CONCERNING THE MINIMUM OF PERMANENTS ON DOUBLY STOCHASTIC CIRCULANTS

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Let \( P_n \) be the permutation matrix such that \((P_n)_{ij} = 1\) if \( j = i + 1 \text{ (mod } n)\). Minc [2] proved that the minimum of the permanent on the collection of \( n \times n \) doubly stochastic circulants \( \alpha I_n + \beta P_n + \gamma P_n^z \) is in \((1/2^n, 1/2^{n-1})\), and if \( n \geq 5 \) then the minimum is not achieved at \((1/3)I_n + (1/3)P_n + (1/3)P_n^z\). This paper proves that if \( n \geq 3 \) then the minimum of such permanents is less than \( 1/2^n - 1 \), and if \( n \in \{3, 4\} \) then this minimum is uniquely achieved at \((1/3)I_n + (1/3)P_n + (1/3)P_n^z\).

Introduction. Let \( n \) be a positive integer, let \( I_n \) denote the \( n \times n \) identity matrix, and let \( P_n \) denote the full cycle permutation matrix such that \((P_n)_{ij} = 1\) if \( j = i + 1 \text{ (mod } n)\). Minc [2] studied the permanent of circulants \( \alpha I_n + \beta P_n + \gamma P_n^z \) and proved the following three theorems:

**Theorem 1.** If \( n \geq 3 \) then

\[
\text{per} (\alpha I_n + \beta P_n + \gamma P_n^z) = \left( \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2} \right)^n + \left( \frac{\beta - \sqrt{\beta^2 + 4\alpha\gamma}}{2} \right)^n + \alpha^n + \gamma^n.
\]

**Theorem 2.** If \( \alpha, \beta, \gamma \) are nonnegative then

\[
\frac{1}{2^n} < \min_{\alpha+\beta+\gamma=1} \text{per} (\alpha I_n + \beta P_n + \gamma P_n^z) \leq \frac{1}{2^{n-1}}.
\]

**Theorem 3.** If \( \alpha, \beta, \gamma \) are nonnegative, \( n \geq 5 \), then

\[
\min_{\alpha+\beta+\gamma=1} \text{per} (\alpha I_n + \beta P_n + \gamma P_n^z) < \text{per} \left( \frac{1}{3} I_n + \frac{1}{3} P_n + \frac{1}{3} P_n^z \right).
\]

**Main Results.** Let \( S = \{ (\alpha, \gamma) | 0 \leq \alpha, 0 \leq \gamma, \alpha + \gamma \leq 1 \} \), and let \( f_n \) denote the function on \( S \) such that

\[
f_n(\alpha, \gamma) = \text{per} (\alpha I_n + (1 - \alpha - \gamma)P_n + \gamma P_n^z).
\]

**Theorem 4.** If \( n \geq 3 \) then \( f_n \) is not minimum on the boundary of \( S \).

**Lemma to Theorem 4.** The minimum of \( f_n \) on the boundary of
S is $2^{n-1}$. If $n$ is even this minimum is achieved only on
\{(1/2, 0), (0, 1/2)\}, and if $n > 1$ and $n$ is odd this minimum is
achieved only on \{(1/2, 0), (1/2, 1/2), (0, 1/2)\}.

**Proof.** The lemma is clearly true in case $n \in \{1, 2\}$. Suppose
$n \geq 3$. Since

\[ f_n(1/2, 0) = f_n(0, 1/2) = \frac{1}{2^{n-1}} < 1 = f_n(1, 0) = f_n(0, 0) f_n(0, 1), \]

then it is sufficient to consider only points belonging to the interior
of the boundary of $S$. The only real number $\alpha$ satisfying
$D_1 f_n(\alpha, 0) = 0$ is 1/2. Therefore, since $f_n(\alpha, \gamma) = f_n(\gamma, \alpha)$, then the
minimum of $f_n$ on \{$(\alpha, \gamma) | \alpha \gamma = 0$\} is $2^{n-1}$. Let $g(\alpha) = f_n(\alpha, 1 - \alpha)$.
If $n$ is even, put $k = n/2$ and observe that $g(\alpha) = (\alpha^k + (1 - \alpha)^k)^2$.
If $n$ is odd then $g(\alpha) = \alpha^n + (1 - \alpha)^n$. In either case, 1/2 is the only
real number $\alpha$ such that $g'(\alpha) = 0$. If $n$ is even then $f_n(1/2, 1/2) =
1/2^{n-2} > 1/2^{n-1}$, and if $n$ is odd then $f_n(1/2, 1/2) = 1/2^{n-1}$.

**Proof of Theorem 4.** By the lemma it is sufficient to show
there is a point $q$ of $S$ so that $f_n(q) < f_n(1/2, 0)$. Observe that
$D_1 f_n(\alpha, \gamma)$ is

\[
\begin{align*}
&\frac{n}{2} \left( \frac{1 - \alpha - \gamma + \sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}}{2} \right)^{n-1} \left( -1 + \frac{-1 + \alpha + 3 \gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}} \right) \\
&+ \frac{n}{2} \left( \frac{1 - \alpha - \gamma - \sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}}{2} \right)^{n-1} \left( -1 - \frac{-1 + \alpha + 3 \gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}} \right) \\
&+ n \alpha^{n-1}.
\end{align*}
\]

Thus $D_1 f_n(1/2, 0) = 0$ and therefore, since $D_1 f_n(\alpha, \gamma) = D_1 f_n(\gamma, \alpha)$, then
(1/2, 0) is a critical point for $f_n$. Now observe that $D_1, i(\alpha, \gamma)$ is

\[
\begin{align*}
&\frac{n}{2} \left[ (n-1) \left( \frac{1 - \alpha - \gamma + \sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}}{2} \right)^{n-2} \left( -1 + \frac{-1 + \alpha + 3 \gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}} \right)^2 \right] \\
&+ \left( \frac{1 - \alpha - \gamma + \sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}}{2} \right)^{n-1} \left( (1 - \alpha - \gamma)^2 + 4 \alpha \gamma - (-1 + \alpha + 3 \gamma)^2 \right) \\
&+ \frac{n}{2} \left[ (n-1) \left( \frac{1 - \alpha - \gamma - \sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}}{2} \right)^{n-2} \left( -1 - \frac{-1 + \alpha + 3 \gamma}{\sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}} \right)^2 \right] \\
&+ \left( \frac{1 - \alpha - \gamma - \sqrt{(1 - \alpha - \gamma)^2 + 4 \alpha \gamma}}{2} \right)^{n-1} \left( (1 - \alpha - \gamma)^2 + 4 \alpha \gamma + (-1 + \alpha + 3 \gamma)^2 \right) \\
&+ n(n-1) \alpha^{n-2}.
\end{align*}
\]

Thus $D_1, i f_n(1/2, 0) = n(n-1)/2^{n-3}$, and since $D_2, i f_n(\alpha, \gamma) = D_1, i(\gamma, \alpha)$
then $D_2, i f_n(1/2, 0) = 0$. Finally, observe that $D_1, i f_n(\alpha, \gamma)$ is
Thus \( D_{1,1} f_n(1/2, 0) = n/2^{n-3} = D_{k,1} f_n(1/2, 0) \).

Let \( H \) denote the Hessian matrix for \( f_n \) at \((1/2, 0)\). \( H \) has characteristic values

\[
\lambda_1 = \frac{n}{2^{n-2}} (n - 1 + \sqrt{(n - 1)^2 + 4})
\]

and

\[
\lambda_2 = \frac{n}{2^{n-2}} (n - 1 - \sqrt{(n - 1)^2 + 4})
\]

Since \( \lambda_2 < 0 < \lambda_1 \) then \((1/2, 0)\) is a saddle point for \( f_n \). Let \( x = (\lambda_2, 1) \) and put \( |x| = \sqrt{\lambda_2^2 + 1} \). By Taylor's theorem there is a positive number \( \delta \) so that if \( |x| < \delta \) then there is a number \( R(x) \) so that

\[
\frac{1}{0!} f_n(1/2, 0) + \frac{1}{1!} \sum_{k=1}^2 (x)_k D_k f_n(1/2, 0) + \frac{1}{2!} \sum_{i,j=1}^2 (x)_i(x)_j D_{i,j} f_n(1/2, 0) + R(x)
\]

and therefore, since \((1/2, 0)\) is a critical point for \( f_n \), and since

\[
Hx^T = \lambda_2 x^T,
\]

then

\[
f_n((1/2, 0) + x) = f_n(1/2, 0) + \lambda_2 |x|^2 + R(x).
\]

Since \( \lambda_2 < 0 \) then there is a positive number \( \omega < \delta \) such that if \( |x| < \omega \) then \( \lambda_2 |x|^2 + R(x) < 0 \), and therefore \( f_n((1/2, 0) + x) < f_n(1/2, 0) \).

Let \( q = (1/2, 0) + \omega |x|^{-1} x \), observe that \( q \in S \) and that \( f_n(q) < f_n(1/2, 0) \).

**Theorem 5.** If \( n \in \{3, 4\} \) then \( f_n \) is minimum, uniquely, at \((1/3, 1/3)\).

**Proof.** In [1] Marcus and Newman proved the van der Waerden
conjecture true in case \( n = 3 \), and hence this theorem is also true in this case. Let \((\alpha, \gamma)\) be a point of \( S \) at which \( f_4 \) is minimum. Observe that \( f_4(\alpha, \gamma) \) is

\[
2\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 2\gamma^4 + 6\gamma^2 - 4\gamma - 20\gamma^2 \\
+ 8\alpha\gamma^3 + 16\alpha^2\gamma^2 + 8\alpha^3\gamma - 20\alpha^2\gamma + 16\alpha\gamma + 1 ,
\]

that \( D_1f_4(\alpha, \gamma) \) is

\[
8\alpha^3 - 12\alpha^2 + 12\alpha - 4 - 20\gamma^2 + 8\gamma^3 + 32\alpha\gamma^2 + 24\alpha^2\gamma - 40\alpha\gamma + 16\gamma ,
\]

and that \( D_2f_4(\alpha, \gamma) \) is

\[
8\gamma^3 - 12\gamma^2 + 12\gamma - 4 - 40\alpha\gamma + 24\alpha\gamma^2 + 32\alpha^2\gamma + 8\alpha^3 - 20\alpha^2 + 16\alpha .
\]

By Theorem 4, \((\alpha, \gamma)\) is not on the boundary of \( S \) and so \( D_1f_4(\alpha, \gamma) = 0 = D_2f_4(\alpha, \gamma) \). Thus \( D_1f_4(\alpha, \gamma) - D_2f_4(\alpha, \gamma) = 0 \) and therefore

\[
(1) \quad (\alpha - \gamma)(2(\alpha + \gamma) - 1 - 2\alpha\gamma) = 0 .
\]

Since \( D_1f_4(\alpha, \alpha) = (\alpha - 1/3)(18\alpha^2 - 12\alpha + 3) \) then the only critical point on the diagonal of \( S \) is \((1/3, 1/3)\). Suppose

\[
(2) \quad f_4(\alpha, \gamma) < f_4(1/3, 1/3)
\]

and observe from (1) that

\[
(3) \quad 2(\alpha + \gamma) - 1 - 2\alpha\gamma = 0 .
\]

Let \( \beta = 1 - \alpha - \gamma \). It follows from (3) that \( \beta^2 = \alpha^2 + \gamma^2 \) and from (2) and (3) that

\[
f_4(\alpha, \gamma) = \beta^4 + 2\beta^2(2\alpha\gamma) + (\alpha^2 + \gamma^2)^2 = 2\beta^2(1 - \beta)^2 < \frac{1}{9} .
\]

Hence \( \beta(1 - \beta) < 1/3\sqrt{2} \) and therefore

\[
(4) \quad \text{either } \beta < \frac{1 - \sqrt{1 - 2\sqrt{2}/3}}{2} \quad \text{or } \beta > \frac{1 + \sqrt{1 - 2\sqrt{2}/3}}{2} .
\]

It also follows from (3) that \( 2\gamma^2 - 2(1 - \beta)\gamma + 1 - 2\beta = 0 \) and therefore, since \( \gamma \) is a real number, then

\[
(5) \quad \beta \geq \sqrt{2} - 1 .
\]

Finally, (3) implies that \( 1 - 2\beta - 2\alpha\gamma = 0 \), and therefore since \( \alpha\gamma \geq 0 \), then

\[
(6) \quad 3 \leq 1/2 .
\]
Inequalities (4), (5) and (6) constitute a contradiction.

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REFERENCES


Received July 3, 1973.

Missouri Southern State College
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John Allen Beachy and William David Blair, On rings with bounded annihilators .......................................................... 1

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