COVERINGS OF A PROJECTIVE ALGEBRAIC MANIFOLD

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Let $M$ be a projective algebraic manifold. Suppose $\pi: D \to M$ is a covering of $M$. If $D$ satisfies $H^1(D, O^*) = 0$, then $D$ is a Stein manifold with $H^2(D, Z) = 0$, where $O^*$ is the sheaf of germs of nowhere-vanishing holomorphic functions and $Z$ is the additive group of integers.

Let $D$ be a domain in $\mathbb{C}^n$ and $\Gamma$ be a discrete subgroup of $\text{Aut}(D)$. It is well-known that if the quotient manifold $D/\Gamma$ is compact, then $D$ is a domain of holomorphy. Recently, Carlson-Harvey [1] showed that if $D$ is a domain in a Stein manifold and $D \to M$ is a covering of a compact Moisheson manifold $M$, then $D$ is a Stein manifold. On the other hand, we showed in [4] that if a pseudoconvex domain $D$ in a projective algebraic manifold satisfies $H^1(D, O^*) = 0$, then $D$ is a Stein manifold with $H^2(D, Z) = 0$.

In this paper, we study the case where a covering of a manifold is not contained in a larger manifold. We shall prove the following:

**Theorem.** Let $M$ be a projective algebraic manifold. Suppose $\pi: D \to M$ is a covering of $M$. If $D$ satisfies $H^1(D, O^*) = 0$, then $D$ is a Stein manifold with $H^2(D, Z) = 0$.

We remark that the condition $H^1(D, O^*) = 0$ cannot be replaced by $H^1(D, O) = 0$, where $O$ is the sheaf of germs of holomorphic functions. To see this it is enough to consider the case $D = M = \mathbb{P}_2(\mathbb{C})$ and $\pi$ is the identity mapping.

**Proof of theorem.** Let $\{V_i\}$ be an open covering of $M$ such that each $V_i$ is a local coordinate neighborhood and is biholomorphic to a connected component $\pi^{-1}(V_i)$. Since $M$ is a projective algebraic manifold, there is a positive line bundle $F$ over $M$. Choosing a suitable refinement $\{U_i\}$ of $\{V_i\}$, we can represent $F$ by a system of transition functions $\{f_{j,k}\}$ and find a Harmitian metric $\{a_j\}$ along the fibers of $F$ which satisfies the following conditions:

(i) Each $a_j$ is a $C^\infty$, real-valued and positive function on $U_j$,

(ii) If $U_j \cap U_k \neq \emptyset$, then we have $a_k = |f_{j,k}|^2 a_j$,

(iii) For every point $P$ in $M$, the Hessian of $-\log a_j$ relative to a local coordinate system $(z_1, \ldots, z_n)$ at $P$. 

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\[ L(-\log a_j; P) = \left( -\frac{\partial^2 \log a_j}{\partial z_\alpha \partial \bar{z}_\beta}(P) \right) \]
\[ (\alpha, \beta = 1, \ldots, n) \]

is positive definite. By the compactness of \( M \), \( M \) has a finite open covering \( \{ U_\alpha : j = 1, \ldots, m \} \).

Since \( U_j \) is biholomorphic to each of the connected components of \( \pi^{-1}(U_j) \), we have the functions \( \{a_j \circ \pi\} \) which satisfies the following conditions:

(i) Each \( a_j \circ \pi \) is a \( C^\infty \), real-valued and positive function on \( \pi^{-1}(U_j) \),

(ii) If \( \pi^{-1}(U_j) \cap \pi^{-1}(U_k) \neq \phi \), then we have \( a_j \circ \pi = |f_{jk} \circ \pi|^2 a_k \circ \pi \),

(iii) \( W(-\log a_j \circ \pi; P) \) is positive at every point \( P \) in \( D \), where

\[ W(\phi; P) = \min \left\{ \sum_{\alpha, \beta} \frac{\partial^2 \phi}{\partial w_\alpha \partial \bar{w}_\beta}(P) \lambda_\alpha \bar{\lambda}_\beta : \sum_\alpha |\lambda_\alpha|^2 = 1, \quad \alpha, \beta = 1, \ldots, n \right\} \]

and \( (w_1, \ldots, w_n) \) is a local coordinate at \( P \).

Since \( U = \{ \pi^{-1}(U_j) \} \) is an open covering of \( D \), \( \{f_{jk} \circ \pi\} \) defines an element of \( H^1(U, O^*) \). By the assumption of \( H^1(D, O^*) = 0 \), there is a cochain \( \{f_j\} \) of \( C^0(U, O^*) \) such that \( f_{jk} \circ \pi = f_k/f_j \). We can define a \( C^\infty \) function \( \phi \) on \( D \) in the following way:

\[ \phi(P) = -\log (a_j \circ \pi(P) |f_j(P)|^2) \]

for \( P \) in \( \pi^{-1}(U_j) \). Since \( M \) is paracompact, \( M \) has a finite open covering \( \{ W_j : j = 1, \ldots, m \} \) with \( W_j \subset U_j \). By the property (iii) there is a positive constant \( C_j \) such that \( W(\phi; P) > C_j \) for \( P \) in \( \pi^{-1}(W_j)(j = 1, \ldots, m) \). Hence we have

\[ W(\phi; P) > C : = \min \{C_j : j = 1, \ldots, m\} \]

for \( P \) in \( D \). We remark that \( D \) is not finitely sheeted, because \( D \) has the strongly plurisubharmonic function \( \phi \).

On the other hand, \( M \) is a projective algebraic manifold, so \( D \) has a real-analytic Kähler metric. Let \( d(P, Q) \) be the distance between \( P \) and \( Q \) measured by the Kähler metric. Let us fix a point \( P_0 \) in \( D \) and define a continuous function \( \psi \) on \( D \) in the following way:

\[ \psi(P) = d(P_0, P) \]

for \( P \) in \( D \). We see that for every \( c > 0 \), the set \( \{ P \in D : \psi(P) < c \} \) is relatively compact in \( D \). Denotes by \( \Gamma(P, \varepsilon) \) the set \( \{ Q \in D : d(P, Q) < \varepsilon \} \), where a positive constant \( \varepsilon \) is chosen so that \( \pi(\Gamma(P, \varepsilon)) \)
is contained in some $U_d$ and $\Gamma(P, \varepsilon)$ is homeomorphic to a hypersphere. We define the following operator $A_\varepsilon$ mapping continuous function $f$ on $D$ into $C^1$ function on $D$:

$$A_\varepsilon f(P) = \frac{1}{V} \int_{\Gamma(P, \varepsilon)} f(Q) \ dv,$$

where $dv$ is the volume element determined by the Kähler metric and $V$ is the volume of $\Gamma(P, \varepsilon)$. We see that the set $\{P \in D: A_\varepsilon \psi(P) < c\}$ is relatively compact in $D$. Let define

$$\psi_1 = A_\varepsilon \psi \text{ and } \psi_2 = A_\varepsilon \psi_1$$
on $D$, then $\psi_2$ is $C^2$ and the set $\{P \in D: \psi_2(P) < c\}$ is also relatively compact in $D$. Let compute the Hessian of $\psi_2$. Since $D$ has a real-analytic Kähler metric, there are a local coordinate $(w_1, \cdots, w_n)$ of $\Gamma(P, \varepsilon)$ and a positive constant $K_1$ such that

$$|\psi(Q) - \psi(Q')|^2 \leq K_1 \sum |w_i - w_i'|^2 + \cdots + |w_n - w_n'|^2$$

for two points $Q = (w_1, \cdots, w_n)$ and $Q' = (w_1', \cdots, w_n')$ in $\Gamma(P, \varepsilon)$ (see [3] Lemma 1). By the compactness of $M$, $K_1$ can be chosen independent of $P$. Choosing $K_1$ large enough if necessary, we have

$$\left| \frac{\partial \psi_1}{\partial w_j}(P) \right| \leq K_1 \quad (j = 1, \cdots, n)$$

and consequently

$$\left| \frac{\partial^2 \psi_2}{\partial w_j \partial \bar{w}_k}(P) \right| \leq K_1 \quad (j, k = 1, \cdots, n)$$

for $P$ in $D$. Therefore a positive constant $K$ can be chosen so that

$$W(\psi_2; P) > -K$$

for $P$ in $D$. Now we define a $C^2$ function $\Phi$ on $D$ in the following way:

$$\Phi(P) = K \cdot \phi(P) + C \cdot \psi_2(P)$$

for $P$ in $D$. Then (1) and (2) induce

$$W(\Phi; P) \geq K \cdot W(\phi; P) + C \cdot W(\psi_2; P) > 0$$

for $P$ in $D$. Hence $\Phi$ is a strongly plurisubharmonic function on $D$ and the set $\{P \in D: \Phi(P) < c\}$ is relatively compact in $D$ for every $c > 0$. Therefore $D$ is a Stein manifold by Narasimhan [2]. Moreover from the exact sequence $0 \rightarrow Z \rightarrow O \rightarrow O^* \rightarrow 0$ we obtain the exact cohomology sequence.
\[ \cdots \longrightarrow H^1(D, O) \longrightarrow H^1(D, O^*) \longrightarrow H^2(D, Z) \longrightarrow H^2(D, O) \longrightarrow \cdots. \]

Since \( H^2(D, O) = 0 \) by the Cartan’s Theorem B and \( H^1(D, O^*) = 0 \) by the assumption, we have \( H^2(D, Z) = 0 \). This completes the proof.

**References**


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