^*, \lambda) \), \( (x, \lambda) \in E_\lambda = E \oplus C \).

On the other hand, a bounded approximate identity \( (:b.a.i.) \) on \( E \) will be a net \( (e_\alpha)_\alpha \), with \( p_\alpha(e_\alpha) \leq 1 \), \( \alpha \in A, i \in I \) and \( \lim p_\alpha(e_\alpha x - x) = 0 = \lim p_\alpha(xe_\alpha - x) \), \( x \in E \), \( \alpha \in A \).

3. Space of representations of a l.m.c. \( * \)-algebra. Let \( E \) be a topological \( * \)-algebra (\( * \)-algebra, which is also topological). Then, by a continuous representation of \( E \) we shall mean a \( * \)-morphism \( \phi \) of \( E \) into \( \mathcal{L}(H_\phi) \), continuous relative to the uniform topology on \( \mathcal{L}(H_\phi) \). In the sequel, \( R(E) \) (resp. \( R'(E) \)) will denote the set of all continuous (resp. continuous, topologically irreducible) representations of \( E \). Note that "equivalence of representations" defines an equivalence relation "\( \sim \)" on \( R(E) \) (and hence on \( R'(E) \) too). In this respect, \( (\phi, \phi') \) in \( R(E) \times R'(E) \) with \( \phi \sim \phi' \) implies \( (\phi, \phi') \) in \( R'(E) \times R'(E) \).

Now, set \( B(E) = R'(E)/\sim \), and denote by \([\phi]\) the respective class of \( \phi \in R'(E) \) in \( B(E) \). In the rest of this section we work out the appropriate material for defining \( B(E) \) as a topological space.

Let \( E \) be a l.m.c. \( * \)-algebra, and \( E' \) its weak topological dual. Then, \( E'_s = \bigcup_a U_\alpha^s \), where \( U_\alpha^s \) is the polar of the neighborhood \( U_\alpha = \{x \in E: p_\alpha(x) \leq 1\} \), \( \alpha \in A \). Thus, if \( \mathcal{P}(E) \) denotes the set of all continuous positive linear forms on \( E \), and \( B(E) \) the non-zero extreme points of \( \mathcal{P}(E) \), we obtain

\[
\mathcal{P}(E) = \bigcup \mathcal{P}_\alpha(E), \quad B(E) = \bigcup \mathcal{B}_\alpha(E)
\]

with \( \mathcal{P}_\alpha(E) = \{f \in \mathcal{P}(E): |f(x)| \leq 1, x \in U_\alpha\} \) and \( \mathcal{B}_\alpha(E) \) the extreme points of \( \mathcal{P}_\alpha(E) \), \( \alpha \in A \). The preceding sets being subsets of \( E'_s \) are considered endowed with the relative topology; moreover, since \( \mathcal{P}_\alpha(E) = \mathcal{P}(E) \cap U_\alpha^s \subset U_\alpha^s \), \( \mathcal{P}_\alpha(E) \) (and therefore \( \mathcal{B}_\alpha(E) \)), \( \alpha \in A \) is an equicontinuous subset of \( \mathcal{P}(E) \).

Furthermore, note that a consequence of (3.1) and [9; Chapt. 1, Lemma 1.2] is that for each \( f \in \mathcal{P}(E) \) there exists \( \alpha \in A \) with \( |f(x)| \leq p_\alpha(x) \) for every \( x \in E \). The next theorem extends an analogous result of [5; Th. 4.1].
THEOREM 3.1. Let $E$ be a l.m.c. $^*$-algebra. Then, for each \( \alpha \in A \)

\[
\mathcal{P}(E/N(p_{\alpha})) = \mathcal{P}_a(E) = \mathcal{P}(E_{\alpha})
\]

within homeomorphisms.

Proof. Let \( \alpha \in A \) and \( \mathcal{P}_a(E) \) the corresponding subspace of \( \mathcal{P}(E) \). Then, for each \( f \in \mathcal{P}_a(E) \), \( N(p_{\alpha}) \subset N(f) \), so that we define \( f_{\alpha} \in \mathcal{P}(E/N(p_{\alpha})) \) by \( f_{\alpha}(x_{\alpha}) = f(x) \), \( x_{\alpha} \in E/N(p_{\alpha}) \), and we denote its extension to \( E_{\alpha} \) also by \( f_{\alpha} \). Thus, the map

\[
\mathcal{P}_a(E) \longrightarrow \mathcal{P}(E/N(p_{\alpha}))(\text{resp. } \mathcal{P}(E_{\alpha})): f \longmapsto f_{\alpha}
\]

is a homeomorphism, the continuity being a consequence of the equicontinuity of \( \mathcal{P}(E_{\alpha}) \), since then the weak topologies \( \sigma((E_{\alpha})', E/N(p_{\alpha})) \), \( \sigma((E_{\alpha})', E_{\alpha}) \) coincide on \( \mathcal{P}(E_{\alpha}) \), \( \alpha \in A \) [3; p. 23, Prop. 5]. \( \square \)

By Theorem 3.1 it is clear that \( \mathcal{P}(E_{\alpha}) \) consists of all continuous positive linear forms on \( E_{\alpha} \) with norm \( \leq 1 \).

COROLLARY 3.1. Let \( E \) be as in Theorem 3.1. Then, for each \( \alpha \in A \)

\[
\mathcal{B}(E/N(p_{\alpha})) = \mathcal{B}_a(E) = \mathcal{B}(E_{\alpha})
\]

within homeomorphisms. \( \square \)

LEMMA 3.2. Let \( E \) be a topological algebra with a b.a.i. \((e_i)_{i \in I}\) in \( I \). Then,

(i) If \( E \) has a continuous multiplication, \((e_i)_{i \in I}\) is a b.a.i.

for \( E \).

(ii) If \( E \) has a continuous involution \( ^* \), \((e_i^*)_{i \in I}\) is a b.a.i.

for \( E \).

(iii) If in particular \( E \) is a l.m.c. \( ^* \)-algebra, then \((e_i^*)_{i \in I} = (e_i + N(p_{\alpha}))_{i \in I} \alpha \in A \) is a b.a.i. for both \( E/N(p_{\alpha}) \) and \( E_{\alpha} \), \( \alpha \in A \).

Proof. For (i) cf. [9; Chapt. 6, Lemma 11.1]. (ii) \((e_i^*)_{i \in I}\) is a bounded net in \( E_{\alpha} \) since \( ^* \) is continuous. Moreover, for each \( x \in E \lim (e_i^* x - x) = \lim (x^* e_i - x^*) = 0^* = 0 \), and similarly \( \lim (x e_i^*) = x \), \( x \in E \). (iii) For each \( \alpha \in A \) define \( e_{\alpha}^* = \pi_{\alpha}(e_i) = e_i + N(p_{\alpha}) \), then \( \hat{p}_{\alpha}(e_{\alpha}^*) = p_{\alpha}(e_i) \leq 1 \), \( i \in I \), \( \alpha \in A \). Furthermore, \( \lim \hat{p}_{\alpha}(x_a e_{\alpha}^* - x_a) = \lim p_{\alpha}(x e_i - x) = 0 \), \( x_a \in E/N(p_{\alpha}) \), \( \alpha \in A \); by the same way \( x_a = \lim (e_{\alpha}^* x_a) \), \( x_a \in E/N(p_{\alpha}) \), \( \alpha \in A \). Hence, \((e_{\alpha}^*)_{i \in I}\) is a b.a.i. for \( E/N(p_{\alpha}) \), \( \alpha \in A \) while this net is also a b.a.i. for \( E_{\alpha} \), \( \alpha \in A \) (ibid.). \( \square \)

LEMMA 3.3. Let \( E \) be a l.m.c. \( ^* \)-algebra with a b.a.i. \((e_i)_{i \in I}\),
and \( f \in \mathcal{S}(E) \). Then,

(i) \( f(x^*) = \overline{f(x)} \), \( x \in E \) (i.e., \( f \) is real or hermitian).

(ii) \( |f(x^*)|^2 \leq \|f_a\|^2 \|f(x^*x)\|, \ x \in E \).

**Proof.** (i) \( f(x^*) = \lim_{i} f(e^*_i x) = [7; p. 27, (1)] \lim_{i} \overline{f(e_i^* x)} = (\text{Lemma } 3.2, \text{ (ii)}) \overline{f(x)}, \ a \in J^I \).

(ii) \( |f(x)|^2 = (\text{Lemma } 3.2, \text{ (ii)}) \lim_{i} \overline{f(e^*_i x)^2} \leq [7; p. 27, (2)] \lim_{i} f(e^*_i x)f(x^*x), \ x \in E \). Now, if \( f_a \) is the element of \( \mathcal{S}(E_a) \) defined by \( f \) as in Theorem 3.1, \( \lim_{i} f(e^*_i x) = (\text{Lemma } 3.2, \text{ (iii)}) \lim_{i} f_a((e^*_i)^* e_i) = [7; \text{Prop. } 2.1.5, \text{ (v)}] \|f_a\|. \) Actually, \( \|f_a\| \leq 1 \), since \( |f_a(x_a)| = |f(x)| \leq 1, \ x \in U_a \).

The above assertion (i) is actually valid for any topological algebra with continuous involution and a not necessarily bounded a.i. Every element \( f \in \mathcal{S}(E) \) satisfying conditions (i), (ii) of Lemma 3.3 is called extendable.

**PROPOSITION 3.4.** Let \( E \) be a l.m.c. \(*\)-algebra with a b.a.i. \((e_i)_{i \in I}\). Then,

(i) Each \( f \in \mathcal{S}(E) \) is uniquely extended to an element \( f_i \in \mathcal{S}(E_i) \) with \( f_i(0, 1) = \|f_a\| \), where \((0, 1)\) denotes the identity element of \( E_i \).

(ii) Each element of \( \mathcal{S}(E_i) \) extending \( f \) bounds \( f_i \).

(iii) If \( Q(E_i) = \{h \in \mathcal{S}(E_i): h(0, 1) = \|h_{|E_i}\|\} \) and an element of \( \mathcal{S}(E_i) \) is bounded by an element of \( Q(E_i) \), it must itself belong to \( Q(E_i) \).

(iv) \( f \in \mathcal{B}(E) \iff f_i \in \mathcal{B}(E_i) \iff \mathcal{f}_i \in \mathcal{B}(E_i) \), where \( E_i \) is the completion of \( E_i \) and \( f_i \) the extension of \( f \) to \( E_i \).

**Proof.** (i) For each \( f \in \mathcal{S}(E) \) define \( f_i: E_i \rightarrow C: (x, \lambda) \mapsto f_i(x, \lambda) = f(x) + \lambda \|f_a\| \), where \( f_a \in \mathcal{S}(E_a) \) (cf. Th. 3.1). Then, \( f_i \in \mathcal{S}(E_i) \) with \( f_i(0, 1) = \|f_a\| \). Moreover, \( |f_i(x, \lambda)| \leq |f(x)| + |\lambda| \leq p_a(x) + |\lambda| = p_a(x, \lambda), \ (x, \lambda) \in E_i \), hence \( f_i \in \mathcal{S}(E_i) \).

(ii) Suppose that \( g \in \mathcal{S}(E_i) \) extends \( f \in \mathcal{S}(E) \). Then, there exists \( \gamma \in A \) with \( g \in \mathcal{S}(E_i) \) and \( f \in \mathcal{S}(E_i) \), hence \( \|g_f\| \geq \|f_i\| \) which yields \( g \geq f_i \).

(iii) Let \( g = h + k \) with \( g \in Q(E_i) \) and \( h, k \in \mathcal{S}(E_i) \). Then, \( g \geq h, k \) and \( h + k = g = (g|_{E_i})_1 = (h|_{E_i})_1 + (k|_{E_i})_1 \). Moreover, \( h(0, 1) \geq (h|_{E_i})(0, 1), \ k(0, 1) \geq (k|_{E_i})(0, 1) \), which implies \( h(0, 1) = (h|_{E_i})(0, 1), \ k(0, 1) = (k|_{E_i})(0, 1) \), that is \( h, k \in Q(E_i) \).

(iv) Let \( f \in \mathcal{B}(E) \) and \( g \in \mathcal{S}(E_i) \) with \( f_i \geq g \). Then, \( f \geq g|_E \), i.e., \( g|_E = \lambda f_i, \ \lambda \in [0, 1] \) and since \( g(0, 1) = \lambda f_i(0, 1) \) by (iii), we conclude \( g = \lambda f_i, \ \lambda \in [0, 1] \).

Conversely, let \( f \in \mathcal{S}(E) \) with \( f_i \in \mathcal{B}(E_i) \) and \( g \in \mathcal{S}(E) \) such
that \( f \geq g \). Then, \( f - g \in \mathcal{P}(E) \), so that \( (f - g) \lambda = f \lambda - g \lambda \in \mathcal{P}(E \lambda) \), i.e., \( f \lambda \geq g \lambda , g \lambda \in \mathcal{P}(E \lambda) \); but then, \( g \lambda = \lambda f \lambda , \lambda \in [0, 1] \), hence also \( g = \lambda f \lambda , \lambda \in [0, 1] \). The second equivalence of (iv) is clear.

\[ \square \]

**Remark 3.4.** For \( E \) as in Proposition 3.4 and \( \phi \in \mathcal{R}(E) \) we define \( \phi_i : E_i \rightarrow \mathcal{C}(H) : (x, \lambda) \mapsto \phi_i(x, \lambda) = \phi(x) + \lambda id_{H_i} \). Then, \( \phi_i \in \mathcal{R}(E \lambda_i) \) and particularly \( \phi \in \mathcal{R}'(E) \Leftrightarrow \phi_i \in \mathcal{R}'(E \lambda_i) \Rightarrow \tilde{\phi_i} \in \mathcal{R}'(\tilde{E_i}) \), where \( \tilde{\phi_i} \) is the extension of \( \phi_i \) to \( \tilde{E_i} \).

Now, if \( f, \tilde{f_i} \) are as in Proposition 3.4, \( L\tilde{f_i} = \{ z \in \tilde{E_i} : \tilde{f_i}(z^*z) = 0 \} \) is a left ideal of \( \tilde{E_i} \) and \( H_i = \tilde{E_i}/L\tilde{f_i} \) is a pre-Hilbert space with inner product \( \langle z + L\tilde{f_i}, w + L\tilde{f_i} \rangle = \tilde{f_i}(w^*z) \), \( w, z \in \tilde{E_i} \). Denote by \( H \) the respective Hilbert space, completion of \( H_i \). Then, one obtains \( \overline{E_i/L\tilde{f_i}} = E_i/L\tilde{f_i} \) since \( \| (e_i, 0) + L\tilde{f_i} - (0, 1) + L\tilde{f_i} \|^2 = f_i((e_i, -1)^*(e_i, -1)) = f(e_i^*e_i) - f(e_i) - \tilde{f}(e_i) + ||f_a|| \rightarrow 0 \) (cf. proof of Lemma 3.3 and note that \( \lim f(e_i) = (\text{Th. } 3.1, \text{ Lemma } 3.2) \lim f_a(e_a^i) = [7; \text{Prop. } 2.1.5, (v)] ||f_a|| \).

On the other hand, \( \overline{E_i/L\tilde{f_i}} = H_i \), hence one finally obtains

\[ (3.2) \quad E_i/L\tilde{f_i} = H_i . \]

In this respect, the following extends [5; Th. 6.1], being actually the analogue in our case of the standard Gel'fand-Naimark-Segal construction.

**Theorem 3.4.** Let \( E \) be a l.m.c. \(*\)-algebra with a b.a.i., and \( f \in \mathcal{P}(E) \). Then, there exists a continuous representation \( \phi_f \) of \( E \) and a cyclic vector \( \xi_f \) of \( \phi_f \) such that \( f(x) = \langle \phi_f(x)(\xi_f), \xi_f \rangle , x \in E \).

**Proof.** For each \( f \in \mathcal{P}(E) \), \( \tilde{f_i} \) belongs to \( \mathcal{P}(\tilde{E_i}) \) (Prop. 3.4), so that [5; Th. 6.1] there exists a continuous representation \( \phi_{\tilde{f_i}} \) of \( \tilde{E_i} \) into \( \mathcal{C}(H) \) and a cyclic vector \( \xi_{\tilde{f_i}} \) of \( \phi_{\tilde{f_i}} \) in \( H \) such that

\[ \tilde{f_i}(z) = \langle \phi_{\tilde{f_i}}(z)(\xi_{\tilde{f_i}}), \xi_{\tilde{f_i}} \rangle , z \in \tilde{E_i} . \]

Thus, if \( \phi_f = \phi_{\tilde{f_i}}|_E \) and \( \xi_f = \xi_{\tilde{f_i}} \in H \), one obtains

\[ f(x) = \langle \phi_f(x)(\xi_f), \xi_f \rangle , x \in E , \]

where \( \xi_f \) is cyclic for \( \phi_f \) as this follows by (3.2) and \( \phi(E)(\xi_f) = E/L\tilde{f_i} \).
Now, given a l.m.c. *-algebra $E$ let, for each $\alpha \in A$

\[(3.3) \quad R_\alpha(E) = \{ \phi \in R(E) : \| \phi(x) \| \leq kp_\alpha(x), \ x \in E \}, \ k > 0 , \]

so that $R(E) = \bigcup_\alpha R_\alpha(E)$. Thus, we can define $\phi_\alpha \in R(E/N(p_\alpha))$ with $\phi_\alpha(x_\alpha) = \phi(x), \ x_\alpha \in E/N(p_\alpha)$, so that if $\phi_\alpha$ denotes also the extension of $\phi_\alpha$ to $E_\alpha$, one has $\| \phi_\alpha(z) \| \leq \dot{p}_\alpha(z), \ z \in E_\alpha$ [7; Prop. 1.8.7]; hence $\| \phi(x) \| \leq p_\alpha(x), \ x \in E$ in such a way that one may assume $k \leq 1$ in (3.3), for each $\phi \in R_\alpha(E)$. Besides, if $R'_\alpha(E) = \{ \phi \in R'(E) : \phi \in R_\alpha(E) \}$ and $\mathcal{B}_\alpha(E) = R'_\alpha(E)/\sim$, we get

\[(3.4) \quad R(E) = \lim_{\alpha \rightarrow \infty} R_\alpha(E), \ R'(E) = \lim_{\alpha \rightarrow \infty} R'_\alpha(E), \ \mathcal{R}(E) = \lim_{\alpha \rightarrow \infty} \mathcal{B}_\alpha(E) , \]

within bijections [4; p. 92].

Now, if $\phi_\alpha \in R'(E_\alpha)$ and $M$ is a closed linear subspace of $H_{\varphi}(=H_{\varphi_\alpha})$ with $\phi(E)(M) \subset M$, then $\phi_\alpha(E_\alpha)(M) \subset M$. Hence, $\phi \in R_\alpha(E) \Rightarrow \phi_\alpha \in R'(E/N(p_\alpha))$ (resp. $R'(E_\alpha)$). Finally, notice that $\phi \sim \psi$ in $R'_\alpha(E)$ implies $\phi_\alpha \sim \psi_\alpha$ in $R'(E_\alpha)$. The above yields the following

**Proposition 3.5.** Let $E$ be a l.m.c. *-algebra. Then,

(i) $R(E/N(p_\alpha)) = R_\alpha(E) = R(E_\alpha), \ \alpha \in A,

(ii) $R'(E/N(p_\alpha)) = R'_\alpha(E) = R'(E_\alpha), \ \alpha \in A,

(iii) $\mathcal{B}(E/N(p_\alpha)) = \mathcal{B}_\alpha(E) = \mathcal{B}(E_\alpha), \ \alpha \in A, \ \text{within bijections.} \square$

The following Banach *-algebras analogue [7; Prop. 2.5.4] extends also Corollary 6.4 of [5].

**Proposition 3.6.** Let $E$ be a l.m.c. *-algebra with a b.a.i. Let also $f \in \mathcal{B}(E)$ and $\phi_f$ the respective element of $R(E)$ (cf. Th. 3.4). Then, $f \in \mathcal{B}(E) \Rightarrow \phi_f \in R'(E)$.

**Proof.** $f \in \mathcal{B}(E)$ implies $\tilde{f}_1 \in \mathcal{B}(E_1)$ (Prop. 3.4, (iv)), so that [5; Cor. 6.4] $\phi_{\tilde{f}_1} \in R'(E_1)$, which implies $\phi_{\tilde{f}_1} = \phi_{\tilde{f}_1}|_{E_1} \in R'(E_1)$ and since $\phi_{\tilde{f}_1} = (\phi_f)_1, \ \phi_f \in R'(E)$ by Rem. 3.4.

Conversely, let $f \in \mathcal{B}(E)$ with $\phi_f \in R'(E)$. Then, $\phi_{\tilde{f}_1} = (\phi_f)_1 \in R'(E_1)$ (Remark 3.4), so that $\phi_{\tilde{f}_1} \in R'(E_1)$, which yields $\tilde{f}_1 \in \mathcal{B}(E_1)$ [5; Cor. 6.4]; hence $f \in \mathcal{B}(E)$ by Proposition 3.4, (iv). \square

Furthermore, one gets the next (cf. also [7; Prop. 2.4.1, (ii)].

**Lemma 3.7.** Let $E$ be a *-algebra and $\phi, \psi$ representations of $E$ into $\mathcal{L}(H_\phi), \mathcal{L}(H_\psi)$ respectively. Let also $\xi$ (resp. $\eta$) be a cyclic vector of $\phi$ (resp. $\psi$), with $\langle \phi(x)(\xi), \xi \rangle = \langle \psi(x)(\eta), \eta \rangle, \ x \in E$. Then, $\phi \sim \psi$ such that there exists a Hilbert space isomorphism $U: H_\phi \rightarrow H_\psi$.
with $U \phi(x) = \psi(x) \circ U$, $x \in E$ and $U(\xi) = \eta$.

Now, regarding Proposition 3.6 we notice that for each $\phi \in R'(E)$ there exists $f \in B(E)$ such that $\phi \sim \phi_f$: Indeed, if $\xi$ is a cyclic vector of $\phi$, the formula $f(x) = \langle \phi(x)(\xi), \xi \rangle$, $x \in E$ defines an element $f$ of $P(E)$. Hence, (Th. 3.4) there exists $\phi_f \in R(E)$ and a cyclic vector $\xi_f$ of $\phi_f$ with $f(x) = \langle \phi_f(x)(\xi_f), \xi_f \rangle$, $x \in E$, so that (Lemma 3.7) $\phi \sim \phi_f$ in $R(E)$, i.e., $\phi_f \in R'(E)$, which by Proposition 3.6 implies $f \in B(E)$. Hence, by Theorem 3.4 and Proposition 3.6 we now define an onto map

$$
(3.5) \quad \delta_E: B(E) \longrightarrow R(E): f \longmapsto \delta_E(f) = [\phi_f].
$$

The set $R(E)$ equipped with the final topology $\tau_{\delta_E}$ induced on it by $\delta_E$, is called the space of representations of $E$.

In the next § 4, under additional conditions for $E$ we prove the openness of the map (3.5).

4. Enveloping algebra of a l.m.c. *-algebra. We define below the enveloping algebra $E(E)$ of a l.m.c. *-algebra $E$ with a b.a.i. It is proved that the representation theory of $E$ is actually reduced to that of $E(E)$ (Th. 4.1), the last algebra having the important "$C^*$-property", hence its significance for the latter theory. On the other hand, by further obtaining under appropriate conditions the openness of the map $\delta_{E(E)}$, we finally get the same property for the map (3.5) (Th. 4.2). Further applications, concerning topological tensor product algebras, will be given elsewhere.

**Lemma 4.1.** Let $E$ be a l.m.c. *-algebra with a b.a.i. Then, for any $x \in E$ and $\alpha \in A$, the following hold true:

(i) $a = b = c = d$, where

$$
a = \sup \{\|\phi(x)\|: \phi \in R_\alpha(E)\}, \quad b = \sup \{\|\phi(x)\|: \phi \in R_\alpha'(E)\},
$$

$$
c = (\sup \{f(x^*x): f \in P_\alpha(E)\})^{1/2}, \quad d = (\sup \{f(x^*x): f \in B_\alpha(E)\})^{1/2},
$$

$x \in E$.

(ii) For each $\alpha \in A$, the map $r_\alpha: E \rightarrow R^+: x \mapsto r_\alpha(x) = d$, defines a submultiplicative semi-norm on $E$, which is *-preserving and has the $C^*$-property.

**Proof.** The proof is an immediate consequence of [7; Prop. 2.7.1] since by Theorem 3.1, Corollary 3.1 and Proposition 3.5, one concludes that
\[ a = \sup \{ \| \phi_a(x_a) \|: \phi_a \in R(E_a) \}, \quad b = \sup \{ \| \phi_a(x_a) \|: \phi_a \in R'(E_a) \}, \]

\[ c = (\sup \{ f_a(x_a^*x_a): f_a \in \mathcal{P}(E_a) \})^{1/2}, \quad d = (\sup \{ f_a(x_a^*x_a): f_a \in \mathcal{B}(E_a) \})^{1/2}. \]

Regarding Lemma 4.1, note that \( b \) also coincides with 
\[ \sup \{ \| \phi(x) \|: [\phi] \in \mathcal{B}_a(E) \}. \]

Furthermore, since \( \| \phi(x) \| \leq p_a(x), \ x \in E \) for each \( \phi \in R_a(E) \), one obtains \( r_a(x) \leq p_a(x) \) for any \( \alpha \in A, \ x \in E \), that is each \( r_a(\alpha \in A) \) is continuous with respect to the given topology of \( E \).

**Definition 4.1.** Let \( E \) be a l.m.c. *-algebra with a b.a.i., and \((E, (r_a))\) the respective l.m.c. C*-algebra defined by Lemma 4.1. Then, the "Hausdorff completion" of the latter, that is the algebra 
\[(4.1) \quad \mathcal{E}(E) = (E, (r_a))/I \]
with \( I = \cap \{ N(r_a): \alpha \in A \} \) a closed 2-sided self-adjoint ideal of \( E \), is called the enveloping algebra of \( E \).

In this regard, cf. also [6; p. 65] concerning Fréchet l.m.c. *-algebras with identity. It is clear that (4.1) provides a complete l.m.c. C*-algebra, whose topology is defined by the family \((\bar{q}_a)\) of submultiplicative semi-norms, extensions of \( q_a, \alpha \in A \) to \( \mathcal{E}(E) \), where 
\[ q_a(x + I) = \inf \{ r_a(x + i): i \in I \}, \ x + I \in (E, (r_a))/I. \]
Moreover, if \((e_j)\) is a b.a.i. for \( E \), the net \((e_j + I)\) is a b.a.i. for \( \mathcal{E}(E) \).

**Remark 4.1.** A given l.m.c. *-algebra \( E \) with a b.a.i. has the C*-property iff \( r_a = p_a \) for each \( \alpha \in A \), that is one has then \( p_a(x) \leq r_a(x), \ \alpha \in A, \ x \in E \). In fact, since \( E \) has the C*-property, each \( E_a \) is a C*-algebra, therefore \( E_a, \ \alpha \in A \) has an isometric representation, say \( \phi_a \), that is \( \| \phi_a(z) \| = \hat{p}_a(z), \ z \in E_a \) (cf. [7; Th. 2.6.1]). But then, \( \| \phi(x) \| = p_a(x), \ x \in E \) with \( \phi \in R_a(E) \) (Prop. 3.5).

Now, it is clear that every complete l.m.c. C*-algebra coincides with its enveloping algebra. In the sequel \( E/I \) will stand for \((E, (r_a))/I\).

**Theorem 4.1.** Let \( E \) be a l.m.c. *-algebra with a b.a.i., and \( \mathcal{E}(E) \) its enveloping algebra with \( \mathcal{B}(\mathcal{E}(E)) \) locally equicontinuous. Then, \( \mathcal{B}(E) = \mathcal{B}(\mathcal{E}(E)) \) and \( \mathcal{E}(E) = \mathcal{B}(\mathcal{E}(E)) \) within homeomorphisms.

**Proof.** If \( f \in \mathcal{B}(E) \) there exists \( \alpha \in A \) with \( f \in \mathcal{B}_a(E) \) and \( |f(x)| \leq r_a(x), \ x \in E \) (Lemma 3.3, (ii)). Thus, we define \( g \in \mathcal{B}(E/I) \)
with \( g(x + I) = f(x), \) \( x + I \in E/I. \) Denoting also by \( g \) the respective element of \( \mathcal{B}(\mathcal{C}(E)) \) we have \( g \in \mathcal{B}(\mathcal{C}(E)) \Rightarrow f \in \mathcal{B}(E). \) Now, the map \( \Psi: \mathcal{B}(\mathcal{C}(E)) \to \mathcal{B}(E): g \mapsto \Psi(g) = f \) with \( f = g \circ \tau, \) where \( \tau: E \to \mathcal{C}(E) \) is the canonical continuous morphism (Def. 4.1), is a continuous bijection. Moreover, the inverse of \( \Psi \) is certainly continuous for the weak topology induced on its range by \( E/I. \) On the other hand, let \( V \) be a neighborhood of \( g \) in \( \mathcal{B}(\mathcal{C}(E)) \) which we may always assume to be equicontinuous by hypothesis. Then, the weak topologies on \( V \) from \( E/I \) and \( \tilde{E}/I = \mathcal{C}(E) \) coincide [3; p. 23, Prop. 5], which proves the continuity of \( \Psi^{-1}. \)

Now, if \( \phi \in R(E), \) there exists \( \alpha \in A \) with \( \phi \in R_\alpha(E) \) and \( N(\tau_\alpha) \subset N(\phi), \) so that one gets \( \phi' \in R(E/I) \) with \( \phi'(x + I) = \phi(x), \) \( x + I \in E/I. \) Thus, preserving the same symbol for the extension of \( \phi \) to \( \mathcal{C}(E) \) we have \( \phi' \in R(\mathcal{C}(E)) \Leftrightarrow \phi \in R'(E), \) so that the map \( r: \mathcal{B}(\mathcal{C}(E)) \to \mathcal{B}(E): [\phi'] \mapsto r([\phi']) = [\phi] \) with \( \phi = \phi' \circ \tau, \) is a homeomorphism as this follows by the relation \( r \circ \delta_{\phi(E)} = \delta_{\phi'} \circ \Psi, \) since \( \delta_{\phi}, \Psi \) are continuous and \( \mathcal{B}(\mathcal{C}(E)) \) has the final topology induced on it by \( \delta_{\phi(E)}, \) an analogous argument being valid for the inverse of \( r. \)

Concerning the above theorem, we note that \( \Psi, r \) are always continuous bijections. Moreover, an element \( \phi \in R(E) \) is non-degenerate iff the element \( \phi' \in \mathcal{B}(\mathcal{C}(E)) \) is non-degenerate, and for any \( (\phi, \phi') \in R(E) \times R(\mathcal{C}(E)) \) the set \( \phi(E) \) is dense in \( \phi'(\mathcal{C}(E)). \)

Regarding the local equicontinuity of \( \mathcal{B}(\mathcal{C}(E)) \) we note that this, is equivalent with that of \( \mathcal{B}(E) \) when for instance, \( \mathcal{C}(E) \) is barrelled (cf., for example, [9; Chapt. III, Cor. 5.31]). In this respect (cf. also Def. 4.2 below as well as the comments following it. 

Now, a topological algebra \( E \) is said to be a \( Q \)-algebra, if the set of its quasi-regular elements is open. If \( E \) is a \( Q \)-algebra, the same holds also true for its respective unital algebra \( E, \) [12; p. 174, I].

**DEFINITION 4.2.** A l.m.c. *-algebra \( E \) with a b.a.i., whose enveloping algebra \( \mathcal{C}(E) \) is barrelled (l.m.c.) \( Q \)-algebra, is called a \( bQ \) l.m.c. *-algebra.

In case \( E \) is a Fréchet l.m.c. *-algebra, \( \mathcal{C}(E) \) is by its definition Fréchet and thus barrelled. However, we still assume that \( \mathcal{C}(E) \) is a \( Q \)-algebra to have the situation provided by Theorem 3 of [8], hence its application to the next result.

**THEOREM 4.2.** Let \( E \) be a \( bQ \) l.m.c. *-algebra with a b.a.i. Then,
δ_E: \mathcal{B}(E) \longrightarrow \mathcal{B}(E)
is a (continuous) open map.

**Proof.** Clearly δ_E is continuous by the definition of the final topology \( \tau_{\delta_E} \) on \( \mathcal{B}(E) \). Now, by [8; Th. 3] \( \mathcal{E}(E) \) is a C*-algebra (cf. also [13; Cor. 5]), and since \( \mathcal{E}(E) \subset \mathcal{E}(E) \) (⊂ means topological algebraic imbedding) \( \mathcal{E}(E) \) becomes also a C*-algebra, so that \( \mathcal{B}(\mathcal{E}(E)) \) is equicontinuous, and \( \delta_{\mathcal{E}(E)} \) open by [7; Th. 3.4.11]. Thus the assertion follows by Theorem 4.1 and the relation \( \delta_E = r_{\delta E} \circ \psi^{-1} \).

In the rest of this section we relate \( \mathcal{E}(E) \) with the decomposition of \( E \) as an inverse limit of Banach algebras [1], [11]. Namely, we give \( \mathcal{E}(E) \) (Th. 4.3) as an inverse limit of the C*-algebras \( \mathcal{E}(E_a), \alpha \in A, \) which are the enveloping algebras of the Banach algebras \( E_a, \alpha \in A, \) corresponding to \( E \). However, we still need the following.

**Lemma 4.3.** Let \( E \) be a l.m.c. *-algebra with a b.a.i. Then,

(4.2) \[ \mathcal{E}(E_a) = \mathcal{E}(E/\mathbb{N}(p_a)) = (E/I)_a = \mathcal{E}(E_a), \alpha \in A, \]
within topological algebraic isomorphisms.

**Proof.** By Definition 4.1 \( \mathcal{E}(E/\mathbb{N}(p_a)) = (E/\mathbb{N}(p_a), t_a)/I_a \) with \( t_a(x_a) = \sup \{ ||\phi_a(x_a) || : \phi_a \in \mathcal{B}(E/\mathbb{N}(p_a)) \} = r_a(x), \ x_a \in E/\mathbb{N}(p_a), \alpha \in A \) (cf. Prop. 3.5 and Lemma 4.1) and \( I_a = N(t_a) \). Moreover, \( t_a \leq p_a, \alpha \in A \), hence \( t_a \) has a unique extension \( \tilde{t}_a \) to \( E_a, \alpha \in A \), so that if \( \tilde{I}_a = N(\tilde{t}_a), \mathcal{E}(E_a) = \overline{\mathcal{E}(E_a, \tilde{t}_a)/\tilde{I}_a}, \alpha \in A \). Now, for \( F_a = (E/\mathbb{N}(p_a), t_a)/I_a \) and \( G_a = (E_a, \tilde{t}_a)/\tilde{I}_a, \alpha \in A, \) consider the map

\[ h_a: F_a \longrightarrow G_a: x_a + I_a \mapsto x_a + \tilde{I}_a, \alpha \in A, \]
which is an algebraic isomorphism into. Then, if \( Q_a, \tilde{Q}_a, \alpha \in A, \) are the norms defining the quotient topologies of \( F_a, G_a, \alpha \in A \) respectively, one gets

\[ Q_a(x_a + I_a) = t_a(x_a) = \tilde{Q}_a(x_a + \tilde{I}_a), \ x_a \in E/\mathbb{N}(p_a), \alpha \in A, \]
which yields \( h_a, \alpha \in A, \) as a topological isomorphism too. Now, since by \( t_a \leq p_a \) Im(\( h_a \)) is dense in \( G_a, \alpha \in A, \) one obtains the first part of the assertion. The last part of the statement is similarly proved. Concerning the 2nd equality in (4.2), if \( M_a = (E/I)/\mathbb{N}(q_a), \alpha \in A, \)
the map

\[ k_a: M_a \longrightarrow F_a: (x + I)_a \mapsto x_a + I_a, \alpha \in A, \]

is a representation of \( E_a, \alpha \in A, \) as a C*-algebra.
is an algebraic isomorphism. In fact, \( k_\alpha, \alpha \in A \) is a topological isomorphism: Namely, \( Q_\alpha(x_\alpha + I_\alpha) \leq \hat{q}_\alpha((x + I)_\alpha) \), which yields the continuity of \( k_\alpha \). Besides, \( k_\alpha^{-1} \) is continuous iff \( \rho: (E/N(p_\alpha), t_\alpha) \to M_\alpha; x_\alpha \mapsto (x + I)_\alpha \) is continuous, which is true since \( \hat{q}_\alpha(\rho(x_\alpha)) \leq r_\alpha(x) = t_\alpha(x_\alpha), \ x_\alpha \in E/N(p_\alpha), \ (\alpha \in A) \).

**Theorem 4.3.** If \( E \) is a l.m.c. \( * \)-algebra with a b.a.i., and \( \mathcal{E}(E) \) its enveloping algebra, then

\[
\mathcal{E}(E) = \lim_\alpha \mathcal{E}(E_\alpha),
\]

within an isomorphism of topological algebras.

**Proof.** \( \mathcal{E}(E) \) is by its definition a complete l.m.c. \( C^* \)-algebra, hence

(4.3)

\[
\mathcal{E}(E) = \lim_\alpha \mathcal{E}(E_\alpha)
\]

within a topological algebraic isomorphism, where \( (\mathcal{E}(E_\alpha)) \) is the inverse system of \( C^* \)-algebras corresponding to \( \mathcal{E}(E) \) [2], [11; Th. 5.1]. Now, (4.3) and Lemma 4.3 yield the assertion.

Theorem 4.3 has a special bearing on a previous result in [6; Th. 4.3] referred to a Fréchet l.m.c. \( * \)-algebra with an identity. On the other hand, by applying categorical language, since \( \mathcal{E} \) preserves continuous morphisms between l.m.c. \( * \)-algebras with b.a.i’s (cf. also Th. 4.1) one may consider \( \mathcal{E} \) as a covariant functor between the categories of the respective algebras \( E \) and \( \mathcal{E}(E) \). Moreover, \( \mathcal{E} \) is continuous (:preserves inverse limits) by Theorem 4.3 restricted to the full subcategory of Banach \( * \)-algebras.

The technique developed hitherto is further applied to the case of topological tensor products [10], by considering \( \mathcal{R}(E \hat{\otimes} F) \) and \( \mathcal{R}(E \hat{\otimes} F) \) with \( E, F \) suitable l.m.c. \( * \)-algebras and \( \tau \) an “admissible” tensor product topology.

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