

Pacific Journal of Mathematics

ON THE RELATION $PQ - QP = -iI$

W. J. PHILLIPS

ON THE RELATION $PQ - QP = -iI$

W. J. PHILLIPS

There is a variety of literature on the relationship between the two equations

$$(1.1) \quad PQ - QP \subset -iI$$

$$(1.2) \quad \exp(itP) \exp(isQ) = \exp(ist) \exp(isQ) \exp(itP), \quad s, t \in R$$

where P and Q are self-adjoint operators on a Hilbert space. Von Neumann has characterized the solutions of (1.2) as those pairs (P, Q) which are unitarily equivalent to a direct sum of a number of copies of the Schrödinger pair (p, q) where p is $-i(d/dx)$ and q is multiplication by x on $L^2(R)$. Hence any pair which satisfies (1.2) and is irreducible in the obvious sense is unitarily equivalent to the Schrödinger pair. It is well known that any pair satisfying (1.2) satisfies (1.1) in the following strong sense:

$$(1.3) \quad \text{there is a dense subspace } \Omega \text{ which is a core for both } P \text{ and } Q, \text{ is invariant under } P \text{ and } Q \text{ and } PQf - QPf = -if \text{ for all } f \text{ in } \Omega.$$

In this paper we construct an uncountable family of irreducible, unitarily inequivalent pairs satisfying (1.3) but not (1.2).

A reducible pair satisfying (1.3) but not (1.2) is given in [4, page 275]. The construction is due to Nelson. It was a detailed examination of this pair which led us to our uncountable family of pairs. In [1], Fuglede constructs a pair (P_F, Q_F) satisfying a stronger version of (1.3) but not (1.2), but the irreducibility of the pair is left as an open question. The techniques in the analysis of our examples are applicable to (P_F, Q_F) and we show that this pair is irreducible. For a related example due to Fuglede, see [2, Example 2].

The operators constructed in [4] and [1] are obtained from pairs (X, Y) of self-adjoint operators satisfying

$$(1.4) \quad \text{there is a dense subspace } \Omega \text{ which is a core for both. } X \text{ and } Y, \text{ is invariant under } X \text{ and } Y \text{ and}$$

$$XYf = YXf \quad \text{for all } f \text{ in } \Omega$$

but not

$$(1.5) \quad \text{the spectral projections of } X \text{ and } Y \text{ commute.}$$

We also construct an uncountable family of unitarily inequivalent pairs of operators satisfying (1.4) but not (1.5). For other such pairs see [4, page 273, Example 1], [1] and [2] and [3, Example 3].

The results in this paper were obtained under the supervision of E. J. Woods. The author wishes to express his gratitude to Professor Woods and Queen's University.

2. The operators. Consider the Hilbert space $H = L^2(R^2)$. The Schrödinger 2-system $\{p_1, p_2; q_1, q_2\}$ is defined by the following strongly continuous unitary groups.

$$(2.1) \quad \begin{cases} (\exp(itq_j)f)(x_1, x_2) = \exp(itx_j)f(x_1, x_2) & j = 1, 2 \\ (\exp(itp_1)f)(x_1, x_2) = f(x_1 + t, x_2) \\ (\exp(itp_2)f)(x_1, x_2) = f(x_1, x_2 + t) . \end{cases}$$

For $\alpha \in A = \{z \in C: |z| = 1\}$ define $\phi_\alpha: R^2 \rightarrow C$ by

$$\phi_\alpha(x_1, x_2) = \begin{cases} \alpha & x_1 \leq 0, x_2 \geq 0 \\ 1 & \text{elsewhere.} \end{cases}$$

Define the self-adjoint operators P_α, Q_α by the following strongly continuous unitary groups.

$$(2.2) \quad \begin{cases} \exp(itP_\alpha) = \exp(itp_1) \\ \exp(itQ_\alpha) = \bar{\phi}_\alpha(q_1, q_2) \exp(itq_1) \exp(itp_2) \phi_\alpha(q_1, q_2) . \end{cases}$$

THEOREM 1. *The family $\{(P_\alpha, Q_\alpha): \alpha \in A\}$ satisfies the following properties:*

(a) *For each α there is a dense subspace*

$$\Omega_\alpha \subset \text{dom } P_\alpha \cap \text{dom } Q_\alpha \quad (\text{dom } X \text{ is the domain of } X)$$

with $P_\alpha \Omega_\alpha \subset \Omega_\alpha, Q_\alpha \Omega_\alpha \subset \Omega_\alpha,$

$P_\alpha|_{\Omega_\alpha}$ and $Q_\alpha|_{\Omega_\alpha}$ are essentially self-adjoint.

(b) *For each $\alpha \neq 1$, the pair (P_α, Q_α) is irreducible.*

(c) *For various α , the pairs (P_α, Q_α) are unitarily inequivalent.*

(d) *For each α , the pair (P_α, Q_α) satisfies (1.1); that is,*

$$P_\alpha Q_\alpha f - Q_\alpha P_\alpha f = -if \quad f \in \text{dom } P_\alpha Q_\alpha \cap \text{dom } Q_\alpha P_\alpha .$$

(e) *For each $\alpha \neq 1$ the pair (P_α, Q_α) does not satisfy (1.2).*

The operators P_α and Q_α are related to the self-adjoint operators X_α and Y_α defined by

$$(2.3) \quad \begin{cases} \exp(itX_\alpha) = \exp(itp_1) \\ \exp(itY_\alpha) = \bar{\phi}_\alpha(q_1, q_2) \exp(itp_2) \phi_\alpha(q_1, q_2) \end{cases}$$

and we have the following result:

THEOREM 2. *The family $\{(X_\alpha, Y_\alpha): \alpha \in A\}$ satisfies properties (a), (b) and (c) of Theorem 1. In addition, for each $f \in (\text{dom } X_\alpha Y_\alpha) \cap (\text{dom } Y_\alpha X_\alpha)$*

$$(d)' X_\alpha Y_\alpha f = Y_\alpha X_\alpha f.$$

REMARKS. Since the pair (X_α, Y_α) is irreducible for $\alpha \neq 1$ it is clear that their spectral projections cannot commute.

Proof of Theorem 1. Let Ω_α be the set of all $f \in L^2(R^2)$ such that

$$(1) f \in C^\infty(R^2 \setminus \{(x_1, x_2): x_2 = 0, x_1 \leq 0\})$$

$$(2) \alpha(D_1^j D_2^k f)(x_1, 0^+) = (D_1^j D_2^k f)(x_1, 0^-) \quad x_1 < 0, \quad j, k = 0, 1, 2, \dots$$

$$(3) f \text{ is compactly supported.}$$

It can be shown that:

$$\text{dom } p_j = \{f \in L^2(R^2): \text{for almost all } x_k \ (k \neq j), (x_j \rightarrow f(x_1, x_2)) \text{ is absolutely continuous with } D_j f \in L^2(R^2)\}$$

$$(p_j f)(x_1, x_2) = -i(D_j f)(x_1, x_2), \quad f \in \text{dom } p_j$$

$$\text{dom } q_j = \{f \in L^2(R^2): ((x_1, x_2) \rightarrow x_j f(x_1, x_2)) \in L^2(R^2)\}$$

$$(q_j f)(x_1, x_2) = x_j f(x_1, x_2), \quad f \in \text{dom } q_j.$$

From these facts it is easily deduced that $\Omega_\alpha \subset \text{dom } P_\alpha \cap \text{dom } Q_\alpha$ and for $f \in \Omega_\alpha$ we have the following formulas for a.e. (x_1, x_2) :

$$(P_\alpha f)(x_1, x_2) = -i(D_1 f)(x_1, x_2)$$

$$(Q_\alpha f)(x_1, x_2) = x_1 f(x_1, x_2) - i(D_2 f)(x_1, x_2).$$

This shows that $P_\alpha \Omega_\alpha \subset \Omega_\alpha$ and $Q_\alpha \Omega_\alpha \subset \Omega_\alpha$. To prove the rest of part (a) we introduce the dense subspaces

$$\Omega'_\alpha = \{f \in \Omega_\alpha: (D_1^j D_2^k f)(0, x_2) = 0 \quad \forall x_2, \forall j, k = 0, 1, \dots\}$$

$$\Omega''_\alpha = \{f \in \Omega_\alpha: (D_1^j D_2^k f)(x_1, 0^\pm) = 0 \quad \forall x_1, \forall j, k = 0, 1, \dots\}.$$

These subspaces are dense for they contain all C^∞ compactly supported functions vanishing in a neighborhood of the axes. Furthermore Ω'_α is invariant under the group $(\exp(itQ_\alpha): t \in \mathbb{Q})$ and Ω''_α is invariant under the group $(\exp(itP_\alpha): t \in \mathbb{R})$. It follows from well-known properties of strongly continuous semigroups (see [1] Theorem VIII.10) that

$$(P_\alpha|_{\Omega''_\alpha})^{**} = P_\alpha \quad \text{and} \quad (Q_\alpha|_{\Omega'_\alpha})^{**} = Q_\alpha,$$

so the same result holds when Ω_α is substituted for Ω'_α and Ω''_α . This proves part (a).

Parts (b), (c) and (e) can be proven by computing the "group commutator"

$$C_\alpha(s, t) = \exp(isP_\alpha) \exp(itQ_\alpha) \exp(-isP_\alpha) \exp(-itQ_\alpha).$$

Using the definitions of P_α and Q_α we have

$$(2.4) \quad C_\alpha(s, t) = \exp(isp_1)\bar{\phi}_\alpha(q_1, q_2) \exp(itq_1) \exp(itp_2)\phi_\alpha(q_1, q_2) \\ \times \exp(-isp_1)\bar{\phi}_\alpha(q_1, q_2) \exp(-itq_1) \exp(-itp_2)\phi_\alpha(q_1, q_2) .$$

It follows from (2.1) that if $\phi: R^2 \rightarrow C$ is a bounded Borel function then

$$(2.5) \quad \begin{cases} \exp(itp_1)\phi(q_1, q_2) = \phi(q_1 + tI, q_2) \exp(itp_1) \\ \exp(itp_2)\phi(q_1, q_2) = \phi(q_1, q_2 + tI) \exp(itp_2) \\ \exp(isp_1) \exp(itq_1) = \exp(ist) \exp(itq_1) \exp(isp_1) . \end{cases}$$

Using (2.5) we can simplify (2.4)

$$C_\alpha(s, t) = \exp(ist)\bar{\phi}_\alpha(q_1 + sI, q_2)\phi_\alpha(q_1 + sI, q_2 + tI) \\ \times \bar{\phi}_\alpha(q_1, q_2 + tI)\phi_\alpha(q_1, q_2) .$$

Now for $s, t \geq 0$ define

$$\psi_\alpha(s, t)(x_1, x_2) = \begin{cases} \alpha & -s < x_1 < 0, 0 < x_2 < t \\ 1 & \text{elsewhere.} \end{cases}$$

Then for $s, t \geq 0$ we have

$$(2.6) \quad C_\alpha(s, -t) = \exp(-ist)\psi_\alpha(s, t)(q_1, q_2) .$$

Part (c) follows immediately from (2.6) for the operators $\psi_\alpha(s, t)(q_1, q_2)$ have different eigenvalues for different values of α .

Part (e) follows immediately as well.

To prove (b), let α_α be the von Neumann algebra generated by the spectral projections of P_α and Q_α . We must show that the commutant of α_α , is $\{\lambda I: \lambda \in C\}$. From (2.6) it is clear that

$$(2.7) \quad \psi_\alpha(s, t)(q_1, q_2) \in \alpha_\alpha \quad s, t \geq 0 .$$

Since $\phi_\alpha(q_1, q_2)$ is the strong limit of $\psi_\alpha(s, t)(q_1, q_2)$ as $s, t \rightarrow \infty$ we have

$$(2.8) \quad \phi_\alpha(q_1, q_2) \in \alpha_\alpha .$$

Using (2.8) and (2.2) we obtain

$$(2.9) \quad \begin{cases} \exp(itp_1) \in \alpha_\alpha \quad \forall t \in R \\ \exp(itq_1) \exp(itp_2) \in \alpha_\alpha \quad \forall t \in R . \end{cases}$$

If $\chi(E)$ denotes the characteristic function of a set $E \subset R^2$ then (2.7) implies that

$$(2.10) \quad \chi([-t, 0] \times [0, s])(q_1, q_2) = \frac{1}{\alpha - 1}(\psi_\alpha(s, t)(q_1, q_2) - I)\alpha_\alpha .$$

Let $a \leq b, c \leq d$, then using (2.9) and (2.10) we have

$$(2.11) \quad \begin{aligned} &\chi([a, b] \times [c, d])(q_1, q_2) \\ &= \exp(-ibp_1) \exp(-icq_1) \exp(-icp_2) \chi([a - b, 0] \times [0, d - c]) \\ &\quad \times (q_1, q_2) \exp(icq_1) \exp(icp_2) \exp(ibp_1) \in \alpha_\alpha. \end{aligned}$$

So α_α must contain the maximal abelian von Neumann algebra $m = \{n(q_1, q_2) : n \in L^\infty(R^2)\}$, since linear combinations of operators of the form (2.11) are weakly dense in m . Now let $B \in \alpha'_\alpha$ (commutant of α_α) then $B \in m' = m$. So there is an $n \in L^\infty(R^2)$ such that $B = n(q_1, q_2)$. Using (2.9) and the method employed in (2.11) we obtain

$$n(q_1 + sI, q_2 + tI) = n(q_1, q_2) \quad \forall s, t.$$

This means that n is a constant almost everywhere. That is $B = \lambda I$ for some $\lambda \in C$. This proves part (b).

For part (d) we first show that if $f, g \in \text{dom } P_\alpha \cap \text{dom } Q_\alpha$ with either f or g vanishing in a neighborhood of $(0, 0)$ then

$$(2.12) \quad (Q_\alpha f, P_\alpha g) - (P_\alpha f, Q_\alpha g) = -i(f, g).$$

To prove (2.12) we use (2.6) to write

$$(2.13) \quad \begin{aligned} &(\exp(itQ_\alpha)f, \exp(isP_\alpha)g) \\ &= \exp(-ist)(\exp(-isP_\alpha)\psi_\alpha(s, t)(q_1, q_2)f, \exp(-itQ_\alpha)g). \end{aligned}$$

If f vanishes in a neighborhood of $(0, 0)$ then we can choose s and t small enough so that (2.13) takes the form

$$(2.14) \quad \begin{aligned} &(\exp(itQ_\alpha)f, \exp(isP_\alpha)g) \\ &= \exp(-ist)(\exp(-isP_\alpha)f, \exp(-itQ_\alpha)g). \end{aligned}$$

If $f, g \in \text{dom } P_\alpha \cap \text{dom } Q_\alpha$ we can differentiate both sides of (2.13) with respect to s and t . The result is (2.12). A similar proof shows that (2.12) holds when g vanishes in a neighborhood of $(0, 0)$ and $f, g \in \text{dom } P_\alpha \cap \text{dom } Q_\alpha$. In particular (2.12) holds when $f \in \text{dom } P_\alpha Q_\alpha \cap \text{dom } Q_\alpha P_\alpha$ and g is C^∞ , compactly supported and vanishes in a neighborhood of the axes. In this case we can write (2.12) as

$$(2.15) \quad (P_\alpha Q_\alpha f - Q_\alpha P_\alpha f + if, g) = 0.$$

Since (2.15) must hold for all such g , a dense subspace of $L^2(R^2)$, part (d) is proven. □

The proof of Theorem 1 can be slightly modified to give a proof of Theorem 2.

3. The pair (P_F, Q_F) . Let H be the Hilbert space $L^2(R)$ and \mathcal{F} the Fourier transform $\mathcal{F}: L^2(R) \rightarrow L^2(R)$. The Schrödinger 1-system $\{p, q\}$ is defined by the following strongly continuous unitary

groups

$$(3.1) \quad \begin{cases} (\exp(itq)f)(x) = \exp(itx)f(x) \\ (\exp(itp)f)(x) = f(x+t). \end{cases}$$

It is well known that the pair (p, q) satisfies (1.2), is irreducible and $\mathcal{F}^{-1}q\mathcal{F} = p$, $\mathcal{F}q\mathcal{F}^{-1} = -p$.

Let $\omega = \sqrt{2\pi}$ and define

$$(3.2) \quad \begin{cases} X = \exp(\omega q), & Y = \exp(-\omega p) \\ P_F = \exp(-i\omega^{-1}X)p \exp(i\omega^{-1}X) \\ Q_F = \exp(-i\omega^{-1}Y)q \exp(i\omega^{-1}Y). \end{cases}$$

Fuglede [1] proves that both pairs, (X, Y) and (P_F, Q_F) , satisfy property (a) (of Theorem 1) and the commutation relations

$$(3.3) \quad (Xf, Yg) = (Yf, Xg) \quad f, g \in \text{dom } X \cap \text{dom } Y$$

$$(3.4) \quad (Q_F f, P_F g) - (P_F f, Q_F g) = -i(f, g) \quad f, g \in \text{dom } P_F \cap \text{dom } Q_F.$$

The pair (X, Y) is shown to be irreducible but the question of irreducibility of (P_F, Q_F) is left open.

THEOREM 3. *The pair (P_F, Q_F) is irreducible.*

Proof. We use the group commutator

$$C(s, t) = \exp(itP_F) \exp(isQ_F) \exp(-itP_F) \exp(-isQ_F).$$

To simplify $C(s, t)$ we first show

$$(3.5) \quad \exp(itP_F) = \exp(ih(t)X) \exp(itp)$$

$$(3.6) \quad \exp(isQ_F) = \exp(ih(s)Y) \exp(isq)$$

where $h(t) = \omega^{-1}(e^{\omega t} - 1)$.

To prove (3.5) let $f \in L^2(R)$ then for a.e. x in R .

$$\begin{aligned} (\exp(itP_F)f)(x) &= (\exp(-i\omega^{-1}X) \exp(itp) \exp(i\omega^{-1}Xf))(x) \\ &= \exp(-i\omega^{-1} \exp(\omega x)) (\exp(i\omega^{-1}X)f)(x+t) \\ &= \exp(-i\omega^{-1} \exp(\omega x) + i\omega^{-1} \exp(\omega x + \omega t)) f(x+t) \\ &= \exp(ih(t) \exp(\omega x)) f(x+t) \\ &= (\exp(ih(t)X) \exp(itp)f)(x) \end{aligned}$$

(3.6) follows by a similar calculation. Using (3.5) and (3.6) we can simplify $C(s, t)$ to

$$C(s, t) = \exp(ist) \exp(ih(t)X) \exp(ih(s)Y) \exp(-ih(t)X) \exp(-ih(s)Y).$$

Let \mathcal{A} be the von Neumann algebra generated by P_F and Q_F .

Letting $s, t \rightarrow -\infty$ in $\exp(-ist)C(s, t)$ we obtain

$$(3.7) \quad \exp(-i\omega^{-1}X) \exp(-i\omega^{-1}Y) \exp(i\omega^{-1}X) \exp(i\omega^{-1}Y) \in \mathcal{A} .$$

Since bounded Borel functions of P_F are in \mathcal{A} ,

$$(3.8) \quad \begin{aligned} \exp(i\omega^{-1} \exp(-\omega P_F)) \\ = \exp(-i\omega^{-1}X) \exp(i\omega^{-1}Y) \exp(i\omega^{-1}X) \in \mathcal{A} . \end{aligned}$$

Multiplying (3.7) and (3.8) we obtain

$$\exp(-i\omega^{-1}Y) \in \mathcal{A} .$$

So for all $t \in R$,

$$\exp(itq) = \exp(i\omega^{-1}Y) \exp(itQ_F) \exp(-i\omega^{-1}Y) \in \mathcal{A} .$$

Similarly $\exp(isp) \in \mathcal{A}$ for all $s \in R$. Since the pair (p, q) is irreducible we see that (P_F, Q_F) is irreducible. \square

We have been unable to decide whether the relations (3.3) and (3.4) hold for the families $\{(X_\alpha, Y_\alpha): \alpha \neq 1\}$ and $\{(P_\alpha, Q_\alpha): \alpha \neq 1\}$ respectively.

REFERENCES

1. B. Fuglede, *On the relation $PQ - QP = -iI$* , Math. Scand., (1967), 79-88.
2. N. S. Poulsen, *On the canonical commutation relations*, Math. Scand., **32** (1973), 112-122.
3. R. T. Powers, *Self-adjoint algebras of unbounded operators*, Commun. Math. Phys., **21** (1971), 85-124.
4. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. I, Academic Press, 1972, N. Y.

Received April 17, 1979 and in revised form April 15, 1980.

QUEEN'S UNIVERSITY
KINGSTON, CANADA K7L 3N6

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)

University of California
Los Angeles, CA 90024

HUGO ROSSI

University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ANDREW OGG

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1981 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

George E. Andrews , <i>The Rogers-Ramanujan reciprocal and Minc's partition function</i>	251
Allan Calder, William H. Julian, Ray Mines, III and Fred Richman , <i>ε-covering dimension</i>	257
Thomas Curtis Craven and George Leslie Csordas , <i>An inequality for the distribution of zeros of polynomials and entire functions</i>	263
Thomas Jones Enright and R. Parthasarathy , <i>The transfer of invariant pairings to lattices</i>	281
Allen Roy Freedman and John Joseph Sember , <i>Densities and summability</i>	293
Robert Heller and Francis Aubra Roach , <i>A generalization of a classical necessary condition for convergence of continued fractions</i>	307
Peter Wilcox Jones , <i>Ratios of interpolating Blaschke products</i>	311
V. J. Joseph , <i>Smooth actions of the circle group on exotic spheres</i>	323
Mohd Saeed Khan , <i>Common fixed point theorems for multivalued mappings</i>	337
Samuel James Lomonaco, Jr. , <i>The homotopy groups of knots. I. How to compute the algebraic 2-type</i>	349
Louis Magnin , <i>Some remarks about C^∞ vectors in representations of connected locally compact groups</i>	391
Mark Mandelker , <i>Located sets on the line</i>	401
Murray Angus Marshall and Joseph Lewis Yucas , <i>Linked quaternionic mappings and their associated Witt rings</i>	411
William Lindall Paschke , <i>K-theory for commutants in the Calkin algebra</i>	427
W. J. Phillips , <i>On the relation $PQ - QP = -iI$</i>	435
Ellen Elizabeth Reed , <i>A class of Wallman-type extensions</i>	443
Sungwoo Suh , <i>The space of real parts of algebras of Fourier transforms</i>	461
Antonius Johannes Van Haagen , <i>Finite signed measures on function spaces</i>	467
Richard Hawks Warren , <i>Identification spaces and unique uniformity</i>	483