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DERIVATIONS OF OPERATOR ALGEBRAS INTO SPACES OF UNBOUNDED OPERATORS

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This paper is to study the spatiality of unbounded derivations in operator algebras. Let $\mathscr{M}$ be a von Neumann algebra (C^*-algebra) on a Hilbert space $\mathfrak{H}$ and δ be an unbounded derivation in $\mathscr{M}$. In this paper, extending δ to a derivation $\hat{\delta}$ of $\mathscr{M}$ into a certain space of unbounded operators, we study the spatiality of δ by investigating the property of $\hat{\delta}$.

1. **Introduction.** Unbounded derivations in operator algebras (C^*-algebras and von Neumann algebras) have recently been investigated by many authors, since they are appeared as infinitesimal generators of strongly continuous one-parameter groups of *-automorphisms on C^*-algebras [see; 12]. In particular, the infinitesimal generator mentioned above is implemented by a symmetric operator by giving some representation of its C^*-algebra on a Hilbert space, and there exist many closed derivations in C^*-algebras which possess such a property [2]. In this point of view, we shall study the spatiality of unbounded derivations in operator algebras (see [2]; Problem). Our method is, roughly speaking, to examine the spatiality of an unbounded derivation δ in an operator algebra $\mathscr{M}$ by extending δ to a derivation of $\mathscr{M}$ into some space of unbounded operators containing $\mathscr{M}$.

Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a *-derivation in $\mathscr{M}$ with σ-strongly dense domain $\mathscr{D}(\delta)$. Let $\mathscr{D}$ be a dense subspace of $\mathfrak{H}$. We introduce various locally convex topologies in the space $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ which is the set of all linear operators T of $\mathscr{D}$ into $\mathfrak{H}$ with $\mathscr{D}(T^*) \supset \mathscr{D}$, and extend δ to a $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ assuming corresponding continuity of δ in these topologies.

We shall then examine under what conditions the continuous $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ with some specified topology is spatial, i.e., there exists an element H of $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ such that $\hat{\delta}(A)\xi = [H, A]\xi = \{HA - AH\}\xi$ for all $A \in \mathscr{M}$ and $\xi \in \mathscr{D}$. We call the dense subspace $\mathscr{D}$ countably dominated by a sequence $\{T_n\}$ of closed operators if $\mathscr{D} = \bigcap_{n=1}^{\infty} \mathscr{D}(T_n)$ and $\|T_n\xi\| \leqq \|T_{n+1}\xi\|$ for each $\xi \in \mathscr{D}$ and $n = 1, 2, \cdots$.

Our first result (Theorem 4.11) shows that if $\mathscr{M}$ is a left von Neumann algebra of a Hilbert algebra $\mathfrak{A}$ with identity and $\mathscr{D}$ is countably dominated by $\{T_n\}$ of closed operators then $\hat{\delta}$ is spatial.

The second purpose of this paper is to show (Theorem 4.15) that if $\mathscr{M}$ has certain property (Definition 4.2) and $\mathscr{D}$ is countably dominated by $\{T_n\}$ of closed operators $\eta\mathscr{M}'$ then $\hat{\delta}$ is a spatial $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$.

2. Spaces of unbounded operators. Let $\mathfrak{H}$ be a Hilbert space with inner product $(\,|\,)$ and let $\mathscr{D}$ be a dense subspace of $\mathfrak{H}$. We denote by $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ (resp. $\mathscr{L}_c(\mathscr{D}, \mathfrak{H})$) the space of all (resp. closable) linear operators of $\mathscr{D}$ into $\mathfrak{H}$ and by $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ the space of operators A in $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ for which there exists the adjoints A^* whose domains $\mathscr{D}(A^*)$ contain $\mathscr{D}$. For each $T \in \mathscr{L}(\mathscr{D}, \mathfrak{H})$ we define

$$\|A\|_T = \sup_{\xi \in \mathscr{D}} \frac{\|A\xi\|}{\|T\xi\|}, \quad A \in \mathscr{L}(\mathscr{D}, \mathfrak{H}),$$

where $(\lambda/0) = \infty$ for $\lambda > 0$ and $(0/0) = 0$,

$$\mathfrak{M}_T = \{A \in \mathscr{L}(\mathscr{D}, \mathfrak{H});\ \|A\|_T < \infty\}$$

and

$$\mathfrak{M}_T^\# = \{A \in \mathscr{L}^\#(\mathscr{D}, \mathfrak{H});\ \|A\|_T < \infty\}.$$

Then it is easily seen that $\mathfrak{M}_T$ is a Banach space equipped with the norm $\|\cdot\|_T$ and $\mathfrak{M}_T^\#$ is a subspace of $\mathfrak{M}_T$.

The following lemma is an immediate consequence of the definitions of the spaces of $\mathfrak{M}_T$ and $\mathfrak{M}_T^\#$.

LEMMA 2.1. *Let T be an element of $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ such that $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$, where $\mathscr{B}(\mathfrak{H})$ denotes the algebra of all bounded linear operators on $\mathfrak{H}$. We set*

$$\mathscr{B}_T = \{\overline{AT^{-1}};\ A \in \mathfrak{M}_T\} \quad \textit{and} \quad \mathscr{B}_T^\# = \{\overline{AT^{-1}};\ A \in \mathfrak{M}_T^\#\}.$$

Then the map $\phi\colon A \to \overline{AT^{-1}}$ is an isometric isomorphism of the Banach space $\mathfrak{M}_T$ onto the Banach space $\mathscr{B}(\mathfrak{H})$.

LEMMA 2.2. *Let $\mathfrak{H}$ be a Hilbert space with inner product $(\,|\,)$. If there exists a sequence $\{T_n\}$ of closed operators on $\mathfrak{H}$ such that*

(1) $\mathscr{D} = \bigcap_{n=1}^\infty \mathscr{D}(T_n)$ *is dense in* $\mathfrak{H}$;

(2) $\|T_n\xi\| \leqq \|T_{n+1}\xi\|$ *for all* $\xi \in \mathscr{D}$ *and* $n = 1, 2, \cdots$, *then* $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H}) = \bigcup_{n=0}^\infty \mathfrak{M}_{T_n}^\#$ *where* $T_0 = I$.

Proof. For each $\xi \in \mathscr{D}$ we set

$$\|\xi\|_{T_n} = \|T_n\xi\| \quad \text{for} \quad n = 0, 1, 2 \cdots.$$

We consider the locally convex topology $t_{\{T_n\}}$ on $\mathscr{D}$ generated by family of the seminorms $\|\cdot\|_{T_n}(n = 0, 1, 2, \cdots)$. Suppose that $\{\xi_k\}$ is a Cauchy sequence in $(\mathscr{D}, t_{\{T_n\}})$. Then we have

$$\lim_{k\, l\to\infty}\|\xi_k - \xi_l\| = 0 \quad \text{and} \quad \lim_{k.l\to\infty}\|T_n\xi_k - T_n\xi_l\| = 0 \qquad \text{for} \quad n = 1, 2, \cdots .$$

Since T_n is a closed operator, it follows that $x \in \mathscr{D}(T_n)$ and $\lim_{k\to\infty} T_n\xi_k = T_n x$ for $n = 1, 2, \cdots$. Hence we have $x \in \bigcap_{n=1}^{\infty}\mathscr{D}(T_n) = \mathscr{D}$ and $\lim_{k\to\infty} T_n\xi_k = T_n x$ for $n = 1, 2, \cdots$. This implies that $(\mathscr{D}, t_{\{T_n\}})$ is a Fréchet space.

Suppose $S \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$. We show that the graph of S: $G(S) \equiv \{\langle\xi, S\xi\rangle; \xi \in \mathscr{D}\}$ is closed in $(\mathscr{D}, t_{\{T_n\}}) \times \mathfrak{H}$. Suppose that a sequence $\{\langle\xi_n, S\xi_n\rangle\}$ in $G(S)$ converges to an element $\langle\xi, y\rangle$ of $\mathscr{D} \times \mathfrak{H}$. It then follows that $\xi_n - \xi \in \mathscr{D}$, $\lim_{n\to\infty}\|\xi_n - \xi\| = 0$ and $\lim_{n\to\infty}\|S(\xi_n - \xi) - (y - S\xi)\| = 0$. Since S is closable, we have $y = S\xi$. This implies that $G(S)$ is closed in $(\mathscr{D}, t_{\{T_n\}}) \times \mathfrak{H}$. By the closed graph theorem it follows that the map $S: (\mathscr{D}, t_{\{T_n\}}) \to \mathfrak{H}$ is continuous. Hence there exist a number n and a constant $\gamma > 0$ such that

$$\|S\xi\| \leqq \gamma\|T_n\xi\| \quad \text{for all} \quad \xi \in \mathscr{D} .$$

Therefore, $S \in \mathfrak{M}_{T_n}^{\#}$. This implies that $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H}) = \bigcup_{n=0}^{\infty}\mathfrak{M}_{T_n}^{\#}$.

DEFINITION 2.3. Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathfrak{H}$. If there exists a sequence $\{T_n\}$ of closed operators in $\mathfrak{H}$ such that $\mathscr{D} = \bigcap_{n=1}^{\infty}\mathscr{D}(T_n)$ and $\|T_n\xi\| \leqq \|T_{n+1}\xi\|$ for all $\xi \in \mathscr{D}$ and $n = 1, 2, \cdots$, then $\mathscr{D}$ is said to be countably dominated by $\{T_n\}$. If there exists a sequence $\{S_n\}$ in $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ such that $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H}) = \bigcup_{n=1}^{\infty}\mathfrak{M}_{S_n}^{\#}$ and $\|S_n\xi\| \leqq \|S_{n+1}\xi\|$ for all $\xi \in \mathscr{D}$ and $n = 1, 2, \cdots$, then $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ is said to be countably dominated by $\{S_n\}$.

REMARK. (1) Lemma 2.2 implies that if a pre-Hilbert space $\mathscr{D}$ is countably dominated then $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ is also countably dominated.

(2) It will be seen, by a simple calculation, that if $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H}) = \bigcup_{n=1}^{\infty}\mathfrak{M}_{S_n}^{\#}$ for $S_n \in \mathscr{L}^{\#}(\mathscr{D}) \equiv \mathscr{L}^{\#}(\mathscr{D}, \mathscr{D})(n = 1, 2, \cdots)$, then $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ is countably dominated.

Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathfrak{H}$. We now introduce some locally convex topologies on $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$. We put

$$P_{\xi,x}(A) = |(A\xi|x)| ,$$
$$P_{\xi}(A) = \|A\xi\| ,$$

where $A \in \mathscr{L}(\mathscr{D}, \mathfrak{H})$, $\xi \in \mathscr{D}$ and $x \in \mathfrak{H}$. The locally convex topology on $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ generated by the seminorms $\{P_{\xi,\eta}(\cdot); \xi, \eta \in \mathscr{D}\}$ (resp.

$\{P_{\xi,x}(\cdot);\ \xi \in \mathscr{D},\ x \in \mathfrak{H}\}$, $\{P_\xi(\cdot);\ \xi \in \mathscr{D}\}$) is said to be the weak topology (resp. quasi-weak topology, strong topology) and is simply denoted by $t_w^{\mathscr{D}}$(resp. $t_{qw}^{\mathscr{D}}$, $t_s^{\mathscr{D}}$).

Let $\mathfrak{H}_\infty$ be the Hilbert direct sum of the Hilbert spaces $\mathfrak{H}_n \equiv \mathfrak{H}(n = 1, 2, \cdots)$ and let

$$\mathscr{D}_\infty(\mathscr{D}) = \{\{\xi_n\} \in \mathfrak{H}_\infty;\ \xi_n \in \mathscr{D} \quad \text{for} \quad n = 1, 2, \cdots \quad \text{and} \quad \sum_{n=1}^{\infty} \| A\xi_n \|^2 < \infty \quad \text{for all} \quad A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})\} .$$

We set

$$P_{\{\xi_n\},\{x_n\}}(A) = \left| \sum_{n=1}^{\infty} (A\xi_n | x_n) \right| ,$$

$$P_{\{\xi_n\}}(A) = \left[\sum_{n=1}^{\infty} \| A\xi_n \|^2 \right]^{1/2} ,$$

where $A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$, $\{\xi_n\} \in \mathscr{D}_\infty(\mathscr{D})$ and $\{x_n\} \in \mathscr{D}_\infty$. We equip $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ with the locally convex topology $t_{\sigma w}^{\mathscr{D}}$(resp. $t_{q\sigma w}^{\mathscr{D}}$, $t_{\sigma s}^{\mathscr{D}}$) induced by the seminorms $\{P_{\{\xi_n\},\{\eta_n\}}(\cdot);\ \{\xi_n\}, \{\eta_n\} \in \mathscr{D}_\infty(\mathscr{D})\}$ (resp. $\{P_{\{\xi_n\},\{x_n\}}(\cdot);\ \{\xi_n\} \in \mathscr{D}_\infty(\mathscr{D}),\ \{x_n\} \in \mathfrak{H}_\infty\}$, $\{P_{\{\xi_n\}}(\cdot);\ \{\xi_n\} \in \mathscr{D}_\infty(\mathscr{D})\}$). The topology $t_{\sigma w}^{\mathscr{D}}$ (resp. $t_{q\sigma w}^{\mathscr{D}}$, $t_{\sigma s}^{\mathscr{D}}$) is said to be the σ-weak topology (resp. quasi-σ-weak topology, σ-strong topology) on $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$.

We next define the uniform topology and the quasi-uniform topology. A subset $\mathfrak{M}$ of $\mathscr{D}$ is said to be $\mathscr{D}$-bounded if

$$\sup_{\xi \in \mathfrak{M}} \| A\xi \| < \infty \quad \text{for each} \quad A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H}) .$$

We then define

$$P_{\mathfrak{M}}(A) = \sup_{\xi\ \eta \in \mathfrak{M}} |(A\xi | \eta)| ,$$

$$P^{\mathfrak{M}}(A) = \sup_{\xi \in \mathfrak{M}} \| A\xi \| ,$$

where $\mathfrak{M}$ is $\mathscr{D}$-bounded and $A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$. The locally convex topology generated by the seminorms $\{P_{\mathfrak{M}}(\cdot);\ \mathfrak{M}$ is $\mathscr{D}$-bounded$\}$ (resp. $\{P^{\mathfrak{M}}(\cdot);\ \mathfrak{M}$ is $\mathscr{D}$-bounded$\}$) is said to be the uniform topology (resp. quasi-uniform topology) on $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ and is simply denoted by $t_u^{\mathscr{D}}$ (resp. $t_{qu}^{\mathscr{D}}$).

We next define the ρ-topology and λ-topology on $\mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$. For each $T \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H})$ we put

$$\rho_T(A) = \sup_{\xi \in \mathscr{D}} \frac{|(A\xi | \xi)|}{\| T\xi \|^2} , \quad A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H}) ,$$

where $(\lambda/0) = \infty$ for $\lambda > 0$ and $0/0 = 0$, and

$$\mathfrak{N}_T^{\#} = \{A \in \mathscr{L}^{\#}(\mathscr{D}, \mathfrak{H});\ \rho_T(A) < \infty\} .$$

Then it is easily seen that $\mathfrak{N}_T^\#$ is a normed space equipped with the norm $\rho_T(\cdot)$ and $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H}) = \bigcup_{T\in\mathscr{L}^\#(\mathscr{D},\mathfrak{H})}\mathfrak{N}_T^\#$. The inductive limit topology on $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ with respect to the normed spaces $\{(\mathfrak{N}_T^\#, \rho_T(\cdot));\ T\in\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})\}$ (resp. $\{(\mathfrak{M}_T^\#, \|\cdot\|_T);\ T\in\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})\}$) is said to be the ρ-topology (resp. λ-topology) on $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ and is denoted by $t_\rho^{\mathscr{D}}$ (resp. $t_\lambda^{\mathscr{D}}$).

Now one may easily see the following lemma by the definitions of the topologies.

LEMMA 2.4. *The relation among the topologies introduced here are as follows*:

$$t_\lambda^{\mathscr{D}} \geqq t_\rho^{\mathscr{D}} \geqq \left\{\begin{matrix} t_u^{\mathscr{D}} & \leqq & & t_{qu}^{\mathscr{D}} \\ \text{\rotatebox{90}{$\leqq$}} & & & \text{\rotatebox{90}{$\leqq$}} \\ t_w^{\mathscr{D}} & \leqq t_{qw}^{\mathscr{D}} & \leqq & t_s^{\mathscr{D}} \\ \text{\rotatebox{90}{$\geqq$}} & & & \text{\rotatebox{90}{$\geqq$}} \\ t_{\sigma w}^{\mathscr{D}} & \leqq t_{q\sigma w}^{\mathscr{D}} & \leqq & t_{\sigma s}^{\mathscr{D}} \end{matrix}\right\} \leqq t_\lambda^{\mathscr{D}}\,,$$

where the symbols $\tau_1 \leqq \tau_2$, $\tau_2 \geqq \tau_1$, $\begin{matrix}\tau_1\\ \text{\rotatebox{90}{$\geqq$}}\\ \tau_2\end{matrix}$ *and* $\begin{matrix}\tau_2\\ \text{\rotatebox{90}{$\leqq$}}\\ \tau_1\end{matrix}$ *mean the topology* τ_2 *is finer than the topology* τ_1.

REMARK. The topologies $t_u^{\mathscr{D}}$ and $t_{qu}^{\mathscr{D}}$ (resp. the topologies $t_\rho^{\mathscr{D}}$ and $t_\lambda^{\mathscr{D}}$) on $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ are generalizations of the uniform topology and quasi-uniform one (resp. the ρ-topology and λ-topology) introduced by G. Lassner [8] (resp. D. Arnal and J. P. Jurzak [1]), for an unbounded operator algebra respectively. We denote by t_u (resp. t_w, t_s, $t_{\sigma w}$, $t_{\sigma s}$) the usual uniform (resp. weak, strong, σ-weak, σ-strong) topology on $\mathscr{B}(\mathfrak{H})$. The relations between the topologies on $\mathscr{B}(\mathfrak{H})$ are as follows: $t_u^{\mathfrak{H}} = t_{qu}^{\mathfrak{H}} = t_\rho^{\mathfrak{H}} = t_\lambda^{\mathfrak{H}} = t_u$, $t_w^{\mathfrak{H}} = t_{qw}^{\mathfrak{H}} = t_w$, $t_s^{\mathfrak{H}} = t_s$, $t_{\sigma w}^{\mathfrak{H}} = t_{q\sigma w}^{\mathfrak{H}} = t_{\sigma w}$ and $t_{\sigma s}^{\mathfrak{H}} = t_{\sigma s}$.

LEMMA 2.5. *Suppose that* $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ *is countably dominated by* $\{T_n\}$ *and* $\mathfrak{N}$ *is a subset of* $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$. *Then the following statements are equivalent*:

(1) $\mathfrak{N}$ *is* $t_\rho^{\mathscr{D}}$*-bounded*;

(2) $\mathfrak{N}$ *is* $t_u^{\mathscr{D}}$*-bounded*;

(3) *there exist a number* n *and a constant* $\gamma > 0$ *such that*

$$|(A\xi\,|\,\xi)| \leqq \gamma\|(I + |\overline{T_n}|)\xi\| \quad \textit{for all} \quad A\in\mathfrak{N} \quad \textit{and} \quad \xi\in\mathscr{D}\,,$$

where $\overline{T_n} = U|\overline{T_n}|$ *is the polar decomposition of* $\overline{T_n}$.

Proof. This is proved in the same way as in ([13] Lemma 2.1).

LEMMA 2.6. *Suppose that* $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{S})$ *is countably dominated by* $\{T_n\}$ *and* $\mathfrak{N}$ *is a subset of* $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{S})$. *Then the following statements are equivalent*:

(1) $\mathfrak{N}$ *is* $t_\lambda^{\mathscr{D}}$*-bounded*;

(2) $\mathfrak{N}$ *is* $t_{qu}^{\mathscr{D}}$*-bounded*;

(3) $\mathfrak{N}$ *is* $t_{\sigma s}^{\mathscr{D}}$*-bounded*;

(4) *there exists a number* n *and a constant* $\gamma > 0$ *such that*

$$\|A\xi\| \leqq \gamma \|(I + |\overline{T}_n|)\xi\| \quad \textit{for all} \quad A \in \mathfrak{N} \quad \textit{and} \quad \xi \in \mathscr{D}\ .$$

Furthermore, if $\mathscr{D} = \bigcap_{T \in \mathscr{L}^\dagger(\mathscr{D}, \mathfrak{S})} \mathscr{D}(\bar{T})$, *then the statements* (1)~(4) *are equivalent to the following statements* (5) *and* (6):

(5) $\mathfrak{N}$ *is* $t_s^{\mathscr{D}}$*-bounded*;

(6) $\mathfrak{N}$ *is* $t_{qw}^{\mathscr{D}}$*-bounded.*

Proof. Since $t_\lambda^{\mathscr{D}} \geqq t_{qu}^{\mathscr{D}}$ and $t_\lambda^{\mathscr{D}} \geqq t_{\sigma s}^{\mathscr{D}}$, one can see the implications (4) ⇒ (1), (1) ⇒ (2) and (1) ⇒ (3). We show the implication (3) ⇒ (4). Suppose that the statement (4) is not true. Then there exists a sequence $\{A_n\}$ in $\mathfrak{N}$ and a sequence $\{\xi_n\}$ of nonzero elements of $\mathscr{D}$ such that

$$\|A_n\xi_n\| \geqq n^2 \|(I + |\overline{T}_n|)\xi_n\| \quad \text{for} \quad n = 1, 2, \cdots .$$

Putting

$$\eta_n = \frac{\xi_n}{n\|(I + |\bar{T}_n|)\xi_n\|} \quad \text{for} \quad n = 1, 2, \cdots ,$$

we have

$$\|A_n\eta_n\| \geqq n \quad \text{and} \quad \|T_n\eta_n\| < \frac{1}{n}\ .$$

We now show $\{\eta_n\} \in \mathscr{D}_\infty(\mathscr{D})$. Since $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{S}) = \bigcup_{n=1}^\infty \mathfrak{M}_{T_n}^\dagger$, it follows that for each $A \in \mathscr{L}^\dagger(\mathscr{D}, \mathfrak{S})$ there exists a number k and a constant $\gamma > 0$ such that

$$\|A\xi\| \leqq \gamma \|T_k\xi\| \quad \text{for all} \quad \xi \in \mathscr{D}\ .$$

Then we have

$$\begin{aligned}
\sum_{n=1}^\infty \|A\eta_n\|^2 &\leqq \gamma \sum_{n=1}^\infty \|T_k\eta_n\|^2 \\
&\leqq \gamma \left\{\sum_{n=1}^{k-1} \|T_k\eta_n\|^2 + \|T_k\eta_k\|^2 + \|T_k\eta_{k+1}\|^2 + \cdots\right\} \\
&\leqq \gamma \left\{\sum_{n=1}^{k-1} \|T_k\eta_n\|^2 + \|T_k\eta_k\|^2 + \|T_{k+1}\eta_{k+1}\|^2 + \cdots\right\}
\end{aligned}$$

$$\leqq \gamma \left\{ \sum_{n=1}^{k-1} \| T_k \eta_n \|^2 + \frac{1}{k^2} + \frac{1}{(k+1)^2} + \cdots \right\}$$
$$< \infty .$$

This means $\{\eta_n\} \in \mathscr{D}_\infty(\mathscr{D})$. Furthermore, we have

$$\sup_{A \in \mathfrak{N}} P_{\{\eta_n\}}(A) = \sup_{A \in \mathfrak{N}} \left[\sum_{n=1}^{\infty} \| A\eta_n \|^2 \right]^{1/2}$$
$$\geqq \| A_n \eta_n \| \geqq n .$$

This contradicts that $\mathfrak{N}$ is $t_{\sigma s}^{\mathscr{D}}$-bounded. This comqletes the proof of the implication (3) $\Rightarrow$ (4).

The implication (2) $\Rightarrow$ (4) is proved in the same way as in ([13] Lemma 2.2).

If $\mathscr{D} = \bigcap_{T \in \mathscr{L}^\dagger(\mathscr{D},\mathfrak{H})} \mathscr{D}(\bar{T})$, the equivalence of the statements (1)$\sim$(6) follows from ([1] Proposition 1.6).

3. Extension of derivations. Let $\mathscr{M}$ be a C^*-algebra (or a von Neumann algebra). A linear map δ: $\mathscr{D}(\delta) \subset \mathscr{M} \to \mathscr{M}$ is said to be a $*$-*derivation in* $\mathscr{M}$ *if it satisfies the following conditions:*

(1) the domain $\mathscr{D}(\delta)$ of δ is a dense $*$-subalgebra of $\mathscr{M}$ (i.e., $\mathscr{D}(\delta)$ is norm-dense if $\mathscr{M}$ is a C^*-algebra, and weak-dense if $\mathscr{M}$ is a von Neumann algebra);

(2) $\delta(AB) = \delta(A)B + A\delta(B)$ for each $A, B \in \mathscr{D}(\delta)$;

(3) $\delta(A^*) = \delta(A)^*$ for each $A \in \mathscr{D}(\delta)$.

We begin with the following lemma.

LEMMA 3.1. *Let $\mathscr{M}$ be a unital C^*-algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation in $\mathscr{M}$ with domain $\mathscr{D}(\delta)$. If there exists a dense subspace $\mathscr{D}$ of $\mathfrak{H}$ such that $\mathscr{M}\mathscr{D} \subset \mathscr{D}$ and δ is a continuous map of $(\mathscr{D}(\delta), t_u)$ into $(\mathscr{M}, t_{qu}^{\mathscr{D}})$, then δ is extended to a continuous linear map $\hat{\delta}$ of $(\mathscr{M}, t_u)$ into $(\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H}), t_{qu}^{\mathscr{D}})$ such that*

(1) $\hat{\delta}(AB)\xi = \hat{\delta}(A)B\xi + A\hat{\delta}(B)\xi$;

(2) $\hat{\delta}(A)^*\xi = \hat{\delta}(A^*)\xi$;

(3) $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$

for each $A, B \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$. Namely, the following diagram holds:

$$\begin{array}{ccc} \hat{\delta};\ (\mathscr{M}, t_u) & \xrightarrow{\textit{continuous}} & (\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H}), t_{qu}^{\mathscr{D}}) \\ \uparrow \quad \cup & & \cup \\ \delta;\ (\mathscr{D}(\delta), t_u) & \xrightarrow[\textit{continuous}]{} & (\mathscr{M}, t_{qu}^{\mathscr{D}}) . \end{array}$$

By Lemma 3.1 we define a derivation of a C^*-algebra into a space of unbounded operators as follows:

DEFINITION 3.2. Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathfrak{H}$ and let $\mathscr{M}$ be a unital C^*-algebra acting on $\mathfrak{H}$ with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. A linear map δ of $\mathscr{M}$ into $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ is said to be a derivation of $\mathscr{M}$ into $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ if

$$\delta(AB)\xi = \delta(A)B\xi + A\delta(B)\xi \quad \text{for each} \quad A, B \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D} .$$

In particular, a derivation δ is said to be a *-derivation if the range of δ is contained in $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ and

$$\delta(A)^*\xi = \delta(A^*)\xi \quad \text{for each} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D} .$$

If a derivation δ of $\mathscr{M}$ into $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ is a continuous map of $(\mathscr{M}, \tau_1)$ into $(\mathscr{L}(\mathscr{D}, \mathfrak{H}), \tau_2)$, where τ_1 and τ_2 are topologies on $\mathscr{M}$ and $\mathscr{L}(\mathscr{D}, \mathfrak{H})$ respectively, then it is said to be $(\tau_1 \to \tau_2)$-continuous.

We also have the following result:

LEMMA 3.3. *Let $\mathscr{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation in $\mathscr{M}$. If δ is $(t_w \to t_{qw}^{\mathscr{D}})$-continuous* (resp. $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$*-continuous), then δ is extended to a $(t_w \to t_{qw}^{\mathscr{D}})$-continuous* (resp. $(t_s \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$*-continuous) $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ satisfying $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$ for each $A \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$.*

DEFINITION 3.4. Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation of a C^*-algebra $\mathscr{M}$ on $\mathfrak{H}$ into $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$. If $\delta(\mathscr{M}) \subset \mathfrak{M}_T^\dagger$ for some $T \in \mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$, then δ is said to be a $*$-derivation of $\mathscr{M}$ into $\mathfrak{M}_T^\dagger$. If there exists an element T of $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ such that $\delta(\mathscr{M}_u)$ is a bounded subspace of the normed space $\mathfrak{M}_T^\dagger$, where $\mathscr{M}_u$ is the set of all unitary operators in $\mathscr{M}$, then δ is said to be quasi-bounded.

LEMMA 3.5. *Let $\mathscr{M}$ be a unital C^*-algebra acting on a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation in $\mathscr{M}$. If there exist a dense subspace $\mathscr{D}$ of $\mathfrak{H}$ and an element T of $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ such that $\mathscr{M}\mathscr{D} \subset \mathscr{D}$ and $\|\delta(A)\|_T \leqq \|A\|$ for all $A \in \mathscr{D}(\delta)$, then δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathfrak{M}_T^\dagger$ satisfying $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$ for each $A \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$.*

We now give some examples of quasi-bounded $*$-derivations.

EXAMPLE 3.6. Let δ be a spatial derivation in a C^*-algebra $\mathscr{M}$ acting on a Hilbert space $\mathfrak{H}$ with domain $\mathscr{D}(\delta)$, i.e., there exists a symmetric operator H on $\mathfrak{H}$ such that $\mathscr{D}(\delta)\mathscr{D}(H) \subset \mathscr{D}(H)$ and $\delta(A)\xi = i[H, A]\xi$ for each $A \in \mathscr{D}(\delta)$ and $\xi \in \mathscr{D}(H)$. If there exists a closed operator $T\eta\mathscr{M}'$ and a constant $\gamma > 0$ such that $\|H\xi\| \leqq \gamma\|T\xi\|$ for all $\xi \in \mathscr{D}(T)$, then δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}(T), \mathfrak{H})$.

2. Let $\mathscr{M}_i$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}_i$ and let δ_i be a bounded $*$-derivation on $\mathscr{M}_i(i = 1, 2, \cdots)$. Let $\mathscr{M}$ be a direct sum of the von Neumann algebras $\mathscr{M}_i$ and let $\mathfrak{H}$ be the direct sum of the Hilbert spaces $\mathfrak{H}_i$. We define

$$\mathscr{D}(\delta) = \left\{A = (A_i) \in \prod_i \mathscr{M}_i;\ A_i \neq 0 \quad \text{for only finite coordinates}\right\},$$
$$\delta(A) = (\delta_i(A_i)), \quad A = (A_i) \in \mathscr{D}(\delta).$$

Then δ is a $*$-derivation in $\mathscr{M}$ with the weakly dense domain $\mathscr{D}(\delta)$, but it is not generally bounded. However, δ is $(t_w \to t_{qw}^{\mathscr{D}})$-continuous (and $(t_s \to t_s^{\mathscr{D}})$, $(t_u \to t_u^{\mathscr{D}})$, $(t_u \to t_{qu}^{\mathscr{D}})$, $(t_u \to t_\lambda^{\mathscr{D}})$, $(t_u \to t_\rho^{\mathscr{D}})$-continuous), where

$$\mathscr{D} = \{(\xi_i) \in \mathfrak{H};\ \xi_i \neq 0 \quad \text{for only finite coordinates}\}.$$

Putting

$$T = (\|\delta_i\| I_i)$$

where $\|\delta_i\|$ is the norm of δ_i and I_i is the identity operator on $\mathfrak{H}_i$, we have

$$\|\delta(A)\xi\| \leqq \|A\|\,\|T\xi\| \quad \text{for each} \quad A \in \mathscr{D}(\delta) \quad \text{and} \quad \xi \in \mathscr{D}.$$

Hence, δ is extended to a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathfrak{M}_T^\#$.

3. Let δ be a $(t_u \to t_w^{\mathscr{D}})$-continuous $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D})(\equiv \mathscr{L}^\#(\mathscr{D}, \mathscr{D}))$. If $\delta(\mathscr{M})$ is a finite dimensional subspace of $\mathscr{L}^\#(\mathscr{D})$, then δ is a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$.

4. Let δ be a $*$-derivation in a C^*-algebra $\mathscr{M}$ acting on a Hilbert space $\mathfrak{H}$. If there exists a densely defined closed operator T on $\mathfrak{H}$ such that $\mathscr{M}\mathscr{D}(T) \subset \mathscr{D}(T)$ and δ is $(t_u \to t_{qu}^{\mathscr{D}(T)})$-continuous (or $(t_u \to t_\lambda^{\mathscr{D}(T)})$-continuous), then δ is extended to a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}(T), \mathfrak{H})$. This follows immediately from Lemma 2.2.

As a slight generalization of Example 3.6, 4 we have the following result:

LEMMA 3.7. *Let $\mathscr{D}$ be a countably dominated subspace in a Hilbert space $\mathfrak{H}$ by a sequence $\{T_n\}$ of closed operators on $\mathfrak{H}$. If δ is a $(t_u \to t_{qu}^{\mathscr{D}})$-continuous (or $(t_u \to t_\lambda^{\mathscr{D}})$, $(t_w \to t_{qw}^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$-continuous) $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$, then δ is quasi-bounded.*

Proof. Suppose that δ is $(t_u \to t_{qu}^{\mathscr{D}})$-continuous. By the continuity of δ, $\delta(\mathscr{M}_1)$ is a bounded subset of $(\mathscr{L}^\#(\mathscr{D}, \mathfrak{H}), t_{qu}^{\mathscr{D}})$, where $\mathscr{M}_1$ is the unit ball of $\mathscr{M}$. It then follows from Lemma 2.4 that $\delta(\mathscr{M}_1)$ is a bounded subset of the normed space $\mathfrak{M}_{I+|T_n|}^\#$ for some n. This implies that δ is quasi-bounded.

4. The spatiality of quasi-bounded $*$-derivations.

Throughout this section we may assume that $\mathscr{D}$ is a dense subspace of a Hilbert space $\mathfrak{H}$ and $\mathscr{M}$ is a unital C^*-algebra with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. Let δ be a quasi-bounded $*$-derivation of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$, i.e., there exists an element T of $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ such that $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$ and $\delta(\mathscr{M}_u)$ is a bounded subset of the normed space $\mathfrak{M}_T^\#$.

LEMMA 4.1. *Suppose that $\mathfrak{M}$ is a subspace of $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. Then the following statements are equivalent:*

(1) *f is a $t_{qw}^{\mathscr{D}}$-continuous linear functional on $\mathfrak{M}$;*

(2) *f is a $t_s^{\mathscr{D}}$-continuous linear functional on $\mathfrak{M}$;*

(3) *$f = \sum_{i=1}^n \omega_{\xi_i, x_i}$ for $\xi_i \in \mathscr{D}$ and $x_i \in \mathfrak{H}$, where $\omega_{\xi,x}(A) = (A\xi | x)$ for $A \in \mathscr{L}(\mathscr{D}, \mathfrak{H})$, $\xi \in \mathscr{D}$ and $x \in \mathfrak{H}$.*

Proof. This is proved in the same way as in ([1] Theorem 1.3).

Let $T \in \mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$ and $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$. Then, by Lemma 2.1 $\mathscr{B}_T^\# \equiv \{\overline{AT^{-1}};\ A \in \mathfrak{M}_T^\#\}$ is a subspace of $\mathscr{B}(\mathfrak{H})$. We denote by $\widetilde{\mathscr{B}}_T^\#$ the t_w-closure of $\mathscr{B}_T^\#$ and denote by $\widetilde{\mathfrak{M}}_T^\#$ the $t_{qw}^{\mathscr{D}}$-closure of $\mathfrak{M}_T^\#$ in $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. Then $\widetilde{\mathscr{B}}_T^\#$ is a weakly closed subspace of $\mathscr{B}(\mathfrak{H})$ and $\widetilde{\mathfrak{M}}_T^\#$ is $t_{qw}^{\mathscr{D}}$-closed subspace of $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. Furthermore, the following lemma is seen by a simple calculation.

LEMMA 4.2. *Let ϕ be the isomorphism of $\mathfrak{M}_T^\#$ onto $\mathscr{B}_T^\#$ in Lemma 2.1. Then ϕ^{-1} is a continuous map of $(\mathscr{B}_T^\#, t_w)$ onto $(\mathfrak{M}_T^\#, t_{qw}^{\mathscr{D}})$, so that it is extended to a continuous linear map $\tilde{\phi}^{-1}$ of $(\widetilde{\mathscr{B}}_T^\#, t_w)$ onto $(\widetilde{\mathfrak{M}}_T^\#, t_{qw}^{\mathscr{D}})$.*

LEMMA 4.3. *Let $\mathfrak{K}$ be a subset of $\mathfrak{M}_T^\#$ and let $\mathfrak{Q}$ be the $t_{qw}^{\mathscr{D}}$-closed convex hull of $\mathfrak{K}$ in $\mathscr{L}(\mathscr{D}, \mathfrak{H})$. If $\mathfrak{K}$ and $\mathfrak{K}^\# \equiv \{A^\# = A^*/\mathscr{D};\ A \in \mathfrak{K}\}$ are bounded in $\mathfrak{M}_T^\#$, where $A^*/\mathscr{D}$ is the restriction of A^* to $\mathscr{D}$, then $\mathfrak{Q}$ is a $t_{qw}^{\mathscr{D}}$-compact subset of $\mathfrak{M}_T^\#$.*

Proof. Let $\mathfrak{K}'$ be the convex hull of $\mathfrak{K}$. Then $\mathfrak{K}'$ and $(\mathfrak{K}')^\#$ are bounded in $\mathfrak{M}_T^\#$. Hence we may assume that $\mathfrak{K}$ is convex. We first show that $\mathfrak{Q}$ is a bounded subset of the normed space $\mathfrak{M}_T^\#$. By the boundedness of $\mathfrak{K}$ and $\mathfrak{K}^\#$ there exists a constant $\gamma > 0$ such that $\|A\|_T \leqq \gamma$ and $\|A^\#\|_T \leqq \gamma$ for all $A \in \mathfrak{K}$. For each $S \in \mathfrak{Q}$ there is a net $\{A_\alpha\}$ in $\mathfrak{K}$ which converges to S with respect to the topology $t_{qw}^{\mathscr{D}}$. It then follows that for each $\xi \in \mathscr{D}$ and $x \in \mathfrak{G}$

$$\begin{aligned} |(S\xi|x)| &= \lim_\alpha |(A_\alpha \xi|x)| \\ &\leqq \overline{\lim_\alpha} \|A_\alpha \xi\| \|x\| \\ &\leqq \gamma \|T\xi\| \|x\| , \end{aligned}$$

so that $\|S\|_T \leqq \gamma$. Furthermore, for each $\xi, \eta \in \mathscr{D}$ we have

$$\begin{aligned} |(S\xi|\eta)| &= \lim_\alpha |(A_\alpha \xi|\eta)| \\ &\leqq \overline{\lim_\alpha} \|A_\alpha^* \eta\| \|\xi\| \\ &\leqq \gamma \|T\eta\| \|\xi\| . \end{aligned}$$

Hence, $\eta \in \mathscr{D}(S^*)$. Thus we have $S \in \mathfrak{M}_T^\#$ and $\|S\|_T \leqq \gamma$.

We show that $\mathfrak{Q}$ is a $t_{qw}^{\mathscr{D}}$-compact subset of $\mathfrak{M}_T^\#$. In fact, $(\tilde{\mathscr{B}}_T^\#)_\gamma \equiv \{X \in \tilde{\mathscr{B}}_T^\#; \|X\| \leqq \gamma\}$ is weakly compact, and so Lemma 4.2 implies that $\tilde{\phi}^{-1}((\tilde{\mathscr{B}}_T^\#)_\gamma)$ is $t_{qw}^{\mathscr{D}}$-compact in $\widetilde{\mathfrak{M}}_T^\#$. Since $\mathfrak{Q}$ is a $t_{qw}^{\mathscr{D}}$-closed subset of $\tilde{\phi}^{-1}((\tilde{\mathscr{B}}_T^\#)_\gamma)$, it follows that $\mathfrak{Q}$ is a $t_{qw}^{\mathscr{D}}$-compact subset of $\mathfrak{M}_T^\#$.

Notation. Let $\mathfrak{K}_\delta$ be a set $\{U^*\delta(U); U \in \mathscr{M}_u\}$ and let $\mathfrak{Q}_\delta$ be the $t_{qw}^{\mathscr{D}}$-closed convex hull of $\mathfrak{K}_\delta$ in $\mathscr{L}(\mathscr{D}, \mathfrak{G})$.

LEMMA 4.4. *$\mathfrak{Q}_\delta$ is a $t_{qw}^{\mathscr{D}}$-compact subset of $\mathfrak{M}_T^\#$.*

Proof. It is easily seen that $\mathfrak{K}_\delta$ and $\mathfrak{K}_\delta^\#$ are bounded subsets of $\mathfrak{M}_T^\#$. Hence, the lemma follows from Lemma 4.3.

Furthermore, one may easily see the following lemma.

LEMMA 4.5. *For each $U \in \mathscr{M}_u$ we define*

$$A_U(S) = U^*SU + U^*\delta(U) \quad \text{for} \quad S \in \mathscr{L}^\#(\mathscr{D}, \mathfrak{G}) .$$

Then;

(1) *A_U is a $t_{qw}^{\mathscr{D}}$-continuous affine map of $\mathscr{L}^\#(\mathscr{D}, \mathfrak{G})$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{G})$;*
(2) *$A_U(V^*\delta(V)) = (VU)^*\delta(VU)$ for each $U, V \in \mathscr{M}_u$;*
(3) *$A_U\mathfrak{Q}_\delta \subset \mathfrak{Q}_\delta$ for each $U \in \mathscr{M}_u$;*
(4) *$A_U A_V = A_{VU}$ for each $U, V \in \mathscr{M}_u$.*

Hence, $G_{\delta,T} \equiv \{A_u; U \in \mathscr{M}_u\}$ is a semigroup of $t_{qw}^{\mathscr{D}}$-continuous affine maps of $\mathfrak{Q}_\delta$ into $\mathfrak{Q}_\delta$.

DEFINITION 4.6. If for each pair of elements $S_1 \neq S_2$ in $\mathfrak{G}_\delta$ the $t_s^{\mathscr{D}}$-closure of $\{A_U(S_1) - A_U(S_2); U \in \mathscr{M}_u\}$ does not contain 0, then $G_{\delta,T}$ is said to be noncontracting.

DEFINITION 4.7. Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathfrak{H}$ and let $\mathscr{M}$ be a C^*-algebra acting on $\mathfrak{H}$ with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. A $*$-derivation (resp. a derivation) δ of $\mathscr{M}$ into $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$(resp. $\mathscr{L}(\mathscr{D}, \mathfrak{H})$) is said to be spatial if there exists an element H of $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$(resp. $\mathscr{L}(\mathscr{D}, \mathfrak{H})$) such that

$$\delta(A)\xi = [H, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D} .$$

PROPOSITION 4.8. *If $G_{\delta,T}$ is noncontracting, then there exists an element S of $\mathfrak{Q}_\delta$ such that*

$$\delta(A)\xi = [S, A]\xi \quad \textit{for all} \quad A \in \mathscr{M} \quad \textit{and} \quad \xi \in \mathscr{D} ;$$

that is, δ is spatial.

Proof. We consider the locally convex space $\mathscr{X} = (\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H}), t_s^{\mathscr{D}})$. By Lemma 4.1 we have $\sigma(\mathscr{X}, \mathscr{X}^*) = t_{qw}^{\mathscr{D}}$, and hence it follows from Lemmas 4.4, 4.5 that $\mathfrak{Q}_\delta$ is a weakly compact subset of $\mathscr{X}$ and $G_{\delta,T}$ is a noncontracting semigroup of weakly continuous affine maps of $\mathfrak{Q}_\delta$ into $\mathfrak{Q}_\delta$. By Ryll-Nardzewski's fixed point theorem [9] there exists an element S_0 of $\mathfrak{Q}_\delta$ such that

$$A_U(S_0) = S_0 \quad \text{for all} \quad U \in \mathscr{M}_U .$$

Hence, putting $S = -S_0$, we have

$$\delta(A)\xi = [S, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D} .$$

COROLLARY 4.9. *Let $\mathscr{D}$ be a countably dominated subspace of a Hilbert space $\mathfrak{H}$ and let $\mathscr{M}$ be a commutative C^*-algebra acting on $\mathfrak{H}$ with $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. Then there does not exist any nonzero $(t_w \to t_{qw}^{\mathscr{D}})$-continuous (or $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$, $(t_u \to t_{qu}^{\mathscr{D}})$, $(t_u \to t_\lambda^{\mathscr{D}})$-continuous) $*$-derivation in $\mathscr{M}$.*

Proof. Suppose that δ is a $*$-derivation which is continuous in one of the above topologies. It then follows from Lemma 3.3 that δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathfrak{M}_T^\#$ where $T \in \mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ and $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$. Since $\mathscr{M}$ is commutative, we can easily see that the semigroup $G_{\hat{\delta},T}$ is noncontracting. Hence it follows from Proposition 4.8 that there exists an element H of $\mathfrak{Q}_{\hat{\delta}}$ such that $\hat{\delta}(A)\xi = [H, A]\xi$ for all $A \in \mathscr{M}$ and $\xi \in \mathscr{D}$. By Lemma 3.3 the elements A and H commute, and so $\hat{\delta} = 0$.

LEMMA 4.10. *Let $\mathfrak{H}$ be the completion of a maximal Hilbert algebra $\mathfrak{A}$ with identity e and let $\mathscr{M}$ be the left von Neumann algebra of $\mathfrak{A}$. Let $\mathscr{D}$ be a dense subspace of $\mathfrak{H}$ such that $e \in \mathscr{D}$ and*

$\mathscr{M}\mathscr{D} \subset \mathscr{D}$ *(for example,* $\mathfrak{A}$ *or the maximal unbounded Hilbert algebra* $L_2^\omega(\mathfrak{A})$ [5]). *If* δ *is a quasi-bounded* $*$*-derivation of* $\mathscr{M}$ *into* $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ *such that* $\overline{\delta(A)}\eta\mathscr{M}$ *for each* $A \in \mathscr{M}$*, then it is spatial.*

Proof. Since δ is quasi-bounded, there is an element T of $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$ such that $T^{-1} \in \mathscr{B}(\mathfrak{H})$ and $\delta(\mathscr{M}_u)$ is a bounded subset of the normed space $\mathfrak{M}_T^\sharp$. It is easily showed that $\mathfrak{A} \subset \mathscr{D}$ and $SB'\xi = B'S\xi$ for all $S \in \mathfrak{D}_\delta$, $B' \in \mathscr{M}'$ and $\xi \in \mathfrak{A}$. This implies that $G_{\delta,T}$ is non-contracting. In fact, for each pair of elements $S_1 \neq S_2$ in $\mathfrak{D}_\delta$ and $U \in \mathscr{M}_u$ we have

$$\begin{aligned} \| U^*(S_1 - S_2)Ue \| &= \| (S_1 - S_2)\overline{\pi'(u)}e \| \\ &= \| \overline{\pi'(u)}(S_1 - S_2)e \| \\ &= \| (S_1 - S_2)e \| \\ &\neq 0 , \end{aligned}$$

where π(resp. π') is the left (resp. right) regular representation of $\mathfrak{A}$ and $U = \overline{\pi(u)}$ for $u \in \mathfrak{A}$. Hence it follows from Proposition 4.8 that δ is spatial.

THEOREM 4.11. *Let* $\mathscr{M}$ *be the left von Neumann algebra of a maximal Hilbert algebra* $\mathfrak{A}$ *with identity* e, $\mathfrak{H}$ *the completion of* $\mathfrak{A}$ *and let* $\mathscr{D}$ *be a countably dominated subspace of* $\mathfrak{H}$ *by a sequence* $\{T_n\}$ *of closed operators such that* $e \in \mathscr{D}$ *and* $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. *If* δ *is a* $(t_w \to t_{qw}^{\mathscr{D}})$*-continuous (or* $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$*-continuous)* $*$*-derivation in* $\mathscr{M}$*, then it can be extended to a spatial* $*$*-derivation* $\hat{\delta}$ *of* $\mathscr{M}$ *into* $\mathscr{L}^\dagger(\mathscr{D}, \mathfrak{H})$.

Proof. This follows from Lemma 3.7 and Lemma 4.10.

We next examine the spatiality of derivations of $\mathscr{M}$ into $\mathfrak{M}_T^\sharp$ when $\bar{T}\eta\mathscr{M}'$ (or $\bar{T}\eta\mathscr{M}$).

Suppose that δ is a derivation of $\mathscr{M}$ into $\mathfrak{M}_T^\sharp$, where $T \in \mathscr{L}_c(\mathscr{D}, \mathfrak{H})$ and $\overline{T^{-1}} \in \mathscr{B}(\mathfrak{H})$. We set

$$\delta_T(A) = \overline{\delta(A)T^{-1}} \quad \text{for} \quad A \in \mathscr{M} .$$

It then follows from Lemma 2.1 that δ_T is a linear map of $\mathscr{M}$ into $\mathscr{B}(\mathfrak{H})$, and so we have the following diagram:

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\delta"] & \mathfrak{M}_T^\sharp \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\ A \arrow[dr, "\delta_T"'] & \delta(A) \arrow[d] \\ & \overline{\delta(A)T^{-1}} \in \mathscr{B}(\mathfrak{H}) . \end{tikzcd}$$

Furthermore, we have the following result, by a simple calculation

LEMMA 4.12. *If* $T \in \mathscr{L}_c(\mathscr{D}, \mathfrak{H})$ *and* $\overline{T^{-1}} \in \mathscr{M}'$, *then the linear map* δ_T *is a derivation of* $\mathscr{M}$ *into* $\mathscr{B}(\mathfrak{H})$.

DEFINITION 4.13. A von Neumann algebra $\mathscr{M}$ on $\mathfrak{H}$ is said to have the property (C) if every derivation δ of $\mathscr{M}$ into $\mathscr{B}(\mathfrak{H})$ is inner; that is, δ is implemented by an element of $\mathscr{B}(\mathfrak{H})$.

We note [3] that if $\mathscr{M}$ is of type I or properly infinite then $\mathscr{M}$ has the property (C).

PROPOSITION 4.14. *Let* $\mathscr{D}$ *be a dense subspace in a Hilbert space* $\mathfrak{H}$ *and let* $\mathscr{M}$ *be a von Neumann algebra on* $\mathfrak{H}$ *with the property* (C) *and* $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. *If* δ *is a* $*$*-derivation of* $\mathscr{M}$ *into* $\mathfrak{M}_T$ *where* $T \in \mathscr{L}_c(\mathscr{D}, \mathfrak{H})$ *and* $T^{-1} \in \mathscr{M}'$, *then there exists an element* B_0 *of* $\mathscr{B}(\mathfrak{H})$ *such that*

$$\delta(A)\xi = [B_0T, A]\xi$$

for all $A \in \mathscr{M}$ *and* $\xi \in \mathscr{D}$, i.e., δ *is spatial.*

Proof. By Lemma 4.12, δ_T is a derivation of $\mathscr{M}$ into $\mathscr{B}(\mathfrak{H})$. Hence it follows by the assumption that there exists an element B_0 of $\mathscr{B}(\mathfrak{H})$ such that

$$\delta_T(A) = [B_0, A] \quad \text{for all} \quad A \in \mathscr{M}.$$

This implies that

$$\delta(A)\xi = [B_0T, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D}.$$

THEOREM 4.15. *Let* $\mathscr{M}$ *be a von Neumann algebra on a Hilbert space* $\mathfrak{H}$ *with the property* (C) *and let* δ *be a* $*$*-derivation in* $\mathscr{M}$. *Suppose that there exists a countably dominated subspace* $\mathscr{D}$ *of* $\mathfrak{H}$ *by a sequence* $\{T_n\}$ *of closed operators* $T_n \eta \mathscr{M}'$ *such that* δ *is* $(t_w \to t_{qw}^{\mathscr{D}})$-*continuous (or* $(t_s \to t_s^{\mathscr{D}})$, $(t_\sigma \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$*-continuous). Then there exists an element* B_0 *of* $\mathscr{B}(\mathfrak{H})$ *and a closed operator* $T \eta \mathscr{M}'$ *such that*

$$\delta(A)\xi = [B_0T, A]\xi \quad \textit{for all} \quad A \in \mathscr{D}(\delta) \quad \textit{and} \quad \xi \in \mathscr{D}.$$

Proof. Since $T_n \eta \mathscr{M}'$ for $n = 1, 2, \cdots$, we have $\mathscr{M}\mathscr{D} \subset \mathscr{D}$. It follows from Lemma 3.3 that δ is extended to a $(t_w \to t_{qw}^{\mathscr{D}})$-continuous $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^\#(\mathscr{D}, \mathfrak{H})$. Furthermore, by Lemma 2.6 $\hat{\delta}$ is quasi-bounded, i.e., $\hat{\delta}(\mathscr{M}) \subset \mathfrak{M}^\#_{I+|T_n|}$ for some n. Hence the theorem follows from Proposition 4.14.

COROLLARY 4.16. *Let $\mathfrak{H}$ be the completion of a Hilbert algebra $\mathfrak{A}$, $\mathscr{M}$ the left von Neumann algebra of $\mathfrak{A}$ and let J be the unitary involution on $\mathfrak{A}$. Suppose that $\mathscr{M}$ has the property* (C) *and there exists a countably dominated subspace $\mathscr{D}$ of $\mathfrak{H}$ by a sequence $\{T_n\}$ of closed operators $T_n \eta \mathscr{M}$ such that $J\mathscr{D} = \mathscr{D}$. If δ is a $(t_w \to t_{qw}^{\mathscr{D}})$-continuous (or $(t_s \to t_s^{\mathscr{D}})$, $(t_{\sigma w} \to t_{q\sigma w}^{\mathscr{D}})$, $(t_{\sigma s} \to t_{\sigma s}^{\mathscr{D}})$-continuous) $*$-derivation in $\mathscr{M}$, then it is extended to spatial derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\dagger}(\mathscr{D}, \mathfrak{H})$.*

Proof. We put

$$T'_n = JT_nJ\ , \quad n = 1, 2, \cdots .$$

It is then proved that $\mathscr{D}$ is countably dominated by the sequence $\{T'_n\}$ of closed operators $T'_n \eta \mathscr{M}'$. Hence the corollary follows from Theorem 4.15.

PROPOSITION 4.17. *Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and let δ be a $*$-derivation in $\mathscr{M}$. If there exists a countably dominated subspace $\mathscr{D}$ of $\mathfrak{H}$ by a sequence $\{T_n\}$ of closed operators $T_n \eta \mathscr{M} \cap \mathscr{M}'$ such that δ is $(t_w \to t_{qw}^{\mathscr{D}})$-continuous, then δ is extended to a spatial $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathscr{L}^{\dagger}(\mathscr{D}, \mathfrak{H})$.*

Proof. By Lemma 3.3 and Lemma 2.6, δ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $\mathscr{M}$ into $\mathfrak{M}_T^{\dagger}$, where $T \in \mathscr{L}^{\dagger}(\mathscr{D}, \mathfrak{H})$ and $\overline{T^{-1}} \in \mathscr{M} \cap \mathscr{M}'$, satisfying $\hat{\delta}(A^*)^*C\xi = C\hat{\delta}(A)\xi$ for each $A \in \mathscr{M}$, $C \in \mathscr{M}'$ and $\xi \in \mathscr{D}$. Since $\mathscr{M}\mathscr{D} \subset \mathscr{D}$ and $\mathscr{M}'\mathscr{D} \subset \mathscr{D}$, we have $\overline{\hat{\delta}(A)} \eta \mathscr{M}$ for each $A \in \mathscr{M}$. Since $T \in \mathscr{M} \cap \mathscr{M}'$, $\hat{\delta}_T$ is a derivation of $\mathscr{M}$ into $\mathscr{M}$. Hence, there exists an element B_0 of $\mathscr{M}$ such that

$$\hat{\delta}_T(A) = [B_0, A] \quad \text{for each} \quad A \in \mathscr{M},$$

so that

$$\hat{\delta}(A)\xi = [B_0T, A]\xi \quad \text{for all} \quad A \in \mathscr{M} \quad \text{and} \quad \xi \in \mathscr{D}\ .$$

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