

# Pacific Journal of Mathematics

**A GENERALIZATION OF A CLASSICAL NECESSARY  
CONDITION FOR CONVERGENCE OF CONTINUED  
FRACTIONS**

ROBERT HELLER AND FRANCIS AUBRA ROACH

# A GENERALIZATION OF A CLASSICAL NECESSARY CONDITION FOR CONVERGENCE OF CONTINUED FRACTIONS<sup>1</sup>

ROBERT HELLER AND F. A. ROACH

**One of the most frequently cited necessary conditions for convergence of continued fractions is the divergence of a particular series. In this paper, we show that convergence of a continued fraction implies divergence of each member of an infinite collection of series.**

We will be concerned with continued fractions which are of, or can be put into (cf. Wall [4], pp. 19-26), the form

$$(1) \quad b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \dots}}}$$

If we let

$$(2) \quad \begin{aligned} A_0 &= b_0, & A_1 &= b_1 A_0 + 1, & B_0 &= 1, & B_1 &= b_1, \\ A_p &= b_p A_{p-1} + A_{p-2}, & \text{and} & & & & & \\ B_p &= b_p B_{p-1} + B_{p-2}, & p &= 2, 3, 4, \dots, \end{aligned}$$

then the  $n$ th approximant of (1) is given by  $A_n/B_n$ . As is customary, we say that (1) converges provided that not infinitely many of the denominators  $B_p$  are zero and  $\{A_p/B_p\}$  converges to a finite limit.

The principal result given in this paper is the following theorem.

**THEOREM.** *Suppose that  $u$  is a complex number,  $v$  is a complex number such that  $-4 < uv \leq 0$ , and  $u = 0$  if  $v = 0$ . If both  $\sum |b_{2p-1} - u|$  and  $\sum |b_{2p} - v|$  converge, then (1) diverges.*

Considering the case where  $u = v$ , we immediately have the following result.

**COROLLARY.** *In order for (1) to converge, it is necessary that for each real number  $k$  between  $-2$  and  $2$ ,  $\sum |b_p - ki|$  diverge.*

If  $k = 0$ , this becomes what is often referred to as von Koch's theorem. According to Perron [2] p. 235, it was first proved by Stern [3] in 1860; additional information concerning the numerators and denominators of the approximants was obtained by von Koch [1] in 1895 (cf. Wall [4], pp. 27-29).

---

<sup>1</sup> This paper is dedicated to the memory of Keith Heller.

The proof is accomplished by establishing that the sequence  $\{B_p\}$  is bounded. This, together with the identity

$$|(A_n/B_n) - (A_{n+1}/B_{n+1})| = 1/|B_n B_{n+1}|$$

implies that (1) is divergent. We will establish the boundedness of this sequence by comparing it with the sequence  $\{D_p\}$  of denominators of the approximants of the periodic continued fraction

$$(3) \quad \frac{1}{u} + \frac{1}{v} + \frac{1}{u} + \frac{1}{v} + \dots$$

which is divergent if and only if  $u$  and  $v$  satisfy the hypothesis. The sequence  $\{D_p\}$  is bounded if  $u = 0$ . If  $uv \neq 0$ , then (3) is equivalent to

$$\frac{z}{u} \left[ \frac{1}{z} + \frac{1}{z} + \frac{1}{z} + \dots \right]$$

where  $z = \sqrt{(|uv|)}i$ . Let  $r$  denote  $[z + \sqrt{(z^2 + 4)}]/2$  and let  $s$  denote  $[z - \sqrt{(z^2 + 4)}]/2$ . Since  $r + s = z$  and  $-rs = 1$ , from (2) we have that for  $p = 2, 3, 4, \dots$ ,

$$D_p = (r + s)D_{p-1} - rsD_{p-2}.$$

From this it follows that  $D_p - rD_{p-1} = s^p$  and  $D_p - sD_{p-1} = r^p$  and hence,

$$(r - s)D_{p-1} = r^p - s^p.$$

Since  $-4 < z^2 < 0$ , we have that  $z^2 + 4$  is a positive number and therefore the complex conjugate of  $r$  is  $[-z + \sqrt{(z^2 + 4)}]/2$ . Thus,  $|r^p| = |s^p| = 1$  and  $2/\sqrt{(z^2 + 4)}$  is a bound for  $|D_p|$ .

Let  $x_p$  denote  $B_p - D_p$ ,  $c_{2p-1}$  denote  $b_{2p-1} - u$ , and  $c_{2p}$  denote  $b_{2p} - v$ . We will now show that for each positive integer  $n$ ,

$$(4) \quad \begin{aligned} x_{2n-1} &= \sum_{p=1}^{2n-1} c_p D_{2n-1-p} B_{p-1} \quad \text{and} \\ x_{2n} &= \sum_{p=1}^{2n} c_p D'_{2n-p} B_{p-1}, \end{aligned}$$

where  $D'_p = (u/v)D_p$  if  $p$  is odd and  $D'_p = D_p$  if  $p$  is even.

Notice that  $x_1 = c_1 D_0 B_0$  and  $x_2 = c_1 D'_1 B_0 + c_2 D'_0 B_1$ . Suppose that for some  $n$ , (4) holds true. From (2) we see that

$$x_{2n+1} = (u + c_{2n+1})B_{2n} + B_{2n-1} - (uD_{2n} + D_{2n-1})$$

which is  $ux_{2n} + x_{2n-1} + c_{2n+1}B_{2n}$ . Replacing  $x_{2n}$  and  $x_{2n-1}$  with the appropriate sums, we have that  $x_{2n+1}$  is

$$\sum_{p=1}^{2n-1} c_p (uD'_{2n-p} + D_{2p-1-p})B_{p-1} + c_{2n} D_1 B_{2p-1} + c_{2p+1} D_0 B_{2p}.$$

This expression can be written as  $\sum_{p=1}^{2n+1} c_p D_{2n+1-p} B_{p-1}$ . In a similar manner, we find that

$$x_{2n+2} = \sum_{p=1}^{2n+2} c_p D'_{2n+2-p} B_{p-1} .$$

So, for each  $n$ , the equations (4) hold true.

We will now show that there exists a nonnegative number  $K$  and a positive number  $M$  such that

$$(5) \quad |x_n| \leq K \prod_{p=1}^{n-1} (1 + M|c_{p+1}|) , \quad n = 2, 3, 4, \dots .$$

Let  $M$  denote a number such that for each  $n$ ,  $|D_n| \leq M$  and  $|D'_n| \leq M$ . From (4),

$$|x_n| \leq M \sum_{p=1}^n |c_p B_{p-1}| \leq M \sum_{p=1}^n |c_p| (|D_{p-1}| + |x_{p-1}|) .$$

By hypothesis,  $\sum |c_p|$  is convergent and hence this sum does not exceed

$$K + \sum_{p=1}^n M |c_p x_{p-1}|$$

where  $K = M^2 \sum_{p=1}^{\infty} |c_p|$ . Since  $x_0 = 0$ , we have that

$$(6) \quad |x_n| \leq K + \sum_{p=1}^{n-1} M |c_{p+1} x_p| , \quad n = 1, 2, 3, \dots ,$$

where  $\sum_{p=1}^0 M |c_{p+1} x_p| = 0$ . Suppose that  $j$  is a positive integer such that for  $n = 1, 2, \dots, j$ , (5) holds true. Then, combining (5) and (6), we have

$$|x_{j+1}| \leq K + \sum_{p=1}^j \left[ M |c_{p+1}| K \prod_{q=1}^{p-1} (1 + M |c_{q+1}|) \right] .$$

The right-hand member of this inequality can be reduced to

$$K \prod_{p=1}^j (1 + M |c_{p+1}|) .$$

Thus by mathematical induction, (5) is established.

Since the series  $\sum M |c_{p+1}|$  is convergent, so is the product  $\prod (1 + M |c_{p+1}|)$ . So, the sequence  $\{x_p\}$  is bounded. Consequently, the sequence  $\{B_p\}$  is bounded and (1) is divergent.

This theorem yields a new necessary condition for convergence of (1) which is considerably stronger than the classical condition. The new condition is not, however, sufficient for the convergence of (1). In fact, even the divergence of both of the series  $\sum |b_{2p-1} - u|$  and  $\sum |b_{2p} - v|$  for every  $u$  and  $v$  satisfying the conditions of the theorem is not sufficient for convergence of (1). This can be seen

by considering the following example. For  $p = 1, 2, 3, \dots$ , let

$$\begin{aligned}b_{3p-2} &= 1, \\b_{3p-1} &= -1 \quad \text{and} \\b_{3p} &= 1.\end{aligned}$$

In this case,  $B_{3p-1} = 0$ ,  $p = 1, 2, 3, \dots$ , but both of the series above are divergent regardless of the values of  $u$  and  $v$ .

#### REFERENCES

1. H. von Koch, *Sur un théorème de Stieltjes et sur les fractions continues*, Bull. Soc. Math. de France, **23** (1895), 23-40.
2. O. Perron, *Die Lehre von den Kettenbrüchen*, 2nd ed., Leipzig, 1929.
3. M. A. Stern, *Lehrbuch der algebraischen Analysis*, Leipzig, 1860.
4. H. S. Wall, *Analytic Theory of Continued Fractions*, Van Nostrand, Princeton, N.J., 1948.

Received September 6, 1979.

MISSISSIPPI STATE UNIVERSITY  
MISSISSIPPI STATE, MS 39762

AND

UNIVERSITY OF HOUSTON  
HOUSTON, TX 77004

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DONALD BABBITT (Managing Editor)

University of California  
Los Angeles, CA 90024

HUGO ROSSI

University of Utah  
Salt Lake City, UT 84112

C. C. MOORE and ANDREW OGG

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University  
Stanford, CA 94305

## ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA  
UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA, RENO  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF HAWAII  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1981 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

<b>George E. Andrews</b> , <i>The Rogers-Ramanujan reciprocal and Minc's partition function</i> .....	251
<b>Allan Calder, William H. Julian, Ray Mines, III and Fred Richman</b> , <i><math>\varepsilon</math>-covering dimension</i> .....	257
<b>Thomas Curtis Craven and George Leslie Csordas</b> , <i>An inequality for the distribution of zeros of polynomials and entire functions</i> .....	263
<b>Thomas Jones Enright and R. Parthasarathy</b> , <i>The transfer of invariant pairings to lattices</i> .....	281
<b>Allen Roy Freedman and John Joseph Sember</b> , <i>Densities and summability</i> .....	293
<b>Robert Heller and Francis Aubra Roach</b> , <i>A generalization of a classical necessary condition for convergence of continued fractions</i> .....	307
<b>Peter Wilcox Jones</b> , <i>Ratios of interpolating Blaschke products</i> .....	311
<b>V. J. Joseph</b> , <i>Smooth actions of the circle group on exotic spheres</i> .....	323
<b>Mohd Saeed Khan</b> , <i>Common fixed point theorems for multivalued mappings</i> .....	337
<b>Samuel James Lomonaco, Jr.</b> , <i>The homotopy groups of knots. I. How to compute the algebraic 2-type</i> .....	349
<b>Louis Magnin</b> , <i>Some remarks about <math>C^\infty</math> vectors in representations of connected locally compact groups</i> .....	391
<b>Mark Mandelker</b> , <i>Located sets on the line</i> .....	401
<b>Murray Angus Marshall and Joseph Lewis Yucas</b> , <i>Linked quaternionic mappings and their associated Witt rings</i> .....	411
<b>William Lindall Paschke</b> , <i><math>K</math>-theory for commutants in the Calkin algebra</i> .....	427
<b>W. J. Phillips</b> , <i>On the relation <math>PQ - QP = -iI</math></i> .....	435
<b>Ellen Elizabeth Reed</b> , <i>A class of Wallman-type extensions</i> .....	443
<b>Sungwoo Suh</b> , <i>The space of real parts of algebras of Fourier transforms</i> .....	461
<b>Antonius Johannes Van Haagen</b> , <i>Finite signed measures on function spaces</i> .....	467
<b>Richard Hawks Warren</b> , <i>Identification spaces and unique uniformity</i> .....	483