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## **MONOTONICITY OF PERMANENTS OF CERTAIN DOUBLY STOCHASTIC MATRICES**

DAVID LONDON

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Let  $p_k(A)$ ,  $k = 1, \dots, n$ , denote the sum of the permanents of all  $k \times k$  submatrices of the  $n \times n$  matrix  $A$ .

We prove that

$$(*) \quad p_k(I_n + P_n) = \frac{n}{n-k} \binom{2n-k-1}{k}, \quad k = 1, \dots, n-1,$$

where  $I_n$  and  $P_n$  are respectively the  $n \times n$  identity matrix and the  $n \times n$  permutation matrix with 1's in positions  $(1, 2), (2, 3), \dots, (n-1, n), (n, 1)$ . Using  $(*)$ , we prove that for  $n \geq 3$  and  $A = (I_n + P_n)/2$ , the functions

$$p_k((1-\theta)J_n + \theta A), \quad k = 2, \dots, n,$$

are strictly monotonic increasing in the interval  $0 \leq \theta \leq 1$ . Here  $J_n$  is the  $n \times n$  matrix all whose entries are equal to  $1/n$ .

Let  $A$  be an  $n \times n$  matrix, let  $p(A)$  be the permanent of  $A$ , let  $p_k(A)$ ,  $k = 1, \dots, n$ , be the sum of the permanents of all  $\binom{n}{k} k \times k$  submatrices of  $A$  and define  $p_0(A) = 1$ . Note that  $p_n(A) = p(A)$ .

Denote by  $\Omega_n$  the set of all  $n \times n$  doubly stochastic matrices, by  $J_n$  the  $n \times n$  matrix all whose entries are equal to  $1/n$ , by  $I_n$  the  $n \times n$  identity matrix and by  $P_n$  the  $n \times n$  permutation matrix with 1's in positions  $(1, 2), (2, 3), \dots, (n-1, n), (n, 1)$ .

The van der Waerden conjecture asserts that if  $A \in \Omega_n$ , then

$$p(A) \geq p(J_n) = \frac{n!}{n^n},$$

with equality if and only if  $A = J_n$ .

A stronger version of this conjecture states that the function

$$p((1-\theta)J_n + \theta A),$$

where  $A$  is any fixed matrix on the boundary of  $\Omega_n$ , is strictly increasing in the interval  $0 \leq \theta \leq 1$ . In [2] the above assertion was proved for  $A = I_n$  and for  $A = (nJ_n - I_n)/(n-1)$ . In [5, p. 158, Problem 8] the problem of finding other matrices  $A$ , for which the above assertion holds, was posed.

In the present paper we prove this assertion for  $A = (I_n + P_n)/2$ . We actually prove a stronger result: for  $n \geq 3$  and  $A = (I_n + P_n)/2$  the functions

$$h_{A,k}(\theta) = p_k((1 - \theta)J_n + \theta A), \quad k = 2, \dots, n,$$

are strictly increasing in the interval  $0 \leq \theta \leq 1$ .

We start with the following lemma.

LEMMA 1. *Let  $n \geq 3$  and let  $A \in \Omega_n$ . If*

$$(1) \quad \frac{p_i(A)}{p_i(J_n)} \leq \frac{p_{i+1}(A)}{p_{i+1}(J_n)}, \quad i = 1, \dots, n - 1,$$

with strict inequality for  $1 \leq i < n - 1$ , then the functions

$$h_{A,k}(\theta) = p_k((1 - \theta)J_n + \theta A), \quad k = 2, \dots, n,$$

are strictly increasing in the interval  $0 \leq \theta \leq 1$ .

*Proof.* By [4, Lemma 2],

$$h_{A,k}(\theta) = p_k(J_n) \sum_{i=0}^k \binom{k}{i} (1 - \theta)^{k-i} \theta^i \frac{p_i(A)}{p_i(J_n)}.$$

Differentiating, we obtain

$$(2) \quad h'_{A,k}(\theta) = k p_k(J_n) \sum_{i=1}^{k-1} \binom{k-1}{i} (1 - \theta)^{k-i-1} \theta^i \left( \frac{p_{i+1}(A)}{p_{i+1}(J_n)} - \frac{p_i(A)}{p_i(J_n)} \right).$$

From (1) and (2) follows that

$$h'_{A,k}(\theta) > 0, \quad k = 2, \dots, n,$$

in  $0 < \theta < 1$ , and so the functions  $h_{A,k}(\theta)$  are strictly increasing in the interval  $0 \leq \theta \leq 1$ .

Doković [1] (see also [3]) conjectured that (1) holds for all  $A \in \Omega_n$ . Lemma 1 shows that if the Doković conjecture holds for a certain matrix  $A \in \Omega_n$ , then the functions  $h_{A,k}(\theta)$ ,  $k = 2, \dots, n$ , are increasing in the interval  $0 \leq \theta \leq 1$ .

To apply Lemma 1 for a given  $A$ ,  $p_k(A)$ ,  $k = 2, \dots, n$ , have to be evaluated. Although the evaluation of  $p_k(A)$  is in general rather difficult, explicit formulas for  $p_k(A)$  are obvious for  $A = I_n$  and can be developed for  $A = (I_n + P_n)/2$ .

For  $A = I_n$ , we get

$$p_k(I_n) = \binom{n}{k}, \quad k = 0, \dots, n.$$

Noting that

$$(3) \quad p_k(J_n) = \binom{n}{k}^2 \frac{k!}{n^k}, \quad k = 0, \dots, n,$$

(1) follows with strict inequality for  $1 \leq i \leq n - 1$ . Hence, for  $n \geq 2$ ,  $p_k((1 - \theta)J_n + \theta I_n)$ ,  $k = 2, \dots, n$ , are strictly increasing in  $0 \leq \theta \leq 1$ . For  $k = n$ , we get the result of Friedland and Minc [2].

To find formulas for  $p_k(I_n + P_n)$ , it is convenient first to bring some combinatorial results.

**LEMMA 2.** *Let  $l$  and  $m$  be positive integers,  $m \leq l$ . The number  $l$  can be represented as a sum of  $m$  positive integers in  $\binom{l-1}{m-1}$  different ways. (Two representations differing in the order of the summands are regarded different.)*

*Proof.* The lemma can be proved easily by induction. We prefer to use power series technique.

Consider

$$\frac{x}{1-x} = \sum_{r=1}^{\infty} x^r, \quad |x| < 1.$$

It is obvious that the requested number of representations is equal to the coefficient of  $x^l$  in the power series of  $[x/(1-x)]^m$ , which is easily found to be equal to  $\binom{l-1}{m-1}$ .

**LEMMA 3.** *Let  $k, l$  and  $n$  be positive integers,  $k < n$ . Then*

$$(4) \quad \sum_{m=1}^{\min(l, n-k)} \binom{l}{m} \binom{n-k-1}{n-k-m} = \binom{n-k+l-1}{n-k},$$

$$(5) \quad \sum_{m=0}^k \binom{n-m-1}{n-k-1} \binom{n-k+m-1}{n-k-1} = \binom{2n-k-1}{k}.$$

*Proof.* We use again power series.

To prove (4), we consider

$$(1+x)^l = \sum_{r=0}^l \binom{l}{r} x^r,$$

$$(1+x)^{n-k-1} = \sum_{r=0}^l \binom{n-k-1}{r} x^r.$$

The sum in the lefthand side of (4) is equal to the coefficient of  $x^{n-k}$  in the power series of  $(1+x)^{n-k+l-1}$ , which is  $\binom{n-k+l-1}{n-k}$ .

To prove (5), we consider

$$\frac{x^{n-k-1}}{(1-x)^{n-k}} = \sum_{r=n-k-1}^{\infty} \binom{r}{n-k-1} x^r, \quad |x| < 1.$$

The sum in the lefthand side of (5) is equal to the coefficient of  $x^{2n-k-2}$  in the power series of  $[x^{n-k-1}/(1-x)^{n-k}]^2$ , which is  $\binom{2n-k-1}{k}$ . The proof of the lemma is completed.

Let  $n$  and  $l$  be positive integers,  $l \leq n$ . Let  $(n_1, \dots, n_l)$ ,  $1 \leq n_1 < n_2 < \dots < n_l \leq n$ , be a  $l$ -combination of  $1, \dots, n$ . Let  $m$  be the number of  $r$ 's,  $r = 1, \dots, l$ , for which  $n_{r+1} \neq n_r + 1$ , where  $n_{l+1}$  is taken as  $n_1$  and  $n + 1$  as  $1$ . We say that the  $l$ -combination  $(n_1, \dots, n_l)$  has  $m$  gaps. Obviously,  $m \leq l$  and  $m + l \leq n$ ; i.e.,  $0 \leq m \leq \min(l, n - l)$ .

Take  $l < n$  and arrange  $1, \dots, n$  in increasing order (clockwise) in a circle. Then the set  $(n_1, \dots, n_l)$  and its complement have the same number of (connected) components. This number is the number  $m$  defined above as the number of gaps of  $(n_1, \dots, n_l)$ .

For example, if  $n = 6$  and  $l = 3$ , then the number of gaps of  $(1, 2, 3)$  and  $(1, 2, 6)$  is 1, of  $(1, 3, 4)$  is 2 and of  $(1, 3, 5)$  is 3. If  $n = l$ , then  $m = 0$ .

We denote by  $\binom{n}{l, m}$  the number of  $l$ -combinations of  $1, \dots, n$  having  $m$  gaps.  $\binom{n}{l, m}$  is thus defined for all nonnegative integers  $l, m, n$  satisfying  $0 < l \leq n, 0 \leq m \leq \min(l, n - l)$ . We also define  $\binom{n}{0, 0} = 1$ . From the definition of  $\binom{n}{l, m}$  follows that

$$\sum_{m=0}^{\min(l, n-l)} \binom{n}{l, m} = \binom{n}{l}.$$

In the following lemma we obtain a formula for  $\binom{n}{l, m}$ .

LEMMA 4. Let  $l, m, n$  be positive integers satisfying  $0 < l \leq n - 1, 0 < m \leq \min(l, n - l)$ . Then

$$(6) \quad \binom{n}{l, m} = \frac{n}{m} \binom{l-1}{m-1} \binom{n-l-1}{m-1}.$$

*Proof.*  $\binom{n}{l, m}$  is equal to the number of  $l$ -combinations of  $1, \dots, n$  with  $m$  gaps. We first find the number of  $l$ -combinations of the form  $(1, n_2, \dots, n_l)$  with  $m$  gaps.

Arrange the numbers  $1, \dots, n$  in a circle and take a  $l$ -combination  $(1, n_2, \dots, n_l)$  with  $m$  gaps. As  $l < n$ , the set  $(1, n_2, \dots, n_l)$  and its complement have each  $m$  components. Let  $m_i$  and  $m'_i, i = 1, \dots, m$ , be the number of elements in the  $i$ th component of  $(1, n_2, \dots, n_l)$  and its complement respectively. We have

$$(7) \quad \begin{cases} \sum_{i=1}^m m_i = l, \\ \sum_{i=1}^m m'_i = n - l. \end{cases}$$

It is obvious that there is a 1 - 1 correspondence between the  $l$ -combinations of the form  $(1, n_2, \dots, n_l)$  with  $m$  gaps and the  $2m$ -tuples  $(m_1, m'_1, \dots, m_m, m'_m)$  of positive integers satisfying (7). By Lemma 2, the number of these  $2m$ -triples is  $\binom{l-1}{m-1} \binom{n-l-1}{m-1}$ . Hence, the number of  $l$ -combinations of the form  $(1, n_2, \dots, n_l)$  with  $m$  gaps is  $\binom{l-1}{m-1} \binom{n-l-1}{m-1}$ .

For each of the numbers  $1, \dots, n$  we get  $\binom{l-1}{m-1} \binom{n-l-1}{m-1}$   $l$ -combinations with  $m$  gaps. Assembling all these combinations, each combination with  $l$  gaps is repeated  $m$  times. Hence, to get the number of these combinations,  $\binom{n-1}{m-1} \binom{n-l-1}{m-1}$  has to be multiplied by  $n$  and divided by  $m$ . Formula (6) is thus proved.

In the following lemma we obtain formulas for  $p_k(I_n + P_n), k = 0, \dots, n$ .

LEMMA 5. *Let  $n \geq 2$ . Then*

$$(8) \quad p_k(I_n + P_n) = \begin{cases} \frac{n}{n-k} \binom{2n-k-1}{k}, & k = 0, \dots, n-1, \\ 2, & k = n. \end{cases}$$

*Proof.* Formula (8) is easily verified for  $k = 0$  and  $k = n$ .

Let  $1 \leq k \leq n-1$ .  $p_k(I_n + P_n)$  is equal to the number of different diagonals of 1's of length  $k$  in  $I_n + P_n$ . (Where diagonals of length  $k$  in the  $n \times n$  matrix  $I_n + P_n$  are defined in the obvious way.) Each such diagonal is composed of  $l$  elements of  $I_n$  and  $k-l$  elements of  $P_n$ .

Let  $(n_1, \dots, n_l)$  be a  $l$ -combination of  $1, \dots, n$  with  $m$  gaps. The number of 1's in  $P_n$  belonging either to the rows  $n_1, \dots, n_l$  or to the columns  $n_1, \dots, n_l$  is  $l+m$ . Hence, the diagonal of length  $l$  consisting of 1's in positions  $(n_1, n_1), (n_2, n_2), \dots, (n_l, n_l)$  can be augmented, using elements of  $P_n$ , to  $\binom{n-l-m}{k-l}$  different diagonals of 1's of length  $k$ . As there are  $\binom{n}{l, m}$   $l$ -combinations with  $m$  gaps, the number of diagonal of length  $k$  which originate in a  $l$ -combination with  $m$  gaps is  $\binom{n}{l, m} \binom{n-l-m}{k-l}$ . Summing up over all possible  $m$  and  $l$ , we obtain

$$p_k(A) = \sum_{l=0}^k \sum_{m=0}^{\min(l, n-k)} \binom{n}{l, m} \binom{n-l-m}{k-l}.$$

Noting that  $\binom{n}{0, 0} = 1$  and, as  $k < n$ ,  $\binom{n}{l, 0} = 0$  for  $l = 1, \dots, k$ , it follows that

$$p_k(A) = \binom{n}{k} + \sum_{l=1}^k \sum_{m=1}^{\min(l, n-k)} \binom{n}{l, m} \binom{n-l-m}{k-l}.$$

Using now Lemma 4, we obtain

$$p_k(A) = \binom{n}{k} + \sum_{l=1}^k \sum_{m=1}^{\min(l, n-k)} \frac{n}{m} \binom{l-1}{m-1} \binom{n-l-1}{m-1} \binom{n-l-m}{k-l}.$$

As

$$\begin{aligned} & \frac{n}{m} \binom{l-1}{m-1} \binom{n-l-1}{m-1} \binom{n-l-m}{k-l} \\ &= \frac{n}{l} \binom{n-l-1}{n-k-1} \binom{l}{m} \binom{n-k-1}{n-k-m}, \end{aligned}$$

it follows that

$$p_k(A) = \binom{n}{k} + n \sum_{l=1}^k \frac{1}{l} \binom{n-l-1}{n-k-1} \sum_{m=1}^{\min(l, n-k)} \binom{l}{m} \binom{n-k-1}{n-k-m},$$

and using (4), we obtain

$$p_k(A) = \binom{n}{k} + n \sum_{l=1}^k \frac{1}{l} \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k}.$$

But

$$\frac{1}{l} \binom{n-k+l-1}{n-k} = \frac{1}{n-k} \binom{n-k+l-1}{n-k-1}.$$

So

$$\begin{aligned} (9) \quad p_k(A) &= \binom{n}{k} + \frac{n}{n-k} \sum_{l=1}^k \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k-1} \\ &= \frac{n}{n-k} \sum_{l=0}^k \binom{n-l-1}{n-k-1} \binom{n-k+l-1}{n-k-1}. \end{aligned}$$

Formula (8) follows from (5) and (9).

We bring now our main result.

**THEOREM.** *Let  $n \geq 3$  and let  $A = (I_n + P_n)/2$ . Then the functions*

$$h_{A,k}(\theta) = p_k((1 - \theta)J_n + \theta A), \quad k = 2, \dots, n,$$

are strictly increasing in the interval  $0 \leq \theta \leq 1$ .

*Proof.* By Lemma 1, it is sufficient to show that

$$(10) \quad \frac{2p_i(I_n + P_n)}{p_i(J_n)} \leq \frac{p_{i+1}(I_n + P_n)}{p_{i+1}(J_n)}, \quad i = 1, \dots, n - 1,$$

with strict inequality for  $1 \leq i < n - 1$ .

For  $i = n - 1$ , (10) holds with equality sign.

For  $i = 1, \dots, n - 2$ , (3) and (8) imply

$$(11) \quad \frac{p_{i+1}(I_n + P_n)}{p_{i+1}(J_n)} = \frac{(i + 1)! [(n - i - 1)!]^2 n^{i+2}}{(n - i - 1)(n!)^2} \binom{2n - i - 2}{i + 1}.$$

From (11) follows

$$\begin{aligned} & \frac{p_{i+1}(I_n + P_n)}{p_{i+1}(J_n)} - \frac{2p_i(I_n + P_n)}{p_i(J_n)} \\ &= \frac{2in^{i+1}(2n - i - 2)! [(n - i - 1)!]^2 (n - i - 1)}{(n!)^2 (2n - 2i - 1)!}. \end{aligned}$$

Hence (10) holds with strict inequality for  $1 \leq i < n - 1$ , and the proof of our theorem is completed.

We note that the theorem holds also for all  $n \times n$  matrices  $A$  which can be obtained from  $(I_n + P_n)/2$  by permutations of rows and columns.

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#### REFERENCES

1. D. Z. Doković, *On a conjecture by van der Waerden*, Mat. Vesnik (19) **4** (1967), 272-276.
2. S. Friedland and H. Minc, *Monotonicity of permanents of doubly stochastic matrices*, Linear and Multilinear Algebra, **6** (1978), 227-231.
3. D. London, *On the Doković conjecture for matrices of rank two*, to appear in Linear and Multilinear Algebra.
4. M. Marcus and H. Minc, *Extensions of classical matrix inequalities*, Linear Algebra and Appl., **1** (1968), 421-444.
5. H. Minc, *Permanents*, Encyclopedia of Mathematics and its Applications, Vol. **6**, Addison-Wesley, 1978.

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