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**CONCERNING THE MINIMUM OF PERMANENTS ON  
DOUBLY STOCHASTIC CIRCULANTS**

GERALD SUCHAN

## CONCERNING THE MINIMUM OF PERMANENTS ON DOUBLY STOCHASTIC CIRCULANTS

GERALD E. SUCHAN

**Let  $P_n$  be the permutation matrix such that  $(P_n)_{ij} = 1$  if  $j = i + 1(\text{mod } n)$ . Minc [2] proved that the minimum of the permanent on the collection of  $n \times n$  doubly stochastic circulants  $\alpha I_n + \beta P_n + \gamma P_n^2$  is in  $(1/2^n, 1/2^{n-1}]$ , and if  $n \geq 5$  then the minimum is not achieved at  $(1/3)I_n + (1/3)P_n + (1/3)P_n^2$ . This paper proves that if  $n \geq 3$  then the minimum of such permanents is less than  $1/2^{n-1}$ , and if  $n \in \{3, 4\}$  then this minimum is uniquely achieved at  $(1/3)I_n + (1/3)P_n + (1/3)P_n^2$ .**

**Introduction.** Let  $n$  be a positive integer, let  $I_n$  denote the  $n \times n$  identity matrix, and let  $P_n$  denote the full cycle permutation matrix such that  $(P_n)_{ij} = 1$  if  $j = i + 1(\text{mod } n)$ . Minc [2] studied the permanent of circulants  $\alpha I_n + \beta P_n + \gamma P_n^2$  and proved the following three theorems:

**THEOREM 1.** *If  $n \geq 3$  then*

$$\begin{aligned} \text{per}(\alpha I_n + \beta P_n + \gamma P_n^2) &= \left( \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2} \right)^n \\ &\quad + \left( \frac{\beta - \sqrt{\beta^2 + 4\alpha\gamma}}{2} \right)^n + \alpha^n + \gamma^n. \end{aligned}$$

**THEOREM 2.** *If  $\alpha, \beta, \gamma$  are nonnegative then*

$$\frac{1}{2^n} < \min_{\alpha+\beta+\gamma=1} \text{per}(\alpha I_n + \beta P_n + \gamma P_n^2) \leq \frac{1}{2^{n-1}}.$$

**THEOREM 3.** *If  $\alpha, \beta, \gamma$  are nonnegative,  $n \geq 5$ , then*

$$\min_{\alpha+\beta+\gamma=1} \text{per}(\alpha I_n + \beta P_n + \gamma P_n^2) < \text{per} \left( \frac{1}{3}I_n + \frac{1}{3}P_n + \frac{1}{3}P_n^2 \right).$$

**MAIN RESULTS.** Let  $S = \{(\alpha, \gamma) \mid 0 \leq \alpha, 0 \leq \gamma, \alpha + \gamma \leq 1\}$ , and let  $f_n$  denote the function on  $S$  such that

$$f_n(\alpha, \gamma) = \text{per}(\alpha I_n + (1 - \alpha - \gamma)P_n + \gamma P_n^2).$$

**THEOREM 4.** *If  $n \geq 3$  then  $f_n$  is not minimum on the boundary of  $S$ .*

**LEMMA TO THEOREM 4.** *The minimum of  $f_n$  on the boundary of*

$S$  is  $1/2^{n-1}$ . If  $n$  is even this minimum is achieved only on  $\{(1/2, 0), (0, 1/2)\}$ , and if  $n > 1$  and  $n$  is odd this minimum is achieved only on  $\{(1/2, 0), (1/2, 1/2), (0, 1/2)\}$ .

*Proof.* The lemma is clearly true in case  $n \in \{1, 2\}$ . Suppose  $n \geq 3$ . Since

$$f_n(1/2, 0) = f_n(0, 1/2) = \frac{1}{2^{n-1}} < 1 = f_n(1, 0) = f_n(0, 0)f_n(0, 1),$$

then it is sufficient to consider only points belonging to the interior of the boundary of  $S$ . The only real number  $\alpha$  satisfying  $D_1 f_n(\alpha, 0) = 0$  is  $1/2$ . Therefore, since  $f_n(\alpha, \gamma) = f_n(\gamma, \alpha)$ , then the minimum of  $f_n$  on  $\{(\alpha, \gamma) | \alpha\gamma = 0\}$  is  $1/2^{n-1}$ . Let  $g(\alpha) = f_n(\alpha, 1 - \alpha)$ . If  $n$  is even, put  $k = n/2$  and observe that  $g(\alpha) = (\alpha^k + (1 - \alpha)^k)^2$ . If  $n$  is odd then  $g(\alpha) = \alpha^n + (1 - \alpha)^n$ . In either case,  $1/2$  is the only real number  $\alpha$  such that  $g'(\alpha) = 0$ . If  $n$  is even then  $f_n(1/2, 1/2) = 1/2^{n-2} > 1/2^{n-1}$ , and if  $n$  is odd then  $f_n(1/2, 1/2) = 1/2^{n-1}$ .

*Proof of Theorem 4.* By the lemma it is sufficient to show there is a point  $q$  of  $S$  so that  $f_n(q) < f_n(1/2, 0)$ . Observe that  $D_1 f_n(\alpha, \gamma)$  is

$$\begin{aligned} & \frac{n}{2} \left( \frac{1 - \alpha - \gamma + \sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}}{2} \right)^{n-1} \left( -1 + \frac{-1 + \alpha + 3\gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}} \right) \\ & + \frac{n}{2} \left( \frac{1 - \alpha - \gamma - \sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}}{2} \right)^{n-1} \left( -1 - \frac{-1 + \alpha + 3\gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}} \right) \\ & + n\alpha^{n-1}. \end{aligned}$$

Thus  $D_1 f_n(1/2, 0) = 0$  and therefore, since  $D_1 f_n(\alpha, \gamma) = D_2 f_n(\gamma, \alpha)$ , then  $(1/2, 0)$  is a critical point for  $f_n$ . Now observe that  $D_{1,1}(\alpha, \gamma)$  is

$$\begin{aligned} & \frac{n}{2} \left[ \frac{(n-1)}{2} \left( \frac{1 - \alpha - \gamma + \sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}}{2} \right)^{n-2} \left( -1 + \frac{-1 + \alpha + 3\gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}} \right)^2 \right. \\ & \left. + \left( \frac{1 - \alpha - \gamma + \sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}}{2} \right)^{n-1} \left( \frac{(1 - \alpha - \gamma)^2 + 4\alpha\gamma - (-1 + \alpha + 3\gamma)^2}{((1 - \alpha - \gamma)^2 + 4\alpha\gamma)^{3/2}} \right) \right] \\ & + \frac{n}{2} \left[ \frac{(n-1)}{2} \left( \frac{1 - \alpha - \gamma - \sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}}{2} \right)^{n-2} \left( -1 - \frac{-1 + \alpha + 3\gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}} \right)^2 \right. \\ & \left. + \left( \frac{1 - \alpha - \gamma - \sqrt{(1 - \alpha - \gamma)^2 + 4\alpha\gamma}}{2} \right)^{n-1} \left( \frac{-(1 - \alpha - \gamma)^2 + 4\alpha\gamma + (-1 + \alpha + 3\gamma)^2}{((1 - \alpha - \gamma)^2 + 4\alpha\gamma)^{3/2}} \right) \right] \\ & + n(n-1)\alpha^{n-2}. \end{aligned}$$

Thus  $D_{1,1} f_n(1/2, 0) = n(n-1)/2^{n-3}$ , and since  $D_{2,2} f_n(\alpha, \gamma) = D_{1,1}(\gamma, \alpha)$  then  $D_{2,2} f_n(1/2, 0) = 0$ . Finally, observe that  $D_{1,2} f_n(\alpha, \gamma)$  is

$$\begin{aligned} & \frac{n}{2} \left[ \frac{(n-1)}{2} \left( \frac{1-\alpha-\gamma+\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2} \right)^{n-2} \left( -1 + \frac{-1+3\alpha+\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}} \right) \right. \\ & \quad \times \left( -1 + \frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}} \right) + \left( \frac{1-\alpha-\gamma+\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2} \right)^{n-1} \\ & \quad \times \left. \left( \frac{3((1-\alpha-\gamma)^2+4\alpha\gamma) - (-1+\alpha+3\gamma)(-1+3\alpha+\gamma)}{((1-\alpha-\gamma)^2+4\alpha\gamma)^{3/2}} \right) \right] \\ & + \frac{n}{2} \left[ \frac{(n-1)}{2} \left( \frac{1-\alpha-\gamma-\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2} \right)^{n-2} \left( -1 - \frac{-1+3\alpha+\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}} \right) \right. \\ & \quad \times \left( -1 - \frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}} \right) + \left( \frac{1-\alpha-\gamma-\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2} \right)^{n-1} \\ & \quad \times \left. \left( \frac{-3((1-\alpha-\gamma)^2+4\alpha\gamma) + (-1+\alpha+3\gamma)(-1+3\alpha+\gamma)}{((1-\alpha-\gamma)^2+4\alpha\gamma)^{3/2}} \right) \right]. \end{aligned}$$

Thus  $D_{1,2}f_n(1/2, 0) = n/2^{n-3} = D_{2,1}f_n(1/2, 0)$ .

Let  $H$  denote the Hessian matrix for  $f_n$  at  $(1/2, 0)$ .  $H$  has characteristic values

$$\lambda_1 = \frac{n}{2^{n-2}}(n - 1 + \sqrt{(n - 1)^2 + 4})$$

and

$$\lambda_2 = \frac{n}{2^{n-2}}(n - 1 - \sqrt{(n - 1)^2 + 4}).$$

Since  $\lambda_2 < 0 < \lambda_1$  then  $(1/2, 0)$  is a saddle point for  $f_n$ . Let  $x = (\lambda_2, 1)$  and put  $|x| = \sqrt{\lambda_2^2 + 1}$ . By Taylor's theorem there is a positive number  $\delta$  so that if  $|x| < \delta$  then there is a number  $R(x)$  so that  $f_n((1/2, 0) + x)$  is

$$\frac{1}{0!}f_n(1/2, 0) + \frac{1}{1!} \sum_{k=1}^2 (x)_k D_k f_n(1/2, 0) + \frac{1}{2!} \sum_{i,j=1}^2 (x)_i (x)_j D_{i,j} f_n(1/2, 0) + R(x)$$

and therefore, since  $(1/2, 0)$  is a critical point for  $f_n$ , and since  $Hx^T = \lambda_2 x^T$ , then

$$f_n((1/2, 0) + x) = f_n(1/2, 0) + \lambda_2 |x|^2 + R(x).$$

Since  $\lambda_2 < 0$  then there is a positive number  $\omega < \delta$  such that if  $|x| < \omega$  then  $\lambda_2 |x|^2 + R(x) < 0$ , and therefore  $f_n((1/2, 0) + x) < f_n(1/2, 0)$ . Let  $q = (1/2, 0) + \omega |x|^{-1} x$ , observe that  $q \in S$  and that  $f_n(q) < f_n(1/2, 0)$ .

**THEOREM 5.** *If  $n \in \{3, 4\}$  then  $f_n$  is minimum, uniquely, at  $(1/3, 1/3)$ .*

*Proof.* In [1] Marcus and Newman proved the van der Waerden

conjecture true in case  $n = 3$ , and hence this theorem is also true in this case. Let  $(\alpha, \gamma)$  be a point of  $S$  at which  $f_4$  is minimum. Observe that  $f_4(\alpha, \gamma)$  is

$$2\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 2\gamma^4 + 6\gamma^2 - 4\gamma - 20\gamma^2 \\ + 8\alpha\gamma^3 + 16\alpha^2\gamma^2 + 8\alpha^3\gamma - 20\alpha^2\gamma + 16\alpha\gamma + 1,$$

that  $D_1f_4(\alpha, \gamma)$  is

$$8\alpha^3 - 12\alpha^2 + 12\alpha - 4 - 20\gamma^2 + 8\gamma^3 + 32\alpha\gamma^2 + 24\alpha^2\gamma - 40\alpha\gamma + 16\gamma,$$

and that  $D_2f_4(\alpha, \gamma)$  is

$$8\gamma^3 - 12\gamma^2 + 12\gamma - 4 - 40\alpha\gamma + 24\alpha\gamma^2 + 32\alpha^2\gamma + 8\alpha^3 - 20\alpha^2 + 16\alpha.$$

By Theorem 4,  $(\alpha, \gamma)$  is not on the boundary of  $S$  and so  $D_1f_4(\alpha, \gamma) = 0 = D_2f_4(\alpha, \gamma)$ . Thus  $D_1f_4(\alpha, \gamma) - D_2f_4(\alpha, \gamma) = 0$  and therefore

$$(1) \quad (\alpha - \gamma)(2(\alpha + \gamma) - 1 - 2\alpha\gamma) = 0.$$

Since  $D_1f_4(\alpha, \alpha) = (\alpha - 1/3)(18\alpha^2 - 12\alpha + 3)$  then the only critical point on the diagonal of  $S$  is  $(1/3, 1/3)$ . Suppose

$$(2) \quad f_4(\alpha, \gamma) < f_4\left(\frac{1}{3}, \frac{1}{3}\right)$$

and observe from (1) that

$$(3) \quad 2(\alpha + \gamma) - 1 - 2\alpha\gamma = 0.$$

Let  $\beta = 1 - \alpha - \gamma$ . It follows from (3) that  $\beta^2 = \alpha^2 + \gamma^2$  and from (2) and (3) that

$$f_4(\alpha, \gamma) = \beta^4 + 2\beta^2(2\alpha\gamma) + (\alpha^2 + \gamma^2)^2 = 2\beta^2(1 - \beta)^2 < \frac{1}{9}.$$

Hence  $\beta(1 - \beta) < 1/3\sqrt{2}$  and therefore

$$(4) \quad \text{either } \beta < \frac{1 - \sqrt{1 - \frac{2\sqrt{2}}{3}}}{2} \quad \text{or } \beta > \frac{1 + \sqrt{1 - \frac{2\sqrt{2}}{3}}}{2}.$$

It also follows from (3) that  $2\gamma^2 - 2(1 - \beta)\gamma + 1 - 2\beta = 0$  and therefore, since  $\gamma$  is a real number, then

$$(5) \quad \beta \geq \sqrt{2} - 1.$$

Finally, (3) implies that  $1 - 2\beta - 2\alpha\gamma = 0$ , and therefore since  $\alpha\gamma \geq 0$ , then

$$(6) \quad 3 \leq 1/2.$$

Inequalities (4), (5) and (6) constitute a contradiction.

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