THE AUTOMORPHISM GROUPS OF SPACES AND FIBRATIONS

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This paper deals with the automorphism group of fibrations \( f: X \to Y \), where \( X \) and \( Y \) are simply connected CW-complexes with either a finite number of homology groups or homotopy groups. It is proved that the automorphism groups of such fibrations are finitely presented, and that in case \( X \) and \( Y \) are \( H_0 \)-spaces the image of the obvious map \( \text{Aut}(f) \to \text{Aut}(H^*(f, Z)) \) has finite index in \( \text{Aut}(H^*(f, Z)) \). It is also proved that in case that \( Y \) belongs to the genus of \( X \), \( \ker(\text{Aut} X \to \text{Aut} X_p) \) is isomorphic to \( \ker(\text{Aut} Y \to \text{Aut} Y_p)(( \_ )_p\text{-localization of } p) \).

Introduction. Let \( X, Y \) be spaces and let \( f: X \to Y \) be a fibration. This work concerns the group \( \text{Aut} X \) of homotopy classes of self equivalences of \( X \) as well as the group \( \text{Aut}(f) \) of homotopy classes of pairs \( (h, k) \in \text{Aut} X \times \text{Aut} Y \) which satisfy \( fh \sim kf \). Throughout this paper all spaces considered are of the homotopy type of nilpotent CW-complexes of finite type, and all, except those which appear in Chapter four, are of the homotopy type of simply connected CW-complexes, which are either finite dimensional or with a finite number of homotopy groups.

We use the notations of Wilkerson [8]. We recall that a space \( X \) is called an \( H_0 \)-space if \( H^*(X, Q) \) is an exterior algebra on odd dimensional generators, that the genus of \( X \) is the set \( G(X) \) of homotopy types of spaces \( Y \) with \( Y_p \approx X_p \) for every prime \( p \), and that the elements \([f']\) of the genus of a fibration \( f: X \to Y \) are equivalence classes of homotopy classes \( f' \) which satisfy: For every prime \( p \) there exist homotopy equivalences \( h_p: X'_p \to X_p, k_p: Y'_p \to Y_p \) satisfying \( f'h_p \sim k_pf'_p \).

Concerning \( \text{Aut} X \) and \( \text{Aut}(f) \) we are interested in the following questions:

(a) Is the group \( \text{Aut}(f) \) finitely presented? i.e., can Theorem B in Wilkerson [8] be generalized to \( \text{Aut}(f) \)?

(b) What is the relation between:

(1) \( \text{Aut} X \) and \( \text{Aut} H^*(X, Z) \) where \( X \) is an \( H_0 \)-space.

(2) \( \text{Aut}(f) \) and \( \text{Aut} H^*(f, Z) \) where \( f \) is an \( H_0 \)-fibration, i.e., \( f \) is a fibration between \( H_0 \)-spaces.

(3) \( \text{Aut} X \) and \( \text{Aut} X' \) where \( X' \) belongs to the genus of \( X \).

(4) \( \text{Aut}(f) \) and \( \text{Aut}(f') \) where \( f' \) belongs to the genus of \( f \).

The answer to question (a) is given by:

**Main Theorem.** Let \( X, Y \) be simply connected CW-complexes
and let $F \to X \to Y$ be a fibration. Then:

(a) $\text{Aut}(f)$ is commensurable with an arithmetic subgroup of $\text{Aut}(f_0)$, where $f_0: X_0 \to Y_0$ is the rationalization of $f$.

(b) $\text{Aut}(f)$ is finitely presented, and

(c) $\text{Aut}(f)$ has only a finite number of finite subgroup up to conjugation.

One of the results of this theorem is:

**Corollary 2.8.** Let $X$ be a simply connected finite CW-complex and let $G \subseteq \text{Aut} X$ be a finitely generated subgroup. If $H_*(X, \mathbb{Z})$ is torsion free then the centralizer of $G$ is finitely presented.

Concerning question (b) we obtain the following interesting results:

**Proposition 3.2.** Let $X, Y$ be $H_0$-spaces and let $f: X \to Y$ be a fibration. Then:

(a) The map $[Y, X] \to \text{Hom}(H_*(Y, \mathbb{Z}), H_*(X, \mathbb{Z})$ is finite to one.

(b) $\text{Im}(\text{Aut} X \to \text{Aut} H_*(X, \mathbb{Z})$ is a subgroup of finite index.

(c) The kernel of the obvious map $\text{Aut}(f) \to \text{Aut} H_*(f, \mathbb{Z})$ is finite and its image is a subgroup of finite index in $\text{Aut} H_*(f, \mathbb{Z})$.

(d) For any pair $(h, k) \in \text{Aut} H_*(f, \mathbb{Z})$ there exists a pair $(\tilde{h}, \tilde{k}) \in \text{Aut}(f)$ and an integer $m$, so that $H^*(h, \mathbb{Z}) = h^m$ and $H^*(k, \mathbb{Z}) = k^m$.

**Proposition 4.6.** Let $X$ be an $H_0$-space either with a finite number of homology groups or with a finite number of homotopy groups. If $H_*(X, \mathbb{Z})$ is torsion free then $\text{Ker}(\text{Aut} X \to \text{Aut} X_q)q \in P$ ($P$ – the set of primes) is a direct product of finite $p$-groups, $p \neq q$.

**Proposition 4.7.** Let $X, Y$ be nilpotent spaces with a finite number of homology groups and let $f: X \to Y$ be a fibration. Then for every prime $p$ and for every fibration $f': X' \to Y'$, which belongs to the genus of $f$, $\text{Ker}(\text{Aut}(f) \to \text{Aut}(f_p))$ is isomorphic to $\text{Ker}(\text{Aut}(f') \to \text{Aut}(f'_p))$.

As a consequence of Propositions 3.6 and 4.7 we obtain:

**Proposition 3.7.** Let $X, \mu_0$ be an $H_0$-space. Suppose $H^*(\mu_0, \mathbb{Q})$ is primitively generated, then the number of equivalence classes of $H$-structure on $X$ for which $H^*(\mu, \mathbb{Q})$ is equivalent to $H^*(\mu_0, \mathbb{Q})$ is finite.

**Corollary 4.9.** Let $X, Y$ be $H_0$-spaces either with a finite number
of homology groups or with a finite number of homotopy groups and let \( f: X \to Y \) be a fibration. Then for every fibration \( f' \) which belongs to the genus of \( f \), \( \text{Ker}(\text{Aut}(f) \to \text{Aut} H_*(f, Z)) \) is isomorphic to \( \text{Ker}(\text{Aut}(f') \to \text{Aut} H_*(f', Z)) \).

**Proposition 4.10.** Let \( f \) and \( f' \) be as in Corollary 4.9. If \( \text{Aut}(f) \) is finite, then \( \text{Aut}(f) \) is isomorphic to \( \text{Aut}(f') \).

The paper is organized as follows:

In section one the relation between automorphism groups and rational equivalence is studied. The main result is proved in section two. In section three, the special properties of \( H_0 \)-spaces and the results of section one are used to draw conclusions on the automorphism groups of \( H_0 \)-spaces and fibrations. In the last section, section four, the relation between automorphism groups and genus is studied.

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1. Automorphism groups and rational equivalence.

**Lemma 1.1.** Let \( X, Y, X', Y' \) be simply connected finite type CW-complexes, \( f: X \to Y, f': X' \to Y' \) be fibrations and \( F \) and \( F' \) be simple CW-complexes with \( \pi_* F \) and \( \pi_* F' \) finite dimensional and finite. Define \( S \) to be the set of homotopy classes of pairs \((\varphi, \psi)\) satisfying:

(a) \( \varphi: X \to X' \) and \( \psi: Y \to Y' \) are maps with homotopy theoretic fibers \( F \) and \( F' \), respectively.

(b) \( f' \varphi \sim f \psi \).

Then \( \text{Aut}(f) \) acts on \( S \) and \( S/\text{Aut}(f) \) is a finite set.

**Proof.** Let \( M \) be the set of triples \((\varphi, \psi, \bar{f})\), where \( F \to \tilde{X} \varphi \to X' \) and \( F' \to \tilde{Y} \psi \to Y' \) are fibrations, \( \bar{f}: \tilde{X} \to \tilde{Y} \) a map and \( \psi \bar{f} \sim f' \varphi \). Define an equivalence relation on \( M \) by: \((\varphi', \psi', \bar{f}') \sim (\varphi, \psi, \bar{f})\) \((\bar{f}': \tilde{X}' \to \tilde{Y}')\) if and only if there exist homotopy equivalences \( \alpha: \tilde{X} \to \tilde{X}', \beta: \tilde{Y} \to \tilde{Y}' \) so that the following diagram homotopy commutes. For any pair \((\varphi, \psi)\) \(\in S\) there is a factorization of \( \varphi \) and \( \psi \) as \( X \xrightarrow{i} X' \xrightarrow{\overset{\phi}{\phi}} X', Y \xrightarrow{j} Y \xrightarrow{\overset{\psi}{\psi}} Y', \) where \( i \) and \( j \) are homotopy equivalences, \( \phi \) and \( \psi \) are fibrations and \( \phi i \sim \varphi, \psi i \sim \psi \). Obviously \( f' \bar{\varphi} \sim \bar{\psi}(jf^{-1}) \) \((f^{-1} = \text{the homotopy inverse of } i)\) and therefore the triple \((\bar{\varphi}, \bar{\psi}, jf^{-1}) \in M\). Changing \((\varphi, \psi)\) within a homotopy class does not vary the equivalence class of the triple \((\bar{\varphi}, \bar{\psi}, jf^{-1})\). Hence \( S \to M \) is well
defined.

Suppose \((\varphi, \psi), (\varphi', \psi') \in S\) and there exists a pair \((\alpha, \beta) \in \text{Aut}(f)\) so that \(\varphi \alpha \sim \varphi'\) and \(\psi \beta \sim \psi'\), then the triples \((\overline{\varphi}, \overline{\psi}, j\beta \alpha^{-1})\) and \((\overline{\varphi}', \overline{\psi}', j'\beta' \alpha^{-1})\) are equivalent in \(M\). Conversely, if the triples \((\overline{\varphi}, \overline{\psi}, j\beta \alpha^{-1})\) and \((\overline{\varphi}', \overline{\psi}', j'\beta' \alpha^{-1})\) are equivalent in \(M\) i.e., if there are homotopy equivalences \(\alpha: X_\varphi \to X_{\varphi'}\) and \(\beta: Y_\psi \to Y_{\psi'}\), so that 


to \(j\beta \alpha^{-1})\), then \(\varphi(i^{-1}\alpha^{-1}i') \sim \varphi'\) and \(\psi(j^{-1}\beta' \alpha^{-1})\) \(j'\beta' \alpha^{-1})\) is well defined and monic. Therefore it is enough to prove that \(M/\sim\) is finite. But by a standard Moore-Postnikov argument any element of \(M\) can be obtained as a sequence of principal fibrations \((\varphi_n, \psi_n)\) with fibers \(K(\pi_n X, n)\) and \(K(\pi_n Y, n)\), so that \(f_n P_n \sim \psi_n f_n\). Hence it suffices to show that for each \(n\) there is a finite number of equivalence classes of such fibrations, where the equivalence relation is defined as in \(M\).

Suppose for the pair of \(k\)-invariants \((k, k') \in H^{n+1}(X_{n-1}, \pi_n X) \times H^{n+1}(Y_{n-1}, \pi_n Y)\) there exists \(f_n: X_n \to Y_n\) so that \(f_{n-1} P_n \sim \psi_n f_n\). Assume also that \(\varphi_n: X_n \to X_{n-1}\) and \(\psi_n: Y_n \to Y_{n-1}\) are fibers of \(k\) and \(k'\), respectively, and there exists \(f'_n: X'_n \to Y'_n\) satisfying \(\psi_n f_n \sim f_{n-1} P_n\).

Consider the following diagram

There exist homotopy equivalences \(\alpha: X_n \to X'_n\), \(\beta: Y'_n \to Y_n\) so \(\varphi_n \alpha \sim \varphi_n\) and \(\psi_n \beta \sim \psi_n\). The map \(\beta f'_n \alpha\) is a lift of \(f_{n-1}\), hence the finiteness of the group \(\pi_n\) (fiber \(\psi\)) and the number of stages implies the finiteness of \(M/\sim\).
LEMMA 1.2. Let \( f: X \rightarrow Y \) and \( f': X' \rightarrow Y' \) be as in Lemma 1.1, and let \( M \) and \( M' \) be simple connected CW-complexes with \( H_\ast(M, \mathbb{Z}) \) and \( H_\ast(M', \mathbb{Z}) \) finite dimensional and finite. Define \( S \) to be the set of homotopy classes of pairs \((\varphi, \psi)\) satisfying:

(a) \( \varphi: X \rightarrow X' \) and \( \psi: Y \rightarrow Y' \) are maps satisfying \( X' \cup \text{Cone} \varphi \) is homotopy equivalent to \( M \) and \( Y' \cup \text{Cone} \psi \) is homotopy equivalent to \( M' \).

(b) \( f' \varphi \sim \psi f \).

Then \( \text{Aut}(f') \) acts on \( S \) and \( S/\text{Aut}(f') \) is a finite set.

Proof. Dual to the proof of 1.1.

THEOREM 1.3. Let \( X, Y, X', Y' \) be simply connected finite type CW-spaces which are either \( H_\ast \)-finite dimensional or \( \pi_\ast \)-finite dimensional, and let \( f: X \rightarrow Y \) and \( f': X' \rightarrow Y' \) be fibrations.

Suppose \( \varphi: X \rightarrow X' \) and \( \psi: Y \rightarrow Y' \) are rational equivalences satisfying \( f' \varphi \sim \psi f \). Then \( \text{Aut}(f) \) and \( \text{Aut}(f') \) are commensurable groups.

Proof. Let \( \Delta(\varphi, \psi) \subseteq \text{Aut}(f) \times \text{Aut}(f') \) be the set of pairs \(((h, k), (h', k')) \in \text{Aut}(f) \times \text{Aut}(f')\) for which the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{\psi} & Y'
\end{array}
\]

commutes, and let \( \text{Stab}(\varphi, \psi, \text{Aut}(f')) \) be the image in \( \text{Aut}(f') \) of the second projection map on \( \Delta(\varphi, \psi) \). We shall show that \( \text{Aut}(f) \) and \( \text{Aut}(f') \) are commensurable with \( \Delta(\varphi, \psi) \).

Let \( S' \) be the set of homotopy classes of pairs of the form \((h' \varphi h, k' \psi k)\) where \((h, k) \in \text{Aut}(f)\) and \((h', k') \in \text{Aut}(f')\). Then \( S' \) is a subset of \( S \) of Lemma 1.1, and hence \( S'/\text{Aut}(f) \) is a finite set. But \( \text{Aut}(f') \) acts on \( S'/\text{Aut}(f) \), i.e., there is a map

\[ \eta: \text{Aut}(f') \longrightarrow \text{Aut}(S'/\text{Aut}(f)) \].

Then the group \( \text{Stab}(\varphi, \psi, \text{Aut}(f')) \) contains the kernel of \( \eta \), and therefore the fact that \( \text{Aut}(S'/\text{Aut}(f)) \) is a finite set implies that \( \text{Stab}(\varphi, \psi, \text{Aut}(f')) \) has finite index in \( \text{Aut}(f') \).
On the other hand, the fact that $\varphi$ and $\psi$ are rational equivalences implies that the kernel of the map $\mathcal{A}(\varphi, \psi) \to \text{Aut}(f)$ is finite. Hence $\mathcal{A}(\varphi, \psi)$ and $\text{Aut}(f')$ are commensurable groups. The proof that $\mathcal{A}(\varphi, \psi)$ and $\text{Aut}(f)$ are commensurable is dual.

**Notation.** For a fibration $f: X \to Y$ denote by $\text{Aut}_X(f)$ the group of homotopy classes of self homotopy equivalences $k: Y \to Y$ satisfying $kf \sim f$, and by $\text{Aut}_Y(f)$ the group of homotopy classes of self homotopy equivalences $h: X \to X$ which satisfy $fh \sim f$.

**Corollary 1.4.** Let $f, f', \varphi$ and $\psi$ be as in Theorem 1.3. Then $\text{Aut}_X(f)$ is commensurable with $\text{Aut}_X(f')$ and $\text{Aut}_Y(f)$ is commensurable with $\text{Aut}_Y(f')$.

**Theorem 1.5.** Let $X, Y, X', Y'$ be simply connected finite type CW-spaces and let $f: X \to Y$ and $f': X' \to Y'$ be fibrations. Suppose $f_0$ is homotopy equivalent to $f'_0$. Then $\text{Aut}(f)$ and $\text{Aut}(f')$ are commensurable groups.

**Proof.** Since $f_0$ is homotopy equivalent to $f'_0$ there exists a commutative diagram.

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X_0 \\
\downarrow f & & \downarrow f_0 \\
Y & \xrightarrow{\psi} & Y_0
\end{array}
$$

where the horizontal maps are rationalizations.

Let $Y''$ be a simply connected CW-complex which satisfies: There exist rational equivalences $\alpha: Y'' \to Y$, $\alpha': Y'' \to Y'$ and $\psi': Y'' \to Y_0$ so that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \xleftarrow{\alpha} & Y'' \\
\downarrow{\varphi} & & \downarrow{\psi''} \\
Y_0 & \xrightarrow{\psi'} & Y'
\end{array}
$$

(By Wilkerson [8] such a space exists.)

Consider the following diagram:
where $X''$ is the pullback of $Y'' \xrightarrow{\alpha} Y' \xrightarrow{f'} X'$ and $\varphi'': X' \to X_0$ is a rationalization, which satisfies $f\varphi'' \sim \psi'' f''$. (The existence of such a rational equivalence follows from the above three diagrams.)

Since $\varphi$ and $\varphi''$ are rationalizations there exists a bouquet of spheres $VS_{n_1}$ and maps $X \xrightarrow{\gamma} VS_{n_1} \xrightarrow{\gamma''} X''$ so that $\pi_* \gamma \otimes Q$ and $\pi_* \gamma'' \otimes Q$ are epimorphisms and $\varphi''\gamma'' \sim \varphi\gamma$. Therefore the commutativity of the two parallelograms and the triangle, in the last diagram, implies that $\psi\alpha f''\gamma'' \sim \varphi f\gamma$. Consequently there exists a map $\delta: VS_{n_1} \to VS_{n_1}$ so that $\alpha f''\gamma'' \sim f\gamma\delta$.

Consider the cofibration $VS_{n_1} \xrightarrow{j} VS_{n_1} \xrightarrow{j} C_2$ where $\text{Im}(\pi_*(\gamma)) = \text{Ker}(\pi_*(\varphi) \otimes Q) = \text{Ker}(\pi_*(\gamma) \otimes \text{Id} \otimes Q)$. There exist maps $\epsilon: C_2 \to X$, $\sigma': C_2 \xrightarrow{\epsilon'} X''$ so that $\epsilon' j \sim \gamma\delta$, $\epsilon'' j \sim \gamma''\delta$ and $\varphi''\epsilon'' \sim \varphi\epsilon$. Consequently the considerations of the previous paragraph imply the existence of a map $\mu: VS_{n_1} \to VS_{n_1}$ and rational equivalences $\phi: C_{2n} \to X$, $\phi': C_{2n} \to X''$ ($C_{2n}$ - the cofibre of $\lambda_1\mu$), so that $\alpha f''\epsilon'' \sim f\phi$. Hence Theorem 1.3 implies that $\text{Aut}(f)$ and $\text{Aut}(f')$ are both commensurable with $\text{Aut}(f''\phi'')$ and therefore they are commensurable.

2. Proof of the main theorem. By Wilkerson [8] there are finitely generated free simplicial $N^e Z$ groups $M$. an $N$. and a map $f.: M. \to N.$ so that $\text{Aut}(f)$ can be identified with the group of loop homotopy equivalence classes of self-equivalences of $f.$, and $\text{Aut}(f_0)$ can be identified with the group of loop homotopy equivalence classes of self-equivalences of $f.: M. \to N.$ Therefore we study here these groups. We denote them by $H\text{Aut}(f)$ and $H\text{Aut}(f_0)$, respectively.

Let $M.$ and $N.$ be finitely generated $N^e Q$ groups. Denote by $\text{Aut}(M.) \times \text{Aut}(N.)$ the group of simplicial automorphisms of $M. \otimes N.$ and by $\text{Aut}(M.), \text{Aut}(N.)$ the set of automorphisms of $M. \otimes N.$ lying over the identity on $N.$ The face maps $d_0, d_1: \text{Aut}(M.) \to \text{Aut}(N.)$ and $d_0', d_1': \text{Aut}(N.) \to \text{Aut}(N')$.

Let $\text{SimpAut}(f.)$ denote the set of simplicial automorphisms of
f_0. Two pair \((h, k), (h', k') \in \text{SimpAut}(f_0)\) are homotopic if and only if \(h' \in d_1d_0^{-1}(h)\) and \(k' \in d_1d_0^{-1}(k)\). Hence

\[
H \Aut(f_0) = \text{SimpAut}(f_0)/ (d_1d_0^{-1}(id) \times d_1d_0^{-1}(id)) \cap \text{Simp Aut}(f_0).
\]

**Proposition 2.1.** Let \(f_0: M_0 \to N_0\) be a simplicial map between finitely generated free simplicial \(N^\circ Q\) groups. There exists an affine group scheme \(G\) over \(Q\), so that \(\text{SimpAut}(f_0)\) can be identified with the \(Q\)-valued points of \(G\).

**Proof.** Similar to the proof of Proposition 9.2 in Wilkerson [8].

**Proposition 2.2.** There is a normal closed subgroup scheme over \(Q, H\) of \(G\), such that \((d_1d_0^{-1}(id) \times d_1d_0^{-1}(id)) \cap \text{SimpAut}(f_0) = H(Q)\).

**Proof.** Since linear algebraic groups are closed under finite cartesian products and finite intersections, the result follows from Proposition 9.3 in Wilkerson [8].

**Proposition 2.3.** Let \(G\) and \(H\) be as defined above. There exists an affine group scheme \(G/H\) over \(Q\), such that \(H \Aut(f_0) = (G/H)(Q) = G(Q)/H(Q)\).

**Proof.** Proposition 9.4. in Wilkerson [8], the discussion above and the fact that a subgroup of a unipotent group is unipotent, implies that \(H\) is unipotent and that \(H \Aut(f_0) = G(Q)/H(Q)\). By Borel [1, 6.8], the quotient of an affine group scheme over \(Q\) by a closed normal subgroup scheme over \(Q\) is again an affine group scheme over \(Q\). That is \(G/H\) exists. The Galois cohomology sequence [Serre] \(1 \to H(Q) \to G(Q) \to G/H(Q) \to H'(\text{Gal}(\overline{Q}, Q), H) \cdots\) is an exact sequence of groups and pointed sets. Hence the fact that \(H\) is unipotent implies that \(H'(\text{Gal}(\overline{Q}, Q), H) = 0\) and the result follows.

**Proposition 2.3'.** Let \(X, Y\) be simply connected finite CW-complexes and let \(f: X \to Y\) be a fibration. Then \(\Aut(f_0)\) is the set of \(Q\)-valued points of a linear algebraic group over \(Q\).

**Proposition 2.4.** Let \(M_0\) and \(N_0\) be finitely generated free simplicial nilpotent groups of class \(c\) and let \(f_0: M_0 \to N_0\) be a simplicial map. Define \(M_L \subseteq M_0(N_L \subseteq N_0)\) to be the intersection of all lattice subgroups of \(M_0(N_0)\) that contain \(M_0(N_0)\).

Then \(f_0\) induces a map \(f_L: M_L \to N_L\) and \(\text{SimpAut}(f_0)\) has finite index in \(\text{Simp Aut}(f_L)\).
Proof. The existence of $f_L$ and the fact that $G = \text{def}(\text{Simp Aut}(M) \times \text{Simp Aut}(N)) \subseteq \text{Simp Aut}(M_L) \times \text{Simp Aut}(N_L)$ is a subgroup of finite index, follows from Wilkerson [8, 8.1 and 8.3]. Hence

$$G \cap \text{Simp Aut}(f_L) \subseteq \text{Simp Aut}(f_L)$$

is a subgroup of finite index and it suffices to prove that $\text{Simp Aut}(f_L) = G \cap \text{Sim Aut}(f_L)$. But this is clear, since $(h, k) \in G \cap \text{Simp Aut}(f_L)$ implies that $h|_M : M \to M$, $k|_N : N \to N$, and $kf_Lh^{-1} = f_L$ and therefore $(h|_M, k|_N) \in \text{Simp Aut}(f_L)$.

Proposition 2.5. Let $X, Y$ be simply connected finite CW-complexes and let $f : X \to Y$ be a fibration. There exist finite CW-complexes $X'$ and $Y'$ so that $H_*(X', Z)$ and $H_*(Y', Z)$ are torsion free and a fibration $f' : X' \to Y'$ so that $\text{Aut}(f')$ and $\text{Aut}(f)$ are commensurable groups.

Proof. By Theorem 1.3 it suffices to prove that there exist rational equivalences $h : X' \to X$ and $k : Y' \to Y$ so that the diagram

$$\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow h & & \downarrow k \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes.

Since $f$ is homotopic to a cellular map we can assume that $f$ is cellular. Suppose there exists a commutative diagram

$$\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow h_n & & \downarrow k_n \\
X & \xrightarrow{f} & Y
\end{array}$$

where $f_n$ is cellular, $h_n, k_n$ are rational equivalences and the groups $H_m(X_n, Z)$ and $H_m(Y_n, Z)$ are torsion free for $m \leq n$.

Let $X_n^{(a)}$, $Y_n^{(a)}$ be the $n$-skeletons of $X_n$ and $Y_n$. Since $f_n$ is cellular $f_n$ induces a map $f_n' : X_n/X_n^{(a)} \to Y_n/Y_n^{(a)}$. Therefore the fact that $H_{n+1}(X, Z) = \pi_{n+1}(X_n/X_n^{(a)})$ and $H_{n+1}(Y, Z) = \pi_{n+1}(Y_n/Y_n^{(a)})$ implies the existence of a commutative diagram ($'$($'$) denotes the torsion subgroup of ($'$)).
Let $X_{n+1}$ and $Y_{n+1}$ be the fibers of the maps

$$X_n \longrightarrow X_n/X_n^{(n)} \longrightarrow K(H_{n+1}X, Z) \longrightarrow K(H_{n+1}X, Z)$$

and

$$Y_n \longrightarrow Y_n/Y_n^{(n)} \longrightarrow K(H_{n+1}Y, Z) \longrightarrow K(H_{n+1}Y, Z)$$

and let $f_{n+1}: X_{n+1} \to Y_{n+1}$ be the induced map. Obviously $X_{n+1}$ is rational equivalent to $X$, $Y_{n+1}$ is rational equivalent to $Y$ and there exists a commutative diagram

$$\begin{array}{ccc}
X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \\
\downarrow h_{n+1} & & \downarrow k_{n+1} \\
X & \xrightarrow{f} & Y
\end{array}$$

where $h_{n+1}$ and $k_{n+1}$ are rational equivalences and $f_{n+1}$ is cellular. By the Serre spectral sequence $H_m(X_{n+1}, Z)$ and $H_m(Y_{n+1}, Z)$ are torsion free for $m \leq n + 1$, and the result follows.

**Proposition 2.6.** Let $M.$ and $N.$ be finitely generated connected minimal simplicial $\mathbb{N}^\ast\mathbb{Z}$ groups and let $f.: M. \to N.$ be a simplicial map. Then $H\text{Aut}(f.)$ is an arithmetic subgroup of $H\text{Aut}(f'_0)$.

**Proof.** Since $\text{SimpAut}(f) \subseteq \text{SimpAut}(f_L)$ is a subgroup of finite index, the theorem follows from Theorem 9.8 in [8] by replacing $N.$ by $f., N_L$ by $f_L$ and $d_2d_0^{-1}(id)$ by $(d_4d_0^{-1}(id) \times d_4d_0^{-1}(id)) \cap \text{SimpAut}(f'_0)$.

**Proof of the main theorem.** By Proposition 2.5 we can assume that $H_*(X, Z)$ and $H_*(Y, Z)$ are torsion free. Hence by Wilkerson [8] $\text{Aut}(f)$ can be calculated as $H\text{Aut}(f.)$ for some $f.: M. \to N.$, where $M.$ and $N.$ are connected minimal free simplicial $\mathbb{N}^\ast\mathbb{Z}$ groups. Therefore $\text{Aut}(f)$ is an arithmetic subgroup of a linear algebraic group, and the result follows from Proposition 10.3 in [8].
COROLLARY 2.7. Let $X, Y$ be simply connected finite CW-complexes and let $f: X \to Y$ be a fibration. Then $\text{Aut}_r(f)$ is finitely presented.

Proof. Similar to the proof of the main theorem.

COROLLARY 2.8. Let $X$ be a simply connected finite CW-complex and let $G \subseteq \text{Aux}_X$ be a finitely generated subgroup. If $H_*(X, Z)$ is torsion free then the centralizer of $G$ is finitely presented.

Proof. Suppose $G$ is generated by $g_1, g_2, \ldots, g_n$. Since the centralizer of $G$ is equal to the centralizer of the set $\{g_1, g_2, \ldots, g_n\}$, the proof is similar to the proof of the main theorem.

3. Commensurability and $H_0$-spaces and fibrations. Let $X, Y$ be $H_0$-spaces and let $f: X \to Y$ be a fibration. In this section we deal with the relation between $\text{Aut}_X$ and $\text{Aux}_*H^*(X, Y)$ and between $\text{Aut}(f)$ and $\text{Aux}_*H^*(f, Z)$. In case $X$ is an $H$-space we draw conclusions on the relation between the $H$-structures on $X$ and the Hopf-algebra structures on $H^*(X, Q)$.

NOTATION. For any $H_0$-space $X$ we denote $K(QH^*(X, Z)/\text{torsion})$ by $K(X)$.

PROPOSITION 3.1. Let $f_1, f_2: X \to Y$ be fibrations. If $\text{rank}(H^*(f_1, Q))$ is equal to $\text{rank}(H^*(f_2, Q))$ then $\text{Aut}(f_1)$ and $\text{Aut}(f_2)$ are commensurable groups.

Proof. Since $\text{rank}(H^*(f_1, Q)) = \text{rank}(H^*(f_2, Q))$ there exist Eilenberg-Maclane spaces $K_1, K_2$ and rational equivalences $\varphi_i: X \to K(X), \psi_i: Y \to K(Y)$ ($i = 1, 2$) so that $K(X) = K \times K_1, K(Y) = K \times K_2$ and the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{f_i} & Y \\
\downarrow{\varphi_i} & & \downarrow{\psi_i} \\
K(X) & = & K \times K_1 \\
 & \xrightarrow{p} & K \\
& \leftarrow{i} & K \times K_2 = K(Y)
\end{array}
\]

Hence $\text{Aut}(f)$ and $\text{Aut}(g)$ are both commensurable with $\text{Aut}(ip)$ and therefore they are commensurable groups.

PROPOSITION 3.2. Let $X, Y$ be $H_0$-spaces and let $f: X \to Y$ be a fibration. Then:
(a) The map \([Y, X] \to \text{Hom}(H^*_*(Y, Z), H^*_*(X, Z))\) is finite to one.
(b) \(\text{Im}(\text{Aut} X \to \text{Aut} H^*(X, Z))\) is a subgroup of finite index.
(c) The kernel of the obvious map \(\eta: \text{Aut}(f) \to \text{Aut} H^*(f, Z)\) is finite and its image is a subgroup of finite index in \(\text{Aut} H^*(f, Z)\).
(d) For any pair \((h, k) \in \text{Aut} H^*(f, Z)\) there exists a pair \((\tilde{h}, \tilde{k}) \in \text{Aut}(f)\) and an integer \(m\), so that \(H^*(\tilde{h}, Z) = h^m\) and \(H^*(\tilde{k}, Z) = k^m\).

Proof. (a) Let \(\varphi: X \to K(X)\) be a rational equivalence which represents generators of \(H^*(X, Z)/\text{torsion}\). Since \(H^*(f, Z) = H^*(g, Z)\) \((f, g: Y \to X)\) implies that \(\varphi f \sim \varphi g\), the result follows from the fact that any map \(h: Y \to K(X)\) has only a finite number of lifts to a map \(\tilde{h}: Y \to X\), which satisfies \(\varphi \tilde{h} \sim h\).

(b) Let \(\varphi: X \to K(X)\) be as in (a). By Wilkerson [8] \(\text{Im}(\text{Aut}(\varphi) \overset{\text{proj}}{\to} \text{Aut}(K(X)) \to \text{Aut}(H^*(K(X), Z)/\text{torsion}) \to \text{Aut}(H^*(X, Z)/\text{torsion})\) is a subgroup of finite index in \(\text{Aut}(H^*(X, Z)/\text{torsion})\). Hence the result follows from the fact that \(\text{Im}(\text{Aut} X \to \text{Aut}(H^*(X, Z)/\text{torsion}))\) contains the image of the above map.

(c) The fact that \(\text{Kern}\) is a finite group follows from part (a).

Let \(G = \text{Im}(\text{Aut} X \times \text{Aut} Y \to \text{Aut} H^*(X, Z) \times \text{Aut} H^*(Y, Z))\). By part (b) \(G\) is a subgroup of finite index in \(\text{Aut} H^*(X, Z) \times \text{Aut} H^*(Y, Z)\), hence \(G \cap \text{Aut} H^*(f, Z) \subseteq \text{Aut} H^*(f, Z)\) is a subgroup of finite index and it suffices to prove that \(\text{Im} \eta \subseteq G \cap \text{Aut} H^*(f, Z)\) is a subgroup of finite index.

Let \((h, k) \in G \cap \text{Aut} H^*(f, Z)\). There exists a pair \((\tilde{h}, \tilde{k}) \in \text{Aut} X \times \text{Aut} Y\) satisfying \(H^*(\tilde{h}, Z) = h\), \(H^*(\tilde{k}, Z) = k\) and \(H^*(\tilde{k}^{-1} f \tilde{h}, Z) = H^*(f, Z)\) — the homotopy inverse of \(\tilde{k}\). Therefore the fact that there is only a finite number of maps \(f_s, f_s', \ldots, f_s^\ast\) which satisfy \(H^*(f_s^\ast, Z) = H^*(f, Z)\) implies that \(\text{Im} \eta \subseteq G \cap \text{Aut} H^*(f, Z)\) is a subgroup of finite index, and the proof of part (c) is complete.

(d) Suppose \((h, k) \in \text{Aut} H^*(f, Z)\). We have to show that there exists an integer \(m\) so that \((h^m, k^m) \in \text{Im}(\text{Aut}(f) \to \text{Aut} H^*(f, Z))\). Since \(h\) and \(k\) are automorphisms, this follows immediately from the fact that \(\text{Im}(\text{Aut}(f) \to \text{Aut} H^*(f, Z))\) is a subgroup of finite index in \(\text{Aut} H^*(f, Z)\).

**Corollary 3.3.** Let \(X\) be an \(H_0\)-space. Suppose \(h, k \in \text{Aut} X\) satisfy \(H^*(h, Z) = H^*(k, Z)\). Then there exists an integer \(m\) so that \(h^m \sim k^m\). Consequently \(h \in \text{Aut} X\) is of finite order if and only if \(H^*(h, Z)\) is.

**Proof.** The pair \((H^*(h, Z), H^*(k, Z)) \in \text{Aut}(H^*(1_x, Z))\), hence the result follows from part (d) in Proposition 3.2.

**Corollary 3.4.** Suppose \(X, Y\) are \(H_0\)-spaces, \(f: X \to Y\) a fibration
and \((h, k) \in \text{Aut}(f)\). Then:
- \(H^*(f, Z)\) is monic and the order of \(h\) is finite implies that the order of \(k\) is finite.
- \(H^*(f, Z)\) is epic and the order of \(k\) is finite implies that the order of \(h\) is finite.

**Proof.** (a) Obviously, the order of \(h\) is finite implies that the order of \(H^*(k, Z)\) is finite. Hence the result follows from Corollary 3.3.

(b) Similar to (a).

**Corollary 3.5.** Let \(X, Y\) and \(f\) be as in Corollary 3.4. Then:
- \(H^*(f, Z)\) is monic and \(\text{Aut} \ X\) is finite implies that \(\text{Aut} \ (f)\) is finite.
- \(H^*(f, Z)\) is epic and \(\text{Aut} \ Y\) is finite implies that \(\text{Aut} (f)\) is finite.

**Proof.** (a) \((h, k_1), (h, k_2) \in \text{Aut}(f)\) and \(H^*(f, Z)\) is monic implies that \(H^*(k_1, Z) = H^*(k_2, Z)\). Therefore the fact that the kernel of the map \(\text{Aut} \ Y \rightarrow \text{Aut} \ H^*(f, Z)\) is finite implies that for each \(h \in \text{Aut} \ X\) there exist, at most, a finite number of \(k \in \text{Aut} \ Y\), so that the pair \((h, k) \in \text{Aut}(f)\). Hence \(\text{Aut}(f)\) is a finite group.

(b) Similar to (a).

In order to draw conclusions from Proposition 3.2 to the case that \(X\) is an \(H\)-space we need the following definitions:

**Definition.** Let \(X\) be an \(H\)-space and let \(\mu_1, \mu_2\) be two \(H\)-structures on \(X\).

(a) We say that \(\mu_1\) is equivalent to \(\mu_2\) if there exists a homotopy equivalence \(h: X \rightarrow X\), so that \(h \mu_1 \sim \mu_2(h \times h)\).

(b) We say that \(H^*(\mu_1, Z)/\text{torsion}\) is equivalent to \(H^*(\mu_2, Z)/\text{torsion}\) if there exists a map \(h \in \text{Aut}(H^*(X, Z)/\text{torsion})\) so that

\[
(h_* \otimes h_*)H^*(\mu_1, Q) = H^*(\mu_2, Q)h_*.
\]

(c) We say that \(H^*(\mu_1, Q)\) is equivalent to \(H^*(\mu_2, Q)\) if there exists a map \(h \in H^*(X, Q)\) so that \((h \otimes h)H^*(\mu_1, Q) = H^*(\mu_2, Q)h\).

**Proposition 3.6.** Let \(X, \mu_0\) be an \(H\)-space. Then the number of equivalence classes of \(H\)-structures \(\mu\) on \(X\), for which \(H^*(\mu, Z)/\text{torsion}\) is equivalent to \(H^*(\mu_0, Z)/\text{torsion}\) is finite.

**Proof.** Let \(\eta: \text{Aut} \ X \rightarrow \text{Aut} H^*(X, Z)/\text{torsion}\) be the obvious map. By Proposition 3.2(b) \(\text{Im} \ \eta \subseteq \text{Aut} H^*(X, Z)/\text{torsion}\) is a subgroup of
finite index. Assume that the index is \( n \) and that \( h_1, h_2, \ldots, h_{n-1} \in \text{Aut}(H^*(H, Z)/\text{torsion}) \) satisfy

\[
\text{Aut}(H^*(X, Z)/\text{torsion}) = \text{Im} \, \eta \cup h_1 \text{Im} \, \eta \cup \cdots \cup h_{n-1} \text{Im} \, \eta.
\]

Let \( \mu_1, \mu_2 \) be \( H \)-structures on \( X \) and let \( h, h' \in \text{Aut}(H^*(X, Z)/\text{torsion}) \) satisfy:

\[
H^*(\mu_0, Q)(h, h)_* = (h_0 h'_0 \otimes (h_0 h'_0)) H^*(\mu_1, Q),
\]

and

\[
H^*(\mu_0, Q)(h, h')_* = (h_0 h'_0 \otimes (h_0 h'_0)) H^*(\mu_2, Q),
\]

where \( h = H^*(\tilde{h}, Z)/\text{torsion} \) and \( h' = H^*(\tilde{h}', Z)/\text{torsion} \). Then:

\[
H^*(\mu_z, Z)/\text{torsion} = H^*(\tilde{h}', \tilde{h}^{-1})\mu_z(\tilde{h}'^{-1} \times \tilde{h}^{-1})/\text{torsion}
\]

is equivalent to an \( H \)-structure \( \mu' \) which satisfies \( H^*(\mu', Z)/\text{torsion} = H^*(\mu, Z)/\text{torsion} \). Consequently the results follows from the fact that for any \( H \)-structure \( \mu \) on \( X \), the number of \( H \)-structures \( \mu' \) which satisfy \( H^*(\mu', Z)/\text{torsion} = H^*(\mu, Z)/\text{torsion} \) is finite (this follows from Proposition 3.2(a)).

**Proposition 3.7.** Let \( X, \mu_0 \) be an \( H \)-space. Suppose \( H^*(\mu_0, Q) \) is primitively generated, then the number of equivalence classes of \( H \)-structures \( \mu \) on \( X \) for which \( H^*(\mu, Q) \) is equivalent to \( H^*(\mu_0, Q) \) is finite.

**Proof.** By Proposition 3.6 the number of equivalence classes of \( H \)-structures \( \mu \) on \( X \), for which \( H^*(\mu, Z)/\text{torsion} \) is equivalent to \( H^*(\mu_0, Z)/\text{torsion} \) is finite. Hence it suffices to prove that the number of equivalence classes of comultiplications \( H^*(\mu, Z)/\text{torsion} (\mu: X \times X \rightarrow X \text{ an } H \text{-structure}) \) for which \( H^*(\mu, Q) \) is equivalent to \( H^*(\mu_0, Q) \) is finite.

Let \( A \) be the set of the comultiplications \( \nu: H^*(X, Z)/\text{torsion} \rightarrow H^*(X, Z)/\text{torsion} \otimes H^*(X, Z)/\text{torsion} \) which satisfy: There exists a multiplication \( \mu: X \times X \rightarrow X \) so that \( \nu = H^*(\mu, Z)/\text{torsion} \) and \( H^*(\nu, Q) \) is equivalent to \( H^*(\mu_0, Q) \). Denote by \( \varphi: A \rightarrow \text{Hom}(H^*(X, Q), H^*(X, Q) \otimes H^*(X, Q)) \) and by \( \eta: \text{Aut}(H^*(X, Z)/\text{torsion}) \rightarrow \text{Aut} H^*(X, Q) \) the obvious maps. Since the kernels of \( \varphi \) and \( \eta \) are finite it suffices to prove that the number of equivalence classes of \( \text{Im} \, \varphi \) relative to the equivalence relation: \( \varphi(\mu_1) \sim \varphi(\mu_2) \) if and only if there exists \( h \in \text{Im} \, \eta \) so that \( \varphi(\mu_1) h = (h \otimes h) \varphi(\mu_2) (\mu_1, \mu_2 \in A) \) is finite.

By Curjel [2, 5.2] the fact that the groups \( \text{Hom}(H^*(X, Z)/\text{torsion}, H^*(X, Z)/\text{torsion}) \) and \( \text{Hom}(H^*(X, Z)/\text{torsion}, H^*(X, Z)/\text{torsion} \otimes H^*(X, Z)/\text{torsion}) \) are finitely generated implies the existence of a basis \( X = \{x_{ij}\} \), of \( PH^*(X, \mu_0, Q) \), so that the matrix of every map
$f \in \text{Hom}(H^*(X, Z)/\text{torsion}, H^*(X, Z)/\text{torsion})$, with respect to this basis, is integral, and the matrix of every map

$g \in \text{Hom}(H^*(X, Z)/\text{torsion}, H^*(X, Z)/\text{torsion} \otimes H^*(X, Z)/\text{torsion})$

with respect to the basis $\{x_{ij} \otimes 1, 1 \otimes x_{ij}\}$ of $PH^*(X \times X, \mu_0, Q)$ is also, integral. In particular the matrix of every map which belongs either to $\text{Im} \psi$ or to $\text{Im} \eta$, with respect to the above bases, is integral. Hence the result follows from the following theorem of Samelson-Leray:

**THEOREM OF SAMELSON-LERAY [3, 3 Exp 2].** Let $A$ be an algebra over the integers. Suppose that $A$ has no generators in even dimensions. Then all the associative comultiplications on $A$ are equivalent.

4. Genus and automorphism. Let $X$ and $Y$ be nilpotent CW-complexes of finite type and let $f: X \to Y$ be a fibration. Denote by $G(X)$ the genus of $X$ and by $G(f)$ the genus of $f$.

In this section we investigate the relations between $\text{Aut} X$ and $\text{Aut} X'$ where $X' \in G(X)$ and between $\text{Aut}(f)$ and $\text{Aut}(f')$ where $f' \in G(f)$.

**NOTATION.** Let $X$ be a nilpotent CW-complex and let $\varphi: X \to X_0$ be a rationalization. For every prime $p$ and for every $h \in \text{Aut} X$ denote by $(h_p)_\varphi$ the localization of $h$ at $p$ with respect to $\varphi$.

**PROPOSITION 4.1.** Let $X$ be a nilpotent CW-complex with a finite number of homology groups and let $p \in P(P - \text{the set of primes})$. If $X' \in G(X)$ then $\text{Ker}(\text{Aut} X \to \text{Aut} X_p)$ is isomorphic to $\text{Ker}(\text{Aut} X' \to \text{Aut} X_p)$.

**Proof.** Let $\varphi: X \to X_0$ and $\varphi': X' \to X_0$ be rationalizations and let $h \in \text{Ker}(\text{Aut} X \to \text{Aut} X_p)$. Since for every prime $p$ and for every localization $\phi_p: X_p \to X_0$, $\phi_p(h_p) \sim 1_{X_0} \phi_p$, there exists a unique map $h' \in \text{Aut} X'$ so that $(h'_p)_\varphi = (h_p)_\varphi$ for every prime $p$ [5, II 5.6]. Obviously $h' \in \text{Ker}(\text{Aut} X' \to \text{Aut} X_p')$. Hence the map $\gamma: \text{Ker}(\text{Aut} X \to \text{Aut} X_p) \to \text{Ker}(\text{Aut} X' \to \text{Aut} X_p)$ defined by $\gamma(h) = h'$ iff $(h'_p)_\varphi = (h_p)_\varphi$ for every prime $p$, is a well defined homomorphism. The same considerations imply the existence of a homomorphism $\gamma': \text{Ker}(\text{Aut} X' \to \text{Aut} X_p') \to \text{Ker}(\text{Aut} X \to \text{Aut} X_p)$ defined by: $\gamma'(k) = k'$ iff $(k'_p)_\varphi = (k_p)_\varphi$ for every prime $p$. Since $\gamma' \gamma$ and $\gamma \gamma'$ are identities $\text{Ker}(\text{Aut} X \to \text{Aut} X_p)$ is isomorphic to $\text{Ker}(\text{Aut} X' \to \text{Aut} X_p')$.

**COROLLARY 4.2.** Let $X$ be an $H_0$-space with either a finite number of homology groups or a finite number of homotopy groups. Then
for every $X' \in G(X)$ $\text{Ker}(\text{Aut } X \to \text{Aut } H^*(X, Q))$ is isomorphic to $\text{Ker}(\text{Aut } X' \to \text{Aut } H^*(X', Q))$.

Proof. Since $X_0 = \Pi K(Q, n_i)$ the result follows from Proposition 4.1.

Corollary 4.3. If $X$ and $X'$ are as in Corollary 4.2 then $\text{Ker}(\text{Aut } X \to \text{Aut } H_*(X, Z))$ is isomorphic to $\text{Ker}(\text{Aut } X' \to \text{Aut } H_*(X', Z))$.

Proof. Let $\eta$ and $\eta'$ be as in the proof of Proposition 4.1. We have to show that $\eta(\text{Ker}(\text{Aut } X \to \text{Aut } H_*(X, Z))) \subseteq \text{Ker}(\text{Aut } X' \to \text{Aut } H_*(X', Z))$ and that $\eta'(\text{Ker}(\text{Aut } X' \to \text{Aut } H_*(X', Z))) \subseteq \text{Ker}(\text{Aut } X \to \text{Aut } H_*(X, Z))$.

Suppose $h \in \text{Ker}(\text{Aut } X \to \text{Aut } H_*(X, Z))$ and $\eta(h) = h'$. The definition of $\eta$ and the fact that for every prime $p H_*(h, Z) \otimes Z_{(p)} = 1(Z_{(p)} = \text{the localization of } Z \text{ at } p)$ imply for every prime $p H_*(h', Z) \otimes Z_{(p)} = 1$. Hence it follows from Hilton-Mislin and Roitberg [5, I. 3.13] that $h' \in \text{Ker}(\text{Aut } X' \to \text{Aut } H^*(X', Z))$. The proof that $\eta'(\text{Ker}(\text{Aut } X' \to \text{Aut } H_*(X', Z))) \subseteq \text{Ker}(\text{Aut } X \to \text{Aut } H_*(X, Z))$ is similar.

Proposition 4.4. Let $X$ and $X'$ be as in 4.2. If $\text{Aut } X$ is finite then $\text{Aut } X$ is isomorphic to $\text{Aut } X'$.

Proof. Let $h \in \text{Aut } X$ and let $\varphi: X \to X_0$ be a rationalization. Since $X$ is an $H_0$-space and $\text{Aut } X$ is finite imply that $\text{Aut } X_0$ is abelian. For every prime $p$ and for every localization $\phi_p: X_p \to X_0$, $\phi_p(h_p)^p \sim (h_0)^p \varphi_p$. Hence the proof is similar to the proof of Proposition 4.1.

Notations. Let $X$ be an $H_0$-space with either a finite number of homology groups or a finite number of homotopy groups, and let $\varphi: X \to K(X)$ ($K(X)$ = $K(QH^*(X, Z)/\text{torsion})$) be a rational equivalence. Denote by:

(a) $X(p, \varphi)$ the space which satisfies: There exists a factorization of $\varphi$ $X \xrightarrow{\varphi(p)} X(p, \varphi) \xrightarrow{\varphi''(p)} K(X)$, where $\varphi'$ is a mod $- p$ equivalence and $\varphi''$ is a mod $P - p$ equivalence. (Such a space exists by [9, 4.3.1]).

(b) $N(X)$ — the least integer which satisfies: For every $n > N(Z)$ either $\pi_n X = 0$ or $H_n X = 0$ ($\pi_n$ — if $X$ has a finite number of homotopy groups, $H_n$ — if $X$ is finite dimensional).

(c) $t$ — the least integer divisible by

$$\prod_{n \leq N(X)} |\text{torsion}(H^*(X, Z))| \cdot |\text{torsion}(\pi_n(\text{fiber } \varphi))|.$$
Lemma 4.5. Let $X$ and $\varphi$ be as in the notations. Then the map $\text{Aut } X(p, \varphi) \to \text{Aut } X_p$ is monic.

Proof. Let $h \in \text{Ker}(\text{Aut } X(p, \varphi) \to \text{Aut } X_p)$. Since $\text{Ker}(\text{Aut } X(p, \varphi) \to \text{Aut } X_p)$ contains $\text{Ker}(\text{Aut } X(p, \varphi) \to \text{Aut } X_0)$ and $X(p, \varphi)$ is an $H_0$-space, $\varphi''(p)h \sim \varphi''(p)$, i.e., for every prime $p$ $h$ is mod-$p$ homotopic to the identity, hence $h$ is homotopic to the identity [5, II 5.3].

Proposition 4.6. Let $X$ be an $H_0$-space either with a finite number of homology groups or with a finite number of homotopy groups. If $H^*(X, \mathbb{Z})$ is torsion free, then $\text{Ker}(\text{Aut } X \to \text{Aut } X_q)(q \in P)$ is a direct product of finite $p$-groups, $p \neq q, p/t$.

Proof. Let $\varphi: X \to K(X)$ be a rational equivalence which represents generators of $H^*(X, \mathbb{Z})$. By Lemma 4.5 $\text{Ker}(\text{Aut } X \to \text{Aut } X_p)$ is isomorphic to $\text{Ker}(\text{Aut } X \to \text{Aut } X_0)$. Hence the fact that $X$ is the pullback of the maps $X(p, \varphi) \xrightarrow{\varphi''(p)} K(X)(p/t)$ [9, 4.7.2] and that $\text{Ker}(\text{Aut } X(p, \varphi) \to \text{Aut } K(X))$ is a finite $p$-group [10, 2.9] implies the result.

Proposition 4.7. Let $X, Y$ be nilpotent spaces with finite number of homology groups and let $f: X \to Y$ be a fibration. Then for every $f' \in \Gamma(f)(f': X' \to Y')$ and for every prime $p$, $\text{Ker}(\text{Aut } f \to \text{Aut } (f_p))$ is isomorphic to $\text{Ker}(\text{Aut } (f') \to \text{Aut } (f'_p))$.

Proof. Let $\varphi: X \to X_0$, $\psi: Y \to Y_0$, $\varphi': X' \to X_0$ and $\psi': Y' \to Y_0$ be rationalizations. Assume that $f_p$ is the localization of $f$ with respect to $\varphi$ and $\psi$ and that $f'_p$ is the localization of $f'$ with respect to $\varphi'$ and $\psi'$. Since $f'_p$ is homotopy equivalent to $f_p$ one can choose decompositions of $\varphi'$ and $\psi'$

$$
\xymatrix{
X' \ar[r]^{\varphi'_p} & X_p \ar[r] & X_0, \\
Y' \ar[r]_{\psi'_p} & Y_p \ar[r] & Y_0
}
$$

so that $f_p \varphi'_p \sim \psi'_p f'$ [6, 2.1.2]. Consequently, the considerations of the proof of Proposition 4.1 imply that for every pair $(h, k) \in \text{Aut } f$ there exists a unique pair $(h', k') \in \text{Aut } f'$, which satisfies $((h_p)_\varphi, (k_p)_\psi) = ((h'_p)_\varphi, (k'_p)_\psi)$ for every prime $p$ and therefore $\text{Ker}(\text{Aut } f \to \text{Aut } (f'_p))$ is isomorphic to $\text{Ker}(\text{Aut } (f') \to \text{Aut } (f'_p))$.

Corollary 4.8. Let $X, Y$ be as in Proposition 4.2 and $f: X \to Y$ be a fibration. Then for every $f' \in \Gamma(f)\text{Ker}(\text{Aut } (f') \to \text{Aut } H^*(f, Q))$ is isomorphic to $\text{Ker}(\text{Aut } (f') \to \text{Aut } H^*(f', Q))$. 


Corollary 4.9. Let $f$ and $f'$ be as in Corollary 4.8. Then
$$\text{Ker}(\text{Aut}(f) \to \text{Aut} H_*(f, Z))$$ is isomorphic to
$$\text{Ker}(\text{Aut}(f') \to \text{Aut} H_*(f', Z)).$$

Proof. Similar to the proof of Corollary 4.3.

Proposition 4.10. Let $f$ and $f'$ be as in Corollary 4.8. If $\text{Aut}(f)$ is finite, then $\text{Aut}(f)$ is isomorphic to $\text{Aut}(f').$

Proof. Let $\varphi: X \to X_0$ and $\psi: Y \to Y_0$ be rationalizations. Since $f_0$ is homotopy equivalent to $f'_0$ one can choose rationalization $\varphi': X' \to X_0$ and $\psi': Y' \to Y_0$ so that $f_0 \varphi' \sim \psi' f'$. Hence the result follows from the fact that $\text{Aut} X_0$ and $\text{Aut} Y_0$ are abelian groups. (The proof is similar to the proof of Proposition 4.7).

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