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**POINTWISE PERIODIC HOMEOMORPHISMS ON CHAINABLE  
CONTINUA**

EDWIN DUDA

## POINTWISE PERIODIC HOMEOMORPHISMS ON CHAINABLE CONTINUA

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**We show that if  $X$  is a chainable continuum with no small indecomposable subcontinua and which admits a monotone map  $\phi$  onto the unit interval such that no point inverse has interior points, then each pointwise periodic mapping  $T$  on  $X$  is periodic and must have period 1 or 2.**

0. Introduction. Beverly Brechner [2] has shown there exists a chainable continuum  $X$  and a periodic homeomorphism  $T$  of  $X$  of period 4. The only other periodic homeomorphisms on chainable continua known at that time were of period 1 and 2. Wayne Lewis [4] has recently shown that for every positive integer  $n$  there exists a nonhereditarily indecomposable chainable continuum  $X$  with a homeomorphism  $T$  of period  $n$ . He observes that  $X$  could be constructed so as to be a pseudo-arc and still have a homeomorphism  $T$  of period  $n$ . Michel Smith and Sam Young [5] have shown that if a chainable continuum admits a homeomorphism of period greater than 2, then the continuum must contain an indecomposable continuum.

We show that if  $X$  is a chainable continuum with no small indecomposable continua and which admits a monotone map  $\phi$  onto the unit interval such that no point inverse has interior points, then each pointwise periodic homeomorphism  $T$  must be periodic and of period 1 or 2.

1. Notation and background. In this note  $X$  will represent a metric continuum and mapping will mean a continuous function. A mapping  $T$  of  $X$  into itself is said to be pointwise periodic if for each  $x \in X$  there exists an integer  $N_x$  such that  $T^{N_x}(x) = x$ , where  $T^{N_x}$  means the composition of  $T$  with itself  $N_x$  times. By a result of R. H. Bing, [1], a hereditarily unicoherent atriodic continuum  $X$  in which no indecomposable continuum has interior points admits a monotone mapping  $\phi$  onto the unit interval  $I = [0, 1]$  and furthermore no point inverse of  $\phi$  has interior points relative to  $X$ . In case  $X$  is hereditarily decomposable he showed that  $X$  is chainable. J. B. Fugate, [3], strengthened this result by showing that if each indecomposable subcontinuum of an atriodic-hereditarily unicoherent continuum  $X$  is chainable, then  $X$  is chainable. In this note we wish to consider  $X$  to be a chainable continuum which has no indecomposable subcontinua with interior points and which has no inde-

composable continua of diameter less than a fixed positive number. By the above  $X$  admits a monotone mapping  $\phi$  onto the unit interval  $I$  and no point inverse of  $\phi$  has interior points. Furthermore E. S. Thomas, [6], has shown that  $X$  has the following property: If  $U$  and  $V$  are disjoint open sets in  $X$ , then there exists an  $x \in U$  and  $y \in V$  and a continuum  $K_{xy}$  irreducible from  $x$  to  $y$  such that the component of  $x$  in  $K_{xy}$  is  $K_{xy} - \{y\}$ .

## 2. Preliminary results.

LEMMA 1. *A pointwise periodic map  $T$  on a continuum  $X$  is a homeomorphism.*

*Proof.* We show that  $T$  is 1-1. Suppose  $T(x) = T(y)$  for  $x, y \in X$ . There exist integers  $N_x$  and  $N_y$  so that  $T^{N_x}(x) = x$  and  $T^{N_y}(y) = y$ . Now  $x = T^{N_x N_y}(x) = T^{N_x N_y - 1}(T(x)) = T^{N_x N_y - 1}(T(y)) = T^{N_y N_x}(y) = y$ .

LEMMA 2. *If  $A \subset X$  and  $T(A) \subset A(T(A) \supset A)$ , then  $T(A) = A$ , where  $T$  is pointwise periodic on  $X$ .*

*Proof.* Suppose  $x \in A$  and  $T(A) \subset A$ . There is an integer  $N_x$  with  $T^{N_x}(x) = x$  and since  $A \supset T(A) \supset T^2(A) \supset \dots \supset T^{N_x}(A) \supset \dots$ ,  $x \in T(A)$ . The case  $T(A) \supset A$  follows from Lemma 1 and the case just proved.

LEMMA 3. *Let  $X$  be a chainable continuum which admits a monotone map  $\phi$  onto the unit interval  $I$  such that no point inverse contains interior points relative to  $X$ . If  $K$  is a continuum in  $X$  which meets  $\phi^{-1}(t_1)$  and  $\phi^{-1}(t_2)$ , then  $K \supset \phi^{-1}(t)$  for all  $t$  between  $t_1$  and  $t_2$ . Furthermore  $X$  is irreducible from any point of  $\phi^{-1}(0)$  to any point of  $\phi^{-1}(1)$ .*

*Proof.* Assume  $t_1 < t < t_2$ . Let  $L = \phi^{-1}[0, t] \cap K$  and  $M = \phi^{-1}[t, 1] \cap K$ . By monotonicity of  $\phi$  and hereditary unicoherence of  $X$ ,  $L$  and  $M$  are continua. The continua  $L$  and  $M$  have a point in common in  $\phi^{-1}(t)$ . If  $\phi^{-1}(t)$  is not contained in  $L \cup M$ , then  $L, M$  and  $\phi^{-1}(t)$  determine a triod in  $X$  which is impossible. Suppose  $K$  is a continuum irreducible from  $x \in \phi^{-1}(0)$  to  $y \in \phi^{-1}(1)$ . Then  $K \supset \phi^{-1}(0, 1)$  by the first part of the lemma and  $\phi^{-1}(0, 1)$  is dense in  $X$  since no point inverse has interior points. Thus,  $X = \overline{\phi^{-1}(0, 1)} \subset K$  or  $X = K$ .

## 3. Main result.

THEOREM. *Let  $T$  be a pointwise periodic mapping on a chain-*

able continuum  $X$  which has no small indecomposable continua. If  $X$  admits a monotone map  $\phi$  onto the unit interval such that no point inverse has interior points relative to  $X$ , then  $T$  is periodic.

*Proof.* Lemma 1 and Lemma 2 imply that for each  $t \in I$ ,  $T(\phi^{-1}(t))$  is contained in  $\phi^{-1}(s)$  for some  $s \in I$ . The mapping  $T$  thus induces a pointwise periodic map  $T_1$  on  $[0, 1] = I$  defined by  $T_1(t) = (\phi T \phi^{-1})(t)$ . By known results  $T_1$  is periodic and either  $T_1 = \text{identity}$  or  $T_1^2 = \text{identity}$  on  $I$ . We assume  $T_1^2 = \text{identity}$  on  $I$ . In this case  $T^2$  maps each  $\phi^{-1}(t)$  into itself.

Let  $x \in \phi^{-1}(0)$  and  $y \in \phi^{-1}(t)$ ,  $t \in (0, 1)$ , and let  $U$  and  $V$  be disjoint open sets containing  $x$  and  $y$  respectively chosen so that  $\phi(U)$  is entirely to the left of  $\phi(V)$  and  $1 \notin \phi(V)$ . By a result of E. S. Thomas, [6], there exists an  $x_1 \in U$  and a  $y_1 \in V$  and a continuum  $K_{x_1 y_1}$  which is irreducible from  $x_1$  to  $y_1$  and the component of  $x_1$  in  $K_{x_1 y_1}$  is  $K_{x_1 y_1} - \{y_1\}$ . Define a new continuum  $K_{x y_1}$  by  $K_{x y_1} = \phi^{-1}[0, \phi(x_1)] \cup K_{x_1 y_1}$ . The continuum  $K_{x y_1}$  is irreducible from  $x$  to  $y_1$  and the component of  $x$  in  $K_{x y_1}$  is the complement of  $y_1$  in  $K_{x y_1}$ . Let  $K_{x y_1} - \{y_1\} = \bigcup_{i=1}^{\infty} K_i$ , where each  $K_i$  is a continuum containing  $x$  and  $K_i \subset K_{i+1}$  for all  $i$ . If  $y_1 \in T^2(K_{x y_1})$ , then  $T^2(K_{x y_1}) \supset K_{x y_1}$  since  $x, y_1 \in T^2(K_{x y_1})$  and  $K_{x y_1}$  is irreducible between  $x$  and  $y_1$ . By Lemma 2,  $T^2(K_{x y_1}) = K_{x y_1}$ . If  $y_1 \notin T^2(K_{x y_1})$  we show this leads to a contradiction. We must have  $T^2(K_i) \subset K_{x y_1}$  for all  $i$ , otherwise there exists an integer  $N$  such that  $T^2(K_i) \not\subset K_{x y_1}$ , and in this case  $H = \overline{K_{x y_1} - \phi^{-1}\phi(y_1)}$ ,  $K = T^2(K_N) \cap \phi^{-1}\phi(y_1)$ , and  $L = K_{x y_1} \cap \phi^{-1}\phi(y_1)$  determine a triod. The containing relation  $T^2(\bigcup_{i=1}^{\infty} K_i) \subset K_{x y_1}$  implies  $T^2(K_{x y_1}) \subset K_{x y_1}$  and again by Lemma 2,  $T^2(K_{x y_1}) = K_{x y_1}$ .

No  $T^2(K_i)$  can contain  $y_1$  otherwise there exists an integer  $N$  with  $T^2(K_N)$  properly containing  $K_{x y_1}$  which implies a contradiction. Therefore  $T^2(y_1) = y_1$  and since  $T^2$  is the identity on a dense set it follows that  $T^2 = \text{identity}$  on  $X$ .

The argument for  $T_1 = \text{identity}$  on  $I$  implying  $T = \text{identity}$  on  $X$  is similar.

**COROLLARY.** *If  $T$  is pointwise periodic and  $X$  is as in the theorem, then either  $T^2 = \text{identity}$  on  $X$  or  $T = \text{identity}$  on  $X$  and the fixed point set is a continuum.*

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Hédi Amara, Groupe des classes et unité fondamentale des extensions  
quadratiques relatives à un corps quadratique imaginaire principal ..... 1

Douglas S. Bridges, On the isolation of zeroes of an analytic function ..... 13

Andrew J. Casson and John L. Harer, Some homology lens spaces which  
bound rational homology balls ..... 23

Z. A. Chanturia, On the absolute convergence of Fourier series of the  
classes  $H^\omega \cap V[v]$  ..... 37

J.-F. Colombeau and Mário Carvalho Matos, On some spaces of entire  
functions defined on infinite-dimensional spaces ..... 63

Edwin Duda, Pointwise periodic homeomorphisms on chainable continua ..... 77

Richard F. Gustafson, A simple genus one knot with incompressible  
spanning surfaces of arbitrarily high genus ..... 81

Fumio Hiai, Masanori Ohya and Makoto Tsukada, Sufficiency, KMS  
condition and relative entropy in von Neumann algebras ..... 99

Ted Hurley, Intersections of terms of polycentral series of free groups and  
free Lie algebras. II ..... 111

Robert Edward Jamison, II, Partition numbers for trees and ordered sets ... 115

R. D. Ketkar and N. Vanaja, A note on FR-perfect modules ..... 141

Michihiko Kikkawa, On Killing-Ricci forms of Lie triple algebras ..... 153

Jorge Lewowicz, Invariant manifolds for regular points ..... 163

Richard W. Marsh, William H. Mills, Robert L. Ward, Howard Rumsey  
and Lloyd Richard Welch, Round trinomials ..... 175

Claude Schochet, Topological methods for  $C^*$ -algebras. I. Spectral  
sequences ..... 193

Yong Sian So, Polynomial near-fields? ..... 213

Douglas Wayne Townsend, Imaginary values of meromorphic functions in  
the disk ..... 225

Kiyoshi Watanabe, Coverings of a projective algebraic manifold ..... 243

Martin Michael Zuckerman, Choosing  $l$ -element subsets of  $n$ -element  
sets ..... 247