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**CHOOSING  $l$ -ELEMENT SUBSETS OF  $n$ -ELEMENT SETS**

MARTIN MICHAEL ZUCKERMAN

# CHOOSING $\ell$ -ELEMENT SUBSETS OF $n$ -ELEMENT SETS

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**We consider axioms that assert the possibility of choosing a subset of an  $n$ -element set. We study the interdependence of these axioms and of the more usual axioms of choice for  $n$ -element sets.**

The discussion takes place within any of the usual systems of set theory without the axiom of choice. Our logical framework is the first-order predicate calculus with identity. Lower case letters stand for natural numbers. Throughout this paper, we let  $n \geq 2$  and  $\ell \geq 1$ . At first, we assume  $n > \ell$ .

Let  $[n]$  be the statement: "For every nonempty set  $X$  of  $n$ -element sets there is a function  $f$  with domain  $X$  such that for each  $A$  in  $X$ ,  $f(A) \in A$ ." Here,  $[n]$  is called the axiom of choice for  $n$ -element sets. (See [1].)

Let  $S(n, \ell)$  be the statement: "For every nonempty set  $X$  of  $n$ -element sets there is a function  $f$  with domain  $X$  such that for each  $A$  in  $X$ ,  $f(A)$  is an  $\ell$ -element subset of  $A$ ."

Let  $T(n, \ell)$  be the statement: "For every nonempty set  $X$  of  $n$ -element sets there is a function  $f$  with domain  $X$  such that for each  $A$  in  $X$ ,  $f(A)$  is a nonempty subset of  $A$  with at most  $\ell$  elements."

Finally, let  $T^*(n, n - 1)$  be the statement: "For every nonempty set  $X$  of  $n$ -element sets there is a function  $f$  with domain  $X$  such that for each  $A$  in  $X$ ,  $f(A) = \langle A_1, A_2 \rangle$ , where  $A_1$  and  $A_2$  are pairwise disjoint nonempty subsets of  $A$  whose union is  $A$ ."

Observe that  $T(n, n - 1)$  asserts the possibility of choosing a nonempty proper subset of each  $n$ -element set, whereas  $T^*(n, n - 1)$  asserts the possibility of ordering the partition thereby obtained. Clearly,  $T(n, n - 1) \leftrightarrow T^*(n, n - 1)$ .

The following relationships are also immediate.

$$\begin{aligned} S(n, \ell) &\longleftrightarrow S(n, n - \ell) \\ S(n, 1) &\longleftrightarrow S(n, n - 1) \longleftrightarrow T(n, 1) \longleftrightarrow [n] \\ \left[ \binom{n}{\ell} \right] &\longrightarrow S(n, \ell) \\ [2^n - 2] &\longrightarrow T(n, n - 1) \end{aligned}$$

For convenience, for  $n \leq \ell$ , let  $S(n, \ell) = S(n, n - 1)$  and  $T(n, \ell) = T(n, n - 1)$ .

Now let  $k, \ell, m, n$  be natural numbers such that  $k \geq 0, \ell \geq 1,$

$m \geq 2$ , and  $n \geq 2$ .

If  $\ell < k$ , then clearly,

$$S(n, \ell) \longrightarrow T(n, \ell) \longrightarrow T(n, k).$$

Theorem 1 generalizes the first of these relationships.

**THEOREM 1.** For  $\ell < n$  and  $n - \ell < m \leq n$ ,

$$S(n, \ell) \longrightarrow T(m, \ell).$$

*Proof.* Let  $X$  be a nonempty set of  $m$ -element sets. For each  $A$  in  $X$ , let  $A'$  consist of the first  $n - m$  natural numbers that are not in  $A$  and let  $A'' = A \cup A'$ . We use  $S(n, \ell)$  to obtain an  $\ell$ -element subset of  $A''$ . At least one element of this subset belongs to  $A$ .

The next two theorems generalize Tarski's result:

$$[kn] \longrightarrow [n].$$

(See [2], p. 99.)

**THEOREM 2.** For  $\ell < n$  and for  $k \geq 0$ ,

$$(S(n, \ell) \wedge [kn + \ell]) \longrightarrow [n].$$

*Proof.* Let  $X$  be a nonempty set of  $n$ -element sets and let  $A \in X$ . We use  $S(n, \ell)$  to choose an  $\ell$ -element subset  $B$  of  $A$ .

If  $k = 0$ , we use  $[\ell]$  to pick an element of  $B$ .

If  $k > 0$ , we use  $[kn + \ell]$  to pick an element of  $(B \times \{0\}) \cup (A \times \{1, 2, \dots, k\})$ . Let  $f(A)$  be the first coordinate of this chosen element.

**THEOREM 3.** Let  $k \geq 1$ .

(a) For  $\ell < n$ ,  $T(kn, \ell) \rightarrow T(n, \ell)$ .

(b) For  $\ell$  not of the form  $jn$  for any  $j \leq k$ ,  $S(kn, \ell) \rightarrow T(n, \ell)$ .

*Proof.* Let  $X$  be a nonempty set of  $n$ -element sets and let  $X' = \{A \times k : A \in X\}$ .

(a) We choose a subset of at most  $\ell$  elements of each  $A \times k$  in  $X'$ . For each such chosen subset, let  $B$  be the set of first coordinates. Then  $B$  is a nonempty subset of  $A$  and has at most  $\ell$  elements.

(b) If  $\ell \geq kn$ , then

$$S(kn, \ell) \longleftrightarrow S(kn, kn - 1) \longleftrightarrow [kn]$$

and

$$[kn] \longrightarrow [n] \longrightarrow T(n, \ell).$$

If  $\ell < kn$ , we choose an  $\ell$ -element subset  $C$  of each  $A \times k$  in  $X'$ . Not every  $a$  in  $A$  appears the same number of times as the first coordinate of a member of  $C$ . Let  $B$  be the set of those  $a$  that appear the maximal number of times in this role. If  $\ell < n$ ,  $B$  is nonempty and has at most  $\ell$  elements. If  $\ell > n$ ,  $B$  is a nonempty proper subset of  $A$ . In both cases, the axiom  $T(n, \ell)$  is realized.

Henceforth, let  $A$  be a nonempty finite set of natural numbers greater than 1. If  $A = \{a_1, a_2, \dots, a_m\}$ , let  $S(A, \ell)$  denote

$$S(a_1, \ell) \wedge S(a_2, \ell) \wedge \dots \wedge S(a_m, \ell).$$

For  $n > \ell$ , we say that  $n$  is an  $A_\ell$ -number if for some  $j \geq 1$  and some  $k$  satisfying  $0 \leq k < \ell$ ,  $jn + k \in A$ . Furthermore, for all  $n \geq 2$  and  $\ell \geq 1$ , we say that  $A$  and  $n$  satisfy condition  $\mathcal{A}_\ell$  if either

- (i)  $n$  is an  $A_\ell$ -number, or both
- (ii)<sub>a</sub>  $n = rp$  for some prime  $p$  in  $A$ , and
- (ii)<sub>b</sub> whenever  $n = n_1 + n_2$  for  $n_1 > \ell$  and  $n_2 > \ell$ , then either  $A$  and  $n_1$  or else  $A$  and  $n_2$  satisfy condition  $\mathcal{A}_\ell$ .

LEMMA. Let  $p$  be a prime and let  $\ell \geq 1$  and  $r \geq 1$ . Then

$$S(p, \ell) \longrightarrow T(rp, rp - 1).$$

(The lemma is Theorem 2(g) of [3]. See also [4].)

THEOREM 4. Let  $A$  be as above, let  $n \geq 2$  and  $\ell \geq 1$ , and suppose  $A$  and  $n$  satisfy condition  $\mathcal{A}_\ell$ . Then  $S(A, \ell) \rightarrow T(n, \ell)$ .

*Proof.* Assume  $A$  and  $n$  satisfy condition  $\mathcal{A}_\ell$ .

If  $n$  is an  $A_\ell$ -number, then for some  $j \geq 1$  and for some  $k$  satisfying  $0 \leq k < \ell$ ,  $S(jn + k, \ell)$  is true. By Theorem 1,  $T(jn, \ell)$  must be true, and by Theorem 3,  $T(n, \ell)$  is true.

If  $n$  is not an  $A_\ell$ -number, then  $n = rp$  for some prime  $p$  in  $A$ . By our hypothesis,  $S(p, \ell)$  is true. By the lemma,  $T(n, n - 1)$  is true.

If  $2 \leq n \leq \ell$ , then  $T(n, n - 1) = T(n, \ell)$ .

If  $\ell < n < 2\ell$ , we use  $T^*(n, n - 1)$  to obtain  $\langle B_1, B_2 \rangle$ , where  $\{B_1, B_2\}$  forms a partition of an element  $B$  of a nonempty set of  $n$ -element sets. At least one of these subsets,  $B_1$  or  $B_2$ , has at most  $\ell$  elements. We choose the first of these with this property. Thus,  $T(n, \ell)$  is true.

Now assume that for  $n' < n$ , whenever  $A$  and  $n'$  satisfy condition  $\mathcal{A}_\ell$ , then  $S(A, \ell) \rightarrow T(n', \ell)$ .

Let  $n \geq 2\ell$  and suppose  $S(A, \ell)$  is true. We use  $T^*(n, n-1)$  to obtain  $\langle B_1, B_2 \rangle$ , as in the preceding paragraph. If one of the subsets  $B_1$  or  $B_2$  of  $B$  has at most  $\ell$  elements, then  $T(n, \ell)$  is realized. Otherwise, one of these subsets has  $n_1$  elements, the other has  $n - n_1$  elements, and both  $n > \ell$  and  $n - n_1 > \ell$ . By (ii)<sub>b</sub>, either  $A$  and  $n_1$  satisfy condition  $\mathcal{A}_\ell$  or else  $A$  and  $n - n_1$  satisfy condition  $\mathcal{A}_\ell$ . By the inductive hypothesis, either  $T(n_1, \ell)$  or  $T(n - n_1, \ell)$  is true. We can therefore choose a nonempty subset of at most  $\ell$  elements of one of the subsets  $B_1$  or  $B_2$  of  $B$ . Thus,  $T(n, \ell)$  is true.

Let  $A$  be as above and let  $n \geq 2$ . Let  $P(A, n)$  be the statement: "For every prime partition of  $n$ , that is, whenever  $n = p_1 + p_2 + \dots + p_k$ , one of these primes is in  $A$ ."

**THEOREM 5.** *Assume  $P(A, n)$ . Then for all  $\ell$ ,  $S(A, \ell) \rightarrow T(n, \ell)$ .*

*Proof.* It suffices to show that if  $P(A, n)$ , then for all  $\ell \geq 1$ ,  $A$  and  $n$  satisfy condition  $\mathcal{A}_\ell$ .

Assume  $P(A, n)$  and let  $\ell \geq 1$ .

If  $n$  is prime and  $n > \ell$ , then  $n$  is an  $A_\ell$ -number. If  $n$  is prime and  $n \leq \ell$ , then (ii)<sub>a</sub> and (ii)<sub>b</sub> of condition  $\mathcal{A}_\ell$  are satisfied.

Suppose that  $n$  is composite and that for all  $k$ ,  $2 \leq k < n$ , whenever  $P(A, k)$ , then  $A$  and  $k$  satisfy condition  $A_\ell$ . By  $P(A, n)$ , each prime factor of  $n$  is in  $A$ . Let  $n = n_1 + n_2$ , where  $n_1 > \ell$  and  $n_2 > \ell$ . Suppose there is a prime partition of  $n_1$  with no summand in  $A$ . Then by  $P(A, n)$ , every prime partition of  $n_2$  has a summand in  $A$ . Thus,  $P(A, n_1)$  or  $P(A, n_2)$ . By the inductive hypothesis, either  $A$  and  $n_1$  or else  $A$  and  $n_2$  satisfy condition  $\mathcal{A}_\ell$ . Therefore,  $A$  and  $n$  satisfy condition  $\mathcal{A}_\ell$ .

Independence results concerning these axioms can be found in [3].

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