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ON SOME SPACES OF ENTIRE FUNCTIONS DEFINED ON INFINITE DIMENSIONAL SPACES

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Let E be a quasi-complete dual nuclear complex locally convex space; we prove that both spaces $\mathscr{H}_{Nb}(E)$ and $\mathscr{H}_{SNb}(E)$ of entire functions of nuclear type on E introduced by Matos and Matos-Nachbin coincide with the space $\mathscr{H}_S(E)$ of the Silva holomorphic functions. As a consequence, well known results of Boland on convolution equations in $\mathscr{H}(E)$ can be obtained as particular cases of results in Matos's Doctoral Dissertation.

Preface. In order to study various problems, authors were led to introduce various adequate spaces of holomorphic functions on locally convex spaces (l.c.s.). For the study of convolution equations and following Gupta [6, 7], Matos introduced in [11, 12] the concept of entire function of nuclear bounded type on any complex l.c.s. E (we denote their space by $\mathscr{H}_{Nb}(E)$). Latter Matos-Nachbin introduced in [13] the concept of Silva entire function of nuclear bounded type (we denote their space by $\mathscr{H}_{SNb}(E)$). On the other side Boland [1, 2] was the first to obtain results in the whole space $\mathscr{H}(E)$ of the entire functions (G-analytic and continuous) but under the assumption that E is a quasi-complete dual nuclear l.c.s.

The aim of this paper is to prove (Th. 3.6 and 3.9) that when E is a quasi-complete dual nuclear l.c.s. both spaces $\mathscr{H}_{Nb}(E)$ and $\mathscr{H}_{SNb}(E)$ coincide with the space $\mathscr{H}_S(E)$ of the Silva holomorphic functions on E. Since in this case $\mathscr{H}(E)$ is dense in $\mathscr{H}_S(E)$ with induced topology and $\mathscr{H}(E)$ coincides with $\mathscr{H}_S(E)$ if E is a strong dual of a nuclear Fréchet space it follows that Boland's results can now be interpreted as consequences of Matos's results, thus providing a clarification and unification of the theory.

First we fix the notations (part 1) and recall definitions (part 2). In part 3 we prove our result above and in part 4 we apply it to interprete Boland's results in term of Matos' results.

1. Notations and terminology.

We use the classical notations of the theory of infinite dimensional holomorphic functions [14]. All the vector spaces considered here are complex.

If E is a locally convex space (l.c.s. for short), E' denotes its continuous dual and $\mathscr{H}(E)$ the space of the holomorphic (i.e., G analytic + continuous) functions on E with the compact open topology τ_0. If E is a normed space $\mathscr{H}_b(E)$ denotes the space of all entire functions of bounded type (i.e., bounded over the bounded subsets of E) with the topology of uniform convergence on the bounded subsets of E.

To state the results in their correct setting and for more simplicity in the proofs we also use the concept of a bornological convex space (b.c.s. for short). A b.c.s. E is an algebraic injective inductive limit of a family of normed spaces $(E_i)_{i\in I}$. We say that $B\subset E$ is a bounded (respectively, strict compact) subset of E if it is contained and bounded (respectively, compact) in some E_i. For more details on the theory of b.c.s. see for instance Hogbé-Nlend [9] [10]. We denote by $E^\times$ the vector space of all linear forms on E which are bounded over the bounded subsets of E; we say that E is separated by its dual $E^\times$ if for every $x\neq 0$ in E there exists an $x'\in E^\times$ with $x'(x)\neq 0$; all the b.c.s. we shall consider are assumed to be separated by their dual.

When E is a l.c.s. we denote also by E the usual (Von Neumann) bornology on E. This will not give rise to any confusion.

If E is a b.c.s. we say that a complex function on it is Silva-holomorphic if its restriction to each E_i is holomorphic for the normed topology of E_i. We denote by $\mathscr{H}_S(E)$ the vector space of all Silva-holomorphic functions on E endowed with the locally convex topology of the uniform convergence on the strict compact subsets of E.

A bornological convex space is nuclear if it may be represented by $E=\varinjlim_{i\in I}E_i$ where the spaces E_i are Banach spaces such that for every $i\in I$ there is $j\in I$ such that $E_i\subset E_j$ and the corresponding injection is nuclear.

2. Recall of some definitions of spaces of entire functions of nuclear bounded type. Let $E=\varinjlim_{i\in I}E_i$ be a bornological convex space. We denote by $\mathscr{B}_E$ the family of all subsets B of E such that B is a closed convex balanced bounded subset of some E_i, $i\in I$. If E is a locally convex space we consider the Von Neumann bornology on E, and then $\mathscr{B}_E$ is considered as the family of all closed convex balanced bounded subsets of E. For $m=1,2,\cdots$ we may consider the cartesian product $E^m=E\times\cdots\times E$ (m times) with the natural bornology induced by the bornology of E. In this case we may take the vector space $\mathscr{L}_b(^mE)$ of all m-linear complex mappings on E^m

which are bounded over each element of $\mathscr{B}_{E^m}$. On $\mathscr{L}_b({}^mE)$ we consider the locally convex topology of the uniform convergence over the element of $\mathscr{B}_{E^m}$. For $m = 0$ we set $\mathscr{L}_b({}^0E)$ as the complex plane with its usual topology. We note that $E^\times = \mathscr{L}_b({}^1E)$. If $m = 0, 1, \cdots$ and $A \in \mathscr{L}_b({}^mE)$ we consider the function $\hat{A}: E \to \boldsymbol{C}$ given by $\hat{A}(x) = A(x, \cdots, x)$ (m times) for every x in E (for $m = 0$ this function is the constant function $\hat{A}(x) = A$ for each x in E). The vector space of all functions $\hat{A}$, as A varies in $\mathscr{L}_b({}^mE)$, is denoted by $\mathscr{P}_b({}^mE)$ and we consider on it the locally convex topology generated by the seminorms $\|\hat{A}\|_B = \sup\{|A(x)|; x \in B\}$ with B varying in $\mathscr{B}_E$. If $\mathscr{L}_{bs}({}^mE)$ denotes the vector subspace of $\mathscr{L}_b({}^mE)$ formed by all symmetric functions, then the natural mapping $A \to \hat{A}$ gives an isomorphism between $\mathscr{L}_{bs}({}^mE)$ and $\mathscr{P}_b({}^mE)$ which is a homeomorphism for the relative topology on $\mathscr{L}_{bs}({}^mE)$. If $m = 1, 2, \cdots$ and $\varphi_1, \cdots, \varphi_m \in E^\times$ then $\varphi_1 \times \cdots \times \varphi_m$ denotes the element of $\mathscr{L}_b({}^mE)$ given by $(\varphi_1 \times \cdots \times \varphi_m)(x_1, \cdots, x_m) = \varphi_1(x_1) \cdots \varphi_m(x_m)$.

If $\varphi_1 = \cdots = \varphi_m = \varphi$ we denote such mapping by φ^m. Let $\mathscr{L}_{bf}({}^mE)$ be the vector subspace of $\mathscr{L}_b({}^mE)$ generated by all mappings $\varphi_1 \times \cdots \times \varphi_m$ with $\varphi_1, \cdots, \varphi_m$ in $E^\times$. We set $\mathscr{L}_{bfs}({}^mE) = \mathscr{L}_{bf}({}^mE) \cap \mathscr{L}_{bs}({}^mE)$. For $m = 0$ we set $\mathscr{L}_{bf}({}^0E) = \mathscr{L}_{bfs}({}^0E) = \boldsymbol{C}$. Let $\mathscr{P}_{bf}({}^mE)$ be the corresponding subspace of $\mathscr{P}_b({}^mE)$ which is isomorphic to $\mathscr{L}_{bfs}({}^mE)$, $m \in \boldsymbol{N}$. We can show easily that $\mathscr{P}_{bf}({}^mE)$ is the set of elements P of $\mathscr{P}_b({}^mE)$ which can be written in the form $P = \sum_{j=1}^n (\varphi_j)^m$, where $\varphi_j \in E^\times$ for $j = 1, \cdots, n$. If $m = 1, 2, \cdots$ and $(E^\times)^m$ denotes the topological cartesian product, we have the continuous m-linear mapping:

$$\begin{array}{ccc} \alpha_m: (E^\times)^m & \longrightarrow & \mathscr{L}_b({}^mE) \\ (\varphi_1, \cdots, \varphi_m) & & \varphi_1 \times \cdots \times \varphi_m . \end{array}$$

Thus there is a unique continuous linear mapping χ_m from the projective tensor product $E^\times \otimes_\pi \cdots \otimes_\pi E^\times$ (m times) into $\mathscr{L}_b({}^mE)$ such that α_m is the same as $\chi_m \circ \Psi_m$, where Ψ_m is the natural m-linear mapping from $(E^\times)^m$ into $E^\times \otimes_\pi \cdots \otimes_\pi E^\times$. The mapping χ_m is injective and its image is $\mathscr{L}_{bf}({}^mE)$. The nuclear topology on $\mathscr{L}_{bf}({}^mE)$ is the locally convex topology generated by all seminorms of the form:

$$\|A\|_{N,B} = \inf\left\{\sum_{j=1}^n \|\varphi_{j1}\|_B \cdots \|\varphi_{jm}\|_B;\ A = \sum_{j=1}^n \varphi_{j1} \times \cdots \times \varphi_{jm};\ \varphi_{ij} \in E^\times\right\}$$

where $\|\varphi_{ij}\| = \sup\{|\varphi_{ij}(x)|; x \in B\}$ and $B \in \mathscr{B}_E$. The nuclear topology on $\mathscr{P}_{bf}({}^mE)$ is the locally convex topology generated by all seminorms of the type

$$\|P\|_{N,B} = \inf\left\{\sum_{j=1}^n \|\varphi_j\|_B;\ P = \sum_{j=1}^n \widehat{(\varphi_j)^m},\ \varphi_j \in E^\times\right\}$$

where $B \in \mathscr{B}_E$. It can be shown that

$$(1) \qquad \|A\|_{N,B} \leqq \|\hat{A}\|_{N,B} \leqq m^m(m!)^{-1}\|A\|_{N,B}$$

for all $A \in \mathscr{L}_{bfs}(^mE)$ and $B \in \mathscr{B}_E$. The nuclear topology on $\mathscr{L}_{bf}(^mE)$ makes this space isomorphic and homeomorphic to $E^\times \otimes_\pi \cdots \otimes_\pi E^\times$ through the mapping χ_m. The mapping χ_m can be extended continuously to "the" completion $E^\times \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E^\times$ of $E^\times \otimes_\pi \cdots \otimes_\pi E^\times$ into $\mathscr{L}_b(^mE)$ (which is complete). This extension will be noted by $\hat{\chi}_m$. We know that $\hat{\chi}_m$ is injective if and only if $E^\times$ has the approximation property. Let $\tilde{\chi}_m$ be the injective mapping from $(E^\times \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E^\times)/\ker \hat{\chi}_m$ into $\mathscr{L}_b(^mE)$. This mapping is continuous and agrees with χ_m on $E^\times \otimes_\pi \cdots \otimes_\pi E^\times$. If we consider on $(E^\times \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E^\times)/\ker \hat{\chi}_m$ the quotient topology and if we denote the image of $\hat{\chi}_m$ by $\mathscr{L}_{bN}(^mE)$, we may consider on $\mathscr{L}_{bN}(^mE)$ the locally convex topology transferred from the quotient through $\tilde{\chi}_m$. Thus $\mathscr{L}_{bN}(^mE)$ is "the" completion of $\mathscr{L}_{bf}(^mE)$ if this space is considered with the nuclear topology. We still denote by $\|\cdot\|_{N,B}$ the seminorm on $\mathscr{L}_{bN}(^mE)$ obtained by continuous extension of the seminorm $\|\cdot\|_{N,B}$ on $\mathscr{L}_{bf}(^mE)$. It can be proved that the image $\mathscr{P}_{bN}(^mE)$ of $\mathscr{L}_{bN}(^mE)$ through the natural mapping $A \to \hat{A}$ of $\mathscr{L}_b(^mE)$ onto $\mathscr{P}_b(^mE)$ is isomorphic to "the" completion of $\mathscr{P}_{bf}(^mE)$ endowed with the nuclear topology. We still denote by $\|\cdot\|_{N,B}$ the continuous extension to $\mathscr{P}_{bN}(^mE)$ of the seminorm $\|\cdot\|_{N,B}$ on $\mathscr{P}_{bf}(^mE)$. If $\mathscr{L}_{bNs}(^mE)$ is $\mathscr{L}_{bN}(^mE) \cap \mathscr{L}_{bs}(^mE)$ the inequalities (1) are true for all A in $\mathscr{L}_{bNs}(^mE)$ and B in $\mathscr{B}_E$. As usual for $m = 0$ we set $\mathscr{L}_{bN}(^0E) = \mathscr{L}_{bNs}(^0E) = C$ and $\|A\|_{N,B} = |A|$ if $A \in \mathscr{L}_{bN}(^0E)$.

DEFINITION 2.1. If $m \in N$ and $A \in \mathscr{L}_{bN}(^mE)$, we call A a Silva nuclear m-linear mapping. If $P \in \mathscr{P}_{bN}(^mE)$, we call P an m-homogeneous Silva nuclear polynomial.

Now we are in position of recalling the definition of a Silva-entire function of nuclear bounded type (see Matos-Nachbin [13]).

DEFINITION 2.2. A complex mapping f on E is said to be Silva entire of nuclear bounded type if

(a) $f \in \mathscr{H}_S(E)$

(b) $\hat{\delta}^m f(0) \in \mathscr{P}_{bN}(^mE)$ for all $m \in N$,

(c) $\lim_{m\to\infty} [(m!)^{-1}\|\hat{\delta}^m f(0)\|_{N,B}]^{1/m} = 0$ for all $B \in \mathscr{B}_E$.

We denote by $\mathscr{H}_{SNb}(E)$ the vector space of all Silva entire functions of nuclear bounded type on E. On $\mathscr{H}_{SNb}(E)$ we consider the natural locally convex topology generated by all seminorms of the type:

$$\|f\|_{N,B,\rho} = \sum_{m=0}^{\infty} \rho^m \|(m!)^{-1}\hat{\delta}^m f(0)\|_{N,B}$$

for ρ varying in the set of all positive real numbers and B in $\mathscr{B}_E$. It is easy to see that this space is complete and that it is a Fréchet space if $\mathscr{B}_E$ contains a countable fundamental system.

If in the previous construction we consider E' replacing $E^\times$ (in case E is a l.c.s.) we get the following spaces:

(1) $\mathscr{L}_f(^mE)$ which is the vector subspace of $\mathscr{L}_{bf}(^mE)$ formed by all the continuous mappings; $\mathscr{L}_{fs}(^mE) = \mathscr{L}_f(^mE) \cap \mathscr{L}_{bs}(^mE)$.

(2) $\mathscr{P}_f(^mE)$ which is the vector subspace of $\mathscr{P}_{bf}(^mE)$ formed by all the continuous polynomials.

(3) $\mathscr{L}_N(^mE)$ which is "the" completion of $\mathscr{L}_f(^mE)$ for the nuclear topology; $\mathscr{L}_{Ns}(^mE) = \mathscr{L}_N(^mE) \cap \mathscr{L}_{bs}(^mE)$.

(4) $\mathscr{P}_N(^mE)$ which is "the" completion of $\mathscr{P}_f(^mE)$ for the nuclear topology.

DEFINITION 2.3. The elements of $\mathscr{L}_N(^mE)$ and $\mathscr{P}_N(^mE)$ are called respectively nuclear m-linear mappings and nuclear m-homogeneous polynomials.

We may now recall the definition of an entire function of nuclear bounded type (see Matos [11] and [12]).

DEFINITION 2.4. A complex mapping f on a l.c.s. E is called entire of nuclear bounded type if:

(1) $f \in \mathscr{H}_S(E)$.

(2) $\hat{\delta}^m f(0) \in \mathscr{P}_N(^mE)$ for every m in N.

(3) $\lim_{m\to\infty} [(m!)^{-1}\|\hat{\delta}^m f(0)\|_{N,B}]^{1/m} = 0$ for every B in $\mathscr{B}_E$.

We denote by $\mathscr{H}_{Nb}(E)$ the complex vector subspace of $\mathscr{H}_{SNb}(E)$ formed by all entire functions of nuclear bounded type on E. In general some of the elements of $\mathscr{H}_{Nb}(E)$ may not be continuous but they may be approximated by other continuous members of $\mathscr{H}_{Nb}(E)$ in the natural topology of $\mathscr{H}_{Nb}(E)$. It can be proved that $\mathscr{H}_{Nb}(E)$ is complete and that it is a Fréchet space if $\mathscr{B}_E$ contains a countable fundamental system. If E is a holomorphically bornological l.c.s. e.g., a Silva or a metrizable locally convex space, then $\mathscr{H}_{SNb}(E) = \mathscr{H}_{Nb}(E) \subset \mathscr{H}(E)$.

REMARK 2.5. When E is a normed space $\mathscr{P}_{bN}(^mE) = \mathscr{P}_N(^mE)$ is formed by continuous polynomials and it is a Banach space (see Gupta [6] and [7]). In this case its norm is given by $\|\cdot\|_{N,B_1(0)}$ where $B_1(0)$ is the closed unit ball of E centered at the origin. In order to simplify the notations we use Gupta's $\|\cdot\|_N$ for this norm. Also

$\mathscr{H}_{SNb}(E) = \mathscr{H}_{Nb}(E)$ is formed by holomorphic functions and it is a Fréchet space. Let us remark that $\mathscr{H}_{Nb}(E)$ is contained in $\mathscr{H}_b(E)$ with continuous inclusion.

Now we define another space of Silva-holomorphic functions on E which is "nuclear of bounded type" in some sense.

DEFINITION 2.6. A function f from E into C is in the so-denoted space $\mathscr{H}S_{Nb}(E)$ if f restricted to E_B is in $\mathscr{H}_{Nb}(E_B)$ for every B in $\mathscr{B}_E$.

In $\mathscr{H}S_{Nb}(E)$ we consider the natural locally convex topology given by the seminorms:

$$\|f\|_{B,\rho} = \sum_{m=0}^{\infty} \rho^m \|(m!)^{-1}\hat{\delta}^m(f \mid E_B)(0)\|_N$$

for every f in $\mathscr{H}S_{Nb}(E)$ and $B \in \mathscr{B}_E$. Since we have

$$\operatorname*{Sup}_{x \in B} \|(m!)^{-1}\hat{\delta}^m(f \mid E_B)(0)(x)\| \leqq \|(m!)^{-1}\hat{\delta}^m(f \mid E_B)(0)\|_{N,B}$$

for each m in N and $B \in \mathscr{B}_E$ it follows that the inclusion of $\mathscr{H}S_{Nb}(E)$ into $\mathscr{H}_S(E)$ is continuous. (This inclusion is even continuous for the topology $\tau_{\omega Se}$ of Paques [15].)

It is a routine matter to prove the following result.

PROPOSITION 2.7. *The space $\mathscr{H}S_{Nb}(E)$ is the projective limit of the Fréchet spaces $\mathscr{H}_{Nb}(E_B)$, $B \in \mathscr{B}_E$, through the restriction mappings*

$$\begin{aligned} r_B \colon \mathscr{H}S_{Nb}(E) &\longrightarrow \mathscr{H}_{Nb}(E_B) \\ f &\longrightarrow f \mid E_B \,. \end{aligned}$$

As consequences of the preceding result we get that $\mathscr{H}S_{Nb}(E)$ is complete and the inclusion mapping from $\mathscr{H}_{SNb}(E)$ into $\mathscr{H}S_{Nb}(E)$ is continuous. Also a subset $\mathcal{X}$ of $\mathscr{H}S_{Nb}(E)$ is bounded if and only if $\mathcal{X}/E_B$ is bounded in $\mathscr{H}_{Nb}(E_B)$ for every $B \in \mathscr{B}_E$.

3. A comparison of these spaces of holomorphic functions. In this section E is a nuclear b.c.s. (separated by its dual) although some weaker hypotheses on E would be enough. It is well known from the properties of factorization of nuclear maps between Banach spaces (Grothendieck [8]) that E may be written as the "bornological" inductive limit of a family E_i of Hilbert spaces (Hogbe-Nlend [9] p. 71).

Let us equip $E^\times$ with the topology of uniform convergence on the bounded subsets of E; $E^\times$ is then a nuclear complete l.c.s. Since $E^\times$ and the spaces E_i have the approximation property all the maps $\hat{\chi}_m$ that we shall use now are injective. We consider now $\mathscr{B}_E$ as the set of the homothetics of the unit balls of these Hilbert spaces E_i.

LEMMA 3.1. *$E^\times = \varprojlim_{B\in\mathscr{B}_E}(E_B)'$ and more precisely $\widehat{(E^\times)}_{\mathring{B}} = (E_B)'$.*

Proof. Let B_i denote the unit ball of E_i, $\mathring{B}_i$ its polar in $E^\times$ and $(E^\times)_{\mathring{B}_i}$ the normed space associated as usual to the 0-neighborhood $\mathring{B}_i$ of $E^\times$. Let r denote the restriction map:

$$\begin{aligned} r\colon E^\times &\longrightarrow E_i' \\ l &\longrightarrow l/E_i\,. \end{aligned}$$

Then $r(E^\times)$ is dense in E_i': if not there is an $Y\in E_i''(=E_i$ here) $Y\neq 0$ in E_i and such that $Y(l)=0$ for every $l\in E^\times$ hence $l(Y)=0$ hence $Y=0$ because $E^\times$ separates E. Furthermore $r(E^\times)$ is in fact identified with $(E^\times)_{\mathring{B}_i}$ hence $\widehat{(E^\times)}_{\mathring{B}_i}$ is isomorphic to E_i'. The lemma follows from the fact that $E^\times = \varprojlim_{B\in\mathscr{B}_E}\widehat{(E^\times)}_{\mathring{B}}$.

LEMMA 3.2. *Let H be a pre-Hilbert space and let $\hat{H}$ be the Hilbert space completion of H. Then for every $m\in N$ we have the algebraic and topological equality:*

$$H^{\hat{\otimes} m}_{\pi} = (\hat{H})^{\hat{\otimes} m}_{\pi}\,.$$

Furthermore the 2 natural norms π on these respective space (defined by $*$ and $**$ below) are exactly equal.

Proof. If $x\in H^{\otimes m}_{\pi}$, $x=\sum_{\text{finite}} T_{i1}\otimes\cdots\otimes T_{im}$ with $T_{ij}\in H$ and by definition:

$$(*)\qquad \|x\|_{H^{\otimes m}_{\pi}} = \inf_{\substack{\text{(over the above}\\ \text{decompositions)}}}\Big\{\sum_{\text{finite}}\|T_{i1}\|_H\cdots\|T_{ij}\|_H\Big\}\,.$$

If $y\in(\hat{H})^{\otimes m}_{\pi}$ $y=\sum_{\text{finite}} T_{i1}\otimes\cdots\otimes T_{im}$ with $T_{ij}\in\hat{H}$ and we define also

$$(**)\qquad \|y\|_{(\hat{H})^{\otimes m}_{\pi}} = \inf_{\substack{\text{(over the above}\\ \text{decompositions)}}}\Big\{\sum_{\text{finite}}\|T_{i1}\|_{\hat{H}}\cdots\|T_{ij}\|_{\hat{H}}\Big\}\,.$$

$x\in H^{\otimes m}_{\pi}\subset(\hat{H})^{\otimes m}_{\pi}$ and clearly

$$(\mathrm{I})\qquad \|x\|_{(\hat{H})^{\otimes m}_{\pi}} \leqq \|x\|_{H^{\otimes m}_{\pi}}\,.$$

Now we are going to prove the converse: For every $\varepsilon>0$ there exists a decomposition $x = \sum_{\text{finite}} T_{i1} \otimes \cdots \otimes T_{im}$ with $T_{ij} \in \hat{H}$ such that

$$(1) \qquad \sum_{\text{finite}} \| T_{i1} \|_{\hat{H}} \cdots \| T_{im} \|_{\hat{H}} \leqq \| x \|_{\hat{H}^{\otimes m}_{\pi}} + \varepsilon .$$

Since H is dense in $\hat{H}$, for every (i, j) there exists a sequence $(\tau^q_{ij})_{q \in N}$ of elements of H converging to T_{ij} in $\hat{H}$.

Let $x_q = \sum_{\text{finite}} \tau^q_{ij} \otimes \cdots \otimes \tau^q_{im} \in H^{\otimes m}$.

Then an easy computation shows that

$$\| x_q - x \|_{\hat{H}^{\otimes m}_{\pi}} \longrightarrow 0 \quad \text{if} \quad q \longrightarrow +\infty$$

and that $(x_q)_{q \in N}$ is a Cauchy sequence in $H^{\otimes m}_{\pi}$. Hence there exists some element X of $H^{\hat{\otimes} m}_{\pi}$ such that $x_q \to X$ in $H^{\hat{\otimes} m}_{\pi}$. We know (from Grothendieck [8] Chap. 1 Prop. 3 p. 38) that $H^{\hat{\otimes} m}_{\pi}$ is injectively contained (via χ_m) in $\mathscr{L}_b(H'^m)$ and since $x_q \to x$ in $\mathscr{L}_b(H'^m)$, $X = x$ in $H^{\hat{\otimes} m}_{\pi} \subset \mathscr{L}_b(H'^m)$. Hence

$$(2) \qquad \| x_q \|_{H^{\hat{\otimes} m}_{\pi}} \longrightarrow \| x \|_{H^{\hat{\otimes} m}_{\pi}} = \| X \|_{H^{\hat{\otimes} m}_{\pi}} .$$

But

$$\| x_q \|_{H^{\hat{\otimes} m}_{\pi}} \leqq \sum_{\text{finite}} \| \tau^q_{i1} \|_H \cdots \| \tau^q_{im} \|_H .$$

The second member tends to $\sum_{\text{finite}} \| T_{i1} \|_{\hat{H}} \cdots \| T_{im} \|_{\hat{H}}$ if $q \to +\infty$. Hence for q large enough, from (1):

$$(3) \qquad \| x_q \|_{H^{\otimes m}_{\pi}} \leqq \| x \|_{\hat{H}^{\otimes m}_{\pi}} + 2\varepsilon .$$

Obviously $\| x_q \|_{H^{\otimes m}_{\pi}} = \| x_q \|_{H^{\hat{\otimes} m}_{\pi}}$ hence from (2):

$$\| x_q \|_{H^{\otimes m}_{\pi}} \longrightarrow \| x \|_{H^{\hat{\otimes} m}_{\pi}} \quad \text{if} \quad q \longrightarrow +\infty .$$

Hence from (3):

$$\| x \|_{H^{\hat{\otimes} m}_{\pi}} \leqq \| x \|_{\hat{H}^{\otimes m}_{\pi}} + 2\varepsilon .$$

Since this is true for every $\varepsilon > 0$ we have in fact:

$$\| x \|_{H^{\hat{\otimes} m}_{\pi}} \leqq \| x \|_{\hat{H}^{\otimes m}_{\pi}} .$$

Since $x \in H^{\otimes m}_{\pi}$

$$\| x \|_{H^{\hat{\otimes} m}_{\pi}} = \| x \|_{H^{\otimes m}_{\pi}}$$

hence:

$$(\mathrm{II}) \qquad \| x \|_{H^{\otimes m}_{\pi}} \leqq \| x \|_{(\hat{H})^{\otimes m}_{\pi}} .$$

From (I) and (II) we have shown that: for every $x \in H^{\otimes m}_{\pi}$

$$\|x\|_{H_\pi^{\otimes m}} = \|x\|_{(\hat{H})_\pi^{\otimes m}} .$$

We recall that we have the situation:

$$H_\pi^{\otimes m} \underset{\substack{\text{with induced}\\ \text{norm}\\ \text{and density}}}{\subset} (\hat{H})_\pi^{\otimes m} \underset{\text{injective}}{\subset} \mathscr{L}_b(H'^m) .$$

Hence $H_\pi^{\otimes m}$ and $(\hat{H})_\pi^{\otimes m}$ have the same completion and the "natural" norms on $H_\pi^{\otimes m}$ and $(\hat{H})_\pi^{\hat{\otimes} m}$ coming from $(*)$ and $(**)$ respectively are equal which proves Lemma 3.2.

PROPOSITION 3.3. *$\mathscr{H}_{SNb}(E) = \mathscr{H}S_{Nb}(E)$ $(=\varprojlim_{i\in I} \mathscr{H}_{Nb}(E_i)$ from 2.7) algebraically and topologically.*

Proof. Clearly $\mathscr{H}_{SNb}(E) \subset \mathscr{H}S_{Nb}(E)$. For the converse let us first prove that $\varprojlim_{i\in I} \mathscr{P}_N({}^mE_i) \subset \mathscr{P}_{bN}({}^mE)$. From the definition of the projective topological tensor product π (Grothendieck [8] p. 31) a basis of o-neighborhoods in $(E^\times)_\pi^{\otimes m}$ is made of the sets $\Gamma(V^{\otimes m})$ where V varies in a basis of o-neighborhoods in $E^\times$ and where Γ denotes the convex hull. Hence:

$$(E^\times)_\pi^{\otimes m} = \varprojlim_V [(E^\times)^{\otimes m}]_{\Gamma(V^{\otimes m})} .$$

But $[(E^\times)^{\otimes m}]_{\Gamma(V^{\otimes m})} = [(E^\times)_V]_\pi^{\otimes m}$ for the same reason hence

$$(E^\times)_\pi^{\otimes m} = \varprojlim_V [(E^\times)_V]_\pi^{\otimes m}$$

hence

$$(E^\times)_\pi^{\hat{\otimes} m} = \varprojlim_V [(E^\times)_V]_\pi^{\hat{\otimes} m} .$$

In the sequel we choose $V = \mathring{B}$ where B is a convex balanced bounded subset of E such that E_B is a Hilbert space. $(E^\times)_V$ is a pre-Hilbert space. From Lemma 3.1 $\widehat{(E^\times)_V} = (E_B)'$ and from Lemma 3.2.

$$[(E^\times)_V]_\pi^{\hat{\otimes} m} = [(E_B)']_\pi^{\hat{\otimes} m} \quad \text{algebraically and topologically} .$$

Hence

$$(E^\times)_\pi^{\hat{\otimes} m} = \varprojlim_{B \in \mathscr{B}_E} [(E_B)']_\pi^{\hat{\otimes} m} \quad \text{algebraically and topologically}$$

which proves that $\varprojlim_{i\in I} \mathscr{P}_N({}^mE_i) = \mathscr{P}_{bN}({}^mE)$.

Now if $f \in \mathscr{H}_{SNb}(E)$ $\delta^m f(o) \in (E^\times)_\pi^{\hat{\otimes} m}$ (via $\hat{\chi}_m$ which is not explicitely written because it is injective) and

$$\|\delta^m f(o)\|_{N,B} = \|\delta^m f(o)\|_{[(E^\times)_{\mathring{B}}]_\pi^{\hat{\otimes} m}} .$$

If $f \in \mathscr{H}S_{Nb}(E)$ $(\delta^m f/E_B)(o) \in [(E_B)']_\pi^{\hat{\otimes} m}$ and

$$\|(\delta^m f/E_B)(o)\|_N = \|\delta^m f/E_B(o)\|_{[(E_B)']^{\hat{\otimes} m}_\pi}$$

Since $(E^\times)_{\mathring{B}}$ is pre-Hilbertian and $(E_B)' = (\widehat{(E^\times)_{\mathring{B}}})$ (with the corresponding norms) the equality of the 2 norms of Lemma 3.2 is enough to conclude the proof (algebraic and topological equality) of Prop. 3.3.

We recall Lemma 3.1 in Colombeau-Matos [3], which is proved there.

LEMMA 3.4. *If E_1 and E_2 are two normed spaces with a linear nuclear mapping j from E_1 into E_2 and if f is in $\mathscr{H}_b(E_2)$, then $f \circ j$ is in $\mathscr{H}_{Nb}(E_1)$. Moreover, the mapping*

$$\Psi\colon \mathscr{H}_b(E_2) \longrightarrow \mathscr{H}_{Nb}(E_1)$$
$$f \longrightarrow f \circ j$$

is continuous.

PROPOSITION 3.5. *If E is a nuclear bornological convex space, then $\mathscr{H}S_{Nb}(E)$ coincides with $\mathscr{H}_S(E)$ algebraically and topologically.*

Proof. We first show that $\mathscr{H}_S(E)$ is contained in $\mathscr{H}S_{Nb}(E)$. Let B_1 be an element of $\mathscr{B}_E$ such that E_{B_1} is a Hilbert space. From the definition of a nuclear bornological vector space there is $B_2 \in \mathscr{B}_E$ such that $B_1 \subset B_2$ and the inclusion mapping $j_1\colon E_{B_1} \to E_{B_2}$ is nuclear. Also there is $B_3 \in \mathscr{B}_E$ with $B_3 \supset B_2$ and the inclusion mapping $j_2\colon E_{B_2} \to E_{B_3}$ nuclear. Hence mB_2 is relatively compact in E_{B_3} for every m in N. Thus if f is in $\mathscr{H}_S(E)$, we have $f\,|\,E_{B_2} \in \mathscr{H}_b(E_{B_2})$. By Lemma 3 $f\,|\,E_{B_1} = (f\,|\,E_{B_2}) \circ j_1$ is in $\mathscr{H}_{Nb}(E_{B_1})$. This implies that $f \in \mathscr{H}S_{Nb}(E)$.

Obviously $\mathscr{H}_S(E) = \varinjlim_{B \in \mathscr{B}_E} \mathscr{H}_b(E_B)$ algebraically and topologically. From the trivial inclusion $\mathscr{H}_{Nb}(E_B) \subset \mathscr{H}_b(E_B)$ and Lemma 3.4 we have:

$$\varprojlim_{B \in \mathscr{B}_E} \mathscr{H}_b(E_B) = \varprojlim_{B \in \mathscr{B}_E} \mathscr{H}_{Nb}(E_B)\,.$$

From 3.3 and 3.5:

THEOREM 3.6. *Let E be a nuclear* b.c.s. *separated by its dual then*:

$$\mathscr{H}_S(E) = \mathscr{H}_{SNb}(E) = \varprojlim_{i \in I} \mathscr{H}_{Nb}(E_i)$$

if $E = \varinjlim_{i \in I} E_i$ (bornologically) where the spaces E_i are Hilbert spaces.

REMARK 3.7. If E is a quasi-complete dual nuclear l.c.s. (i.e., its strong dual E' is a nuclear l.c.s.) it is well known and easy to prove that the Von Neumann bornology of E is a nuclear bornology.

LEMMA 3.8. *Let E be a quasi-complete dual nuclear* l.c.s. *Then $\mathscr{P}_N({}^mE) = \mathscr{P}_{bN}({}^mE)$ for every $m \in N$ with the equality of the "natural" norms on these spaces.*

Proof. Via χ_m we have:

$$\mathscr{P}_{bN}({}^mE) = (E^\times)_\pi^{\hat{\otimes} m} = \lim_{\substack{\longleftarrow \\ B \in \mathscr{B}_E}} (E_B)'^{\hat{\otimes} m}_\pi$$

from the proof of Prop. 3.3.

$$\mathscr{P}_N({}^mE) = (E')_\pi^{\hat{\otimes} m} = \lim_{\substack{\longleftarrow \\ B}} ((E')_{\mathring{B}})_\pi^{\hat{\otimes} m} = \lim_{\substack{\longleftarrow \\ B}} (\widehat{(E')_{\mathring{B}}})_\pi^{\hat{\otimes} m}$$

from Lemma 3.2.

But the same proof as in Lemma 3.1 shows that $(\hat{E}')_{\mathring{B}} = (E_B)'$ hence $(E')_\pi^{\hat{\otimes} m} = \lim_{\substack{\longleftarrow \\ B}} [(E_B)']_\pi^{\hat{\otimes} m}$ and hence $(E^\times)_\pi^{\hat{\otimes} m} = (E')_\pi^{\hat{\otimes} m}$. Furthermore the equalities of the sets of the "natural" norms on these spaces (defined in §2) comes from Lemma 3.2.

An immediate consequence of Lemma 3.8 is:

THEOREM 3.9. *Let E be a quasi-complete dual nuclear* l.c.s. *Then $\mathscr{H}_{Nb}(E) = \mathscr{H}_{SNb}(E)$ algebraically and topologically.*

4. Application to Boland's theorems.

The first results of existence and approximation of solutions of convolution equations in all the space $\mathscr{H}(E)$ when E is infinite dimensional were obtained by Boland in [1], and remain the basic results in this topic. We prove here that via the results of §3 they appear as a consequence of previous Matos's results for $\mathscr{H}_{Nb}(E)$ in [11] [12].

4.1. *Application to Boland's existence theorem.* Let E be a *DFN* space (strong dual of a nuclear Fréchet space) then it follows from Th. 3.6 that $\mathscr{H}(E) = \mathscr{H}_S(E) = \mathscr{H}_{SNb}(E) = \mathscr{H}_{Nb}(E)$.

Hence the well known Existence theorem of Boland [1] [2] for the convolution equations in $\mathscr{H}(E)$ appears now as a consequence of the similar result in Matos [11] [12].

4.2. *Application to Boland's approximation theorem.* Let E be a quasi-complete dual nuclear l.c.s. It is shown in Colombeau-Meise-

Perrot [4] that $\mathscr{H}(E)$ is dense in $\mathscr{H}_S(E)$ and (the compact subsets of E are strict compact) these two last spaces have also the same convolution operators hence, from Th. 3.6 and 3.9, $\mathscr{H}_{Nb}(E)$ and $\mathscr{H}(E)$ have the same convolution operators. Now let $f \in \mathscr{H}(E)$ hence $f \in \mathscr{H}_S(E)$ hence from 3.6 and 3.9 $f \in \mathscr{H}_{Nb}(E)$. Now if $\mathscr{O}$ is a convolution operator on $\mathscr{H}(E)$ and f is in $\mathscr{H}_{Nb}(E)$ such that $\mathscr{O}f = 0$ the Approximation theorem in Matos [11] [12] implies that f may be approximated by exponential polynomials in the kernel of $\mathscr{O}$. Hence we get Boland's approximation theorem [1] [2].

REMARK 4.3. A more general approximation theorem is given in Colombeau-Perrot [5] in the case E is only a nuclear b.c.s. separated by its dual. General existence and approximation theorems are also given in Colombeau-Matos [3] and Matos Nachbin [13].

REMARK 4.4. It is also immediate to check that Boland's result [1] on the Fourier Borel transform is also a consequence of the corresponding Matos result in [11].

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