

# Pacific Journal of Mathematics

## **THE ROGERS-RAMANUJAN RECIPROCAL AND MINC'S PARTITION FUNCTION**

GEORGE E. ANDREWS

# THE ROGERS-RAMANUJAN RECIPROCAL AND MINC'S PARTITION FUNCTION

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**The reciprocals of the Rogers-Ramanujan identities are considered, and it is shown that the results yield identities for restricted compositions. The same technique is applied to obtain a generating function for partitions previously treated by H. Minc.**

1. Introduction. The celebrated Rogers-Ramanujan identities were first presented in their analytic form as follows:

$$(1.1) \quad 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \dots$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})};$$

$$(1.2) \quad 1 + \frac{q^9}{1-q} + \frac{q^8}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \dots$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

The fascinating story of their discovery by L. J. Rogers [8] and their subsequent rediscovery by S. Ramanujan (see [5; p. 91] and I. J. Schur [9] has been told many times [1; Ch. 7], [2; Ch. 3], [5; Ch. 6]. P. A. MacMahon [6] and I. J. Schur [9] observed that (1.1) and (1.2) are equivalent to the following assertions in additive number theory:

**THEOREM R<sub>1</sub>.** *The number of partitions of  $n$  into parts that differ by at least 2 equals the number of partitions of  $n$  into parts of the forms  $5m + 1$  and  $5m + 4$ .*

**THEOREM R<sub>2</sub>.** *The number of partitions of  $n$  into parts that differ by at least 2 and contain no ones equals the number of partitions of  $n$  into parts of the forms  $5m + 2$  and  $5m + 3$ .*

Apart from Schur's two ingenious proofs in [9], all other proofs effectively rely on establishing the following two variable result:

$$(1.3) \quad F_1(z) \equiv 1 + \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(1-q)(1-q^2) \dots (1-q^n)}$$

$$= \left\{ \prod_{n=1}^{\infty} \frac{1}{(1-zq^n)} \right\} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(zq)_{n-1} (1-zq^{2n}) (-z^2)^n q^{n(5n-1)/2}}{(q)_n} \right\}$$

where  $(A)_n = (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1})$ ,  $(A)_0 = 1$ .

The reciprocal of  $F_1(-zq^{-1})$  was utilized by Carlitz and Riordan [4; p. 386, eq. (10.7)] in their work on  $q$ -analogs of two element lattice permutation numbers; however they give no indication that in fact  $1/F_1(-z)$  is the generating function for certain simply restricted compositions. In another paper Carlitz [3] treats classes of restricted compositions which he calls "up-down" and "down-up" partitions. These he shows are generated by reciprocals of  $q$ -analogs of the Olivier functions. In fact arguments similar to those given by Carlitz may be utilized to prove the following assertion.

**THEOREM 1.** *Let  $C_d(m, n)$  denote the number of representations of  $n$  in the form*

$$n = c_1 + c_2 + \cdots + c_m, \quad \text{where } 1 \leq c_{i+1} \leq c_i + d.$$

Then for  $d \geq 0$ ,

$$(1.4) \quad \sum_{m, n \geq 0} C_d(m, n) z^m q^n = \frac{1}{F_d(-z)},$$

where

$$F_d(z) = \sum_{n=0}^{\infty} \frac{q^{d\binom{n}{2} + \binom{n+1}{2}}}{(q)_n}.$$

We note that  $C_0(m, n)$  is just the number of partitions of  $n$  into  $m$  parts and (1.4) reduces to a well-known generating function identity [1; p. 16] since

$$(1.6) \quad F_0(z) = \prod_{n=1}^{\infty} (1 + zq^n), \quad [1; \text{p. 19}].$$

Let us call a representation of  $n$  of the form  $c_1 + c_2 + \cdots + c_m$  where  $1 \leq c_{i+1} \leq c_i + 1$  a *restricted composition*, and let  $K_e(j; n)$  (resp.  $K_o(j; n)$ ) denote the number of restricted compositions with each  $c_i \geq j$  and with an even (resp. odd) number of parts. Also let  $L_e(j; n)$  (resp.  $L_o(j; n)$ ) denote the number of partitions of  $n$  into an even (resp. odd) number of parts each  $\equiv \pm j \pmod{5}$ . Then equations (1.1) and (1.2) together with Theorem 1 imply:

**THEOREM 2.** *For all  $n \geq 0$ ,*

$$(1.7) \quad K_e(1; n) - K_o(1; n) = L_e(1; n) - L_o(1; n);$$

$$(1.8) \quad K_e(2; n) - K_o(2; n) = L_e(2; n) - L_o(2; n).$$

Both Theorems 1 and 2 will be proved in § 2. In § 3 we apply

these methods to H. Minc's partition function  $\nu(1, n)$ , the number of representations of  $n$  in the form  $n = 1 + c_1 + c_2 + \dots + c_m$  where  $1 = c_0$  and  $c_{i+1} \leq 2c_i$  for  $0 \leq i \leq m - 1$ . Minc [7] reduced an enumeration problem in groupoids to the determination of  $\nu(1, n)$ , and he provided a recurrence whereby  $\nu(1, n)$  could be computed. We shall present the generating function for  $\nu(1, n)$ :

**THEOREM 3.**

$$\sum_{n=1}^{\infty} \nu(1, n)q^n = \frac{q}{\sum_{j=0}^{\infty} \frac{(-1)^j q^{2^{j+1}-j-2}}{(1-q)(1-q^3)(1-q^7) \dots (1-q^{2^{j-1}})}}.$$

**2. The Rogers-Ramanujan reciprocal.** We begin by proving Theorem 1. From the definition of  $C_d(m, n)$  we see that

$$\begin{aligned} \sum_{n \geq 0} C_d(m, n)q^n &\equiv \gamma_m = \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \sum_{c_3=1}^{c_2+d} \dots \sum_{c_m=1}^{c_{m-1}+d} q^{c_1+c_2+\dots+c_m} \\ &= \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \dots \sum_{c_{m-1}=1}^{c_{m-2}+d} q^{c_1+c_2+\dots+c_{m-1}} \frac{(q - q^{c_{m-1}+d+1})}{(1-q)} \\ &= \frac{q}{1-q} \gamma_{m-1} - \frac{q^{d+1}}{1-q} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \dots \sum_{c_{m-2}=1}^{c_{m-3}+d} q^{c_1+c_2+\dots+c_{m-2}} \frac{(q^2 - q^{2c_{m-2}+2d+2})}{(1-q^2)} \\ &= \frac{q}{1-q} \gamma_{m-1} - \frac{q^{d+3}}{(1-q)(1-q^2)} \gamma_{m-2} \\ (2.1) \quad &+ \frac{q^{3d+3}}{(1-q)(1-q^2)} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \dots \sum_{c_{m-3}=1}^{c_{m-4}+1} q^{c_1+c_2+\dots+c_{m-3}} \frac{(q^3 - q^{3c_{m-3}+3d+3})}{(1-q^3)} \\ &= \frac{q}{1-q} \gamma_{m-1} - \frac{q^{d+3}}{(1-q)(1-q^2)} \gamma_{m-2} + \frac{q^{3d+6}}{(1-q)(1-q^2)(1-q^3)} \gamma_{m-3} \\ &- \frac{q^{6d+6}}{(1-q)(1-q^2)(1-q^3)} \sum_{c_1=1}^{\infty} \sum_{c_2=1}^{c_1+d} \dots \sum_{c_{m-4}=1}^{c_{m-5}+d} q^{c_1+\dots+c_{m-4}} \frac{(q^4 - q^{4c_{m-4}+4d+4})}{(1-q^4)} \\ &= \vdots \end{aligned}$$

Thus applying mathematical induction we may rigorously establish that the above iterative process yields

$$(2.2) \quad \sum_{j=0}^m \gamma_{m-j} \frac{(-1)^j q^{d \binom{j}{2} + \binom{j+1}{2}}}{(q)_j} = \begin{cases} 0 & \text{if } m > 0 \\ 1 & \text{if } m = 0. \end{cases}$$

Hence (2.2) is equivalent to

$$(2.3) \quad \sum_{m=0}^{\infty} \gamma_m z^m \sum_{n=0}^{\infty} \frac{(-1)^n q^{d \binom{n}{2} + \binom{n+1}{2}}}{(q)_n} = 1.$$

Consequently by (2.3),

$$\begin{aligned}
 \sum_{n, m \geq 0} C_d(m, n) z^m q^n &= \sum_{m \geq 0} \gamma_m z^m \\
 (2.4) \qquad \qquad \qquad &= \frac{1}{F_d(-z)}.
 \end{aligned}$$

Therefore Theorem 1 is established.

As we remarked in the introduction, Theorem 2 follows immediately from Theorem 1 and the Rogers-Ramanujan identities. Namely

$$\begin{aligned}
 \sum_{n=0}^{\infty} (K_e(1; n) - K_0(1; n)) q^n & \\
 &= \sum_{n, m \geq 0} C_1(m, n) (-1)^m q^n \quad (\text{by definition}) \\
 (2.5) \qquad \qquad \qquad &= \frac{1}{F_1(1)} \quad (\text{by Theorem 1}) \\
 &= \prod_{n=0}^{\infty} (1 - q^{5n+1})(1 - q^{5n+4}) \quad (\text{by (1.1)}) \\
 &= \sum_{n=0}^{\infty} (L_e(1; n) - L_0(1; n)) q^n.
 \end{aligned}$$

Equation (1.7) follows immediately from (2.5) when we compare coefficients of  $q^n$  in the extreme terms. Similarly for (1.8) we see that

$$\begin{aligned}
 \sum_{n=0}^{\infty} (K_e(2; n) - K_0(2; n)) q^n & \\
 &= \sum_{n, m \geq 0} C_1(m, n) (-q)^m q^n \\
 &= \frac{1}{F_1(q)} \\
 &= \prod_{n=0}^{\infty} (1 - q^{5n+2})(1 - q^{5n+3}) \\
 &= \sum_{n=0}^{\infty} (L_e(2; n) - L_0(2; n)) q^n.
 \end{aligned}$$

**3. Minc's partition function.** If  $\mu_m$  denotes the generating function for Minc's partitions with  $m$  parts then as in § 2:

$$\begin{aligned}
 \mu_m &= \sum_{c_1=1}^2 \sum_{c_2=1}^{2c_1} \dots \sum_{c_{m-1}=1}^{2c_{m-2}} q^{1+c_1+c_2+\dots+c_m} \\
 (3.1) \qquad &= \sum_{c_1=1}^2 \sum_{c_2=1}^{2c_1} \dots \sum_{c_{m-1}=1}^{2c_{m-2}} q^{1+c_1+c_2+\dots+c_{m-2}} \frac{(q - q^{2c_{m-1}+1})}{(1 - q)} \\
 &= \frac{q}{1 - q} \mu_{m-1} - \frac{q}{1 - q} \sum_{c_1=1}^2 \dots \sum_{c_{m-2}=1}^{2c_{m-3}} q^{1+c_1+\dots+c_{m-2}} \frac{(q^3 - q^{6c_{m-2}+3})}{(1 - q^3)} \\
 &= \frac{q}{1 - q} \mu_{m-1} - \frac{q^4}{(1 - q)(1 - q^3)} \mu_{m-2}
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{q^4}{(1-q)(1-q^3)} \sum_{c_1=1}^3 \cdots \sum_{c_{m-3}=1}^{2c_{m-4}} q^{1+c_1+\cdots+c_{m-3}} \frac{(q^7 - q^{14c_{m-3}+7})}{(1-q^7)} \\
 &= \vdots
 \end{aligned}$$

As before applying mathematical induction we may rigorously establish that the above iterative process yields

$$(3.2) \quad \sum_{i=0}^m \mu_{m-i} \frac{(-1)^j q^{1+3+7+\cdots+(2^j-1)}}{(1-q)(1-q^3)(1-q^7) \cdots (1-q^{2^j-1})} = \begin{cases} 0 & \text{for } m > 0 \\ q & \text{for } m = 0. \end{cases}$$

Therefore as in Theorem 1

$$\sum_{n=1}^{\infty} \nu(1, n) q^n = \sum_{m=0}^{\infty} \mu_m = \frac{q}{\sum_{j=0}^{\infty} \frac{(-1)^j q^{1+3+7+\cdots+(2^j-1)}}{(1-q)(1-q^3)(1-q^7) \cdots (1-q^{2^j-1})}}$$

and this is clearly seen to be equivalent to Theorem 3 once we recall that  $\sum_{j=0}^s (2^j - 1) = 2^{s+1} - s - 2$ .

**4. Conclusion.** The method here could obviously be applied more generally; for example, the role of 2 in Minc's partitions could clearly be played by any positive integer  $k$ . Of course similar methods are used by Carlitz [3] to treat up-down and down-up partitions. After first discovering Theorem 1, I had hoped that it might be possible to find similar results in general for

$$\frac{1}{f_{\mathcal{C}}(-z, q)}$$

where  $f_{\mathcal{C}}(z, q)$  is the two variable generating function for the linked partition ideal  $\mathcal{C}$  (see [1; Ch. 8] for an explanation of linked partition ideals). Unfortunately the coefficients are not even positive in general.

There is a natural way of providing a common generalization of Theorems 1 and 3. Namely the difference conditions bounding  $c_{i+1}$  can be extended to  $1 \leq c_{i+1} \leq d + a_0 c_i + a_1 c_{i-1} + \cdots + a_j c_{i-j}$ . For example the generating function for representations of  $n$  of the form

$$n = 1 + 1 + c_1 + c_2 + \cdots + c_m$$

subject to  $c_{-1} = c_0 = 1$  and  $c_{i+1} \leq c_i + c_{i-1}$  is

$$\sum_{n=0}^{\infty} \frac{q^2}{(1 - q^{u_1-1})(1 - q^{u_2-1}) \cdots (1 - q^{u_n-1})} \frac{q^{u_1+\cdots+u_n-n} (-1)^n}{1}$$

where  $u_i$  are shifted Fibonacci numbers  $u_1 = 2$ ,  $u_2 = 3$ ,  $u_n = u_{n-1} + u_{n-2}$  for  $n > 2$ . In general the Fibonacci exponent  $u_i - 1$  in the generating function will be replaced by the sum of the 1st  $i$  terms of the recurrent sequence arising from the recurrence  $c_{n+1} = d + a_0 c_n + a_1 c_{n-1} + \cdots + a_j c_{n-j}$ .

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George E. Andrews, The Rogers-Ramanujan reciprocal and Minc's partition function .....	251
Allan Calder, William H. Julian, Ray Mines, III and Fred Richman, $\varepsilon$ -covering dimension .....	257
Thomas Curtis Craven and George Leslie Csordas, An inequality for the distribution of zeros of polynomials and entire functions .....	263
Thomas Jones Enright and R. Parthasarathy, The transfer of invariant pairings to lattices .....	281
Allen Roy Freedman and John Joseph Sember, Densities and summability .....	293
Robert Heller and Francis Aubra Roach, A generalization of a classical necessary condition for convergence of continued fractions .....	307
Peter Wilcox Jones, Ratios of interpolating Blaschke products .....	311
V. J. Joseph, Smooth actions of the circle group on exotic spheres .....	323
Mohd Saeed Khan, Common fixed point theorems for multivalued mappings .....	337
Samuel James Lomonaco, Jr., The homotopy groups of knots. I. How to compute the algebraic 2-type .....	349
Louis Magnin, Some remarks about $C^\infty$ vectors in representations of connected locally compact groups .....	391
Mark Mandelker, Located sets on the line .....	401
Murray Angus Marshall and Joseph Lewis Yucas, Linked quaternionic mappings and their associated Witt rings .....	411
William Lindall Paschke, $K$ -theory for commutants in the Calkin algebra .....	427
W. J. Phillips, On the relation $PQ - QP = -iI$ .....	435
Ellen Elizabeth Reed, A class of Wallman-type extensions .....	443
Sungwoo Suh, The space of real parts of algebras of Fourier transforms .....	461
Antonius Johannes Van Haagen, Finite signed measures on function spaces .....	467
Richard Hawks Warren, Identification spaces and unique uniformity .....	483