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**A CHARACTERIZATION OF THE ADJOINT L -KERNEL OF
SZEGŐ TYPE**

SABUROU SAITOH

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Let G be a bounded regular region in the complex plane and $\hat{L}(z, u)$ the adjoint L -kernel of Szegö kernel function $\hat{K}(z, \bar{u})$ on G . Then, for any analytic function $h(z)$ on G with a finite Dirichlet integral, it is shown that the equation

$$\begin{aligned} & \frac{1}{\pi} \iint_G |h'(z)|^2 dx dy \\ & = \int_{\partial G} \int_{\partial G} |(h(z_1) - h(z_2)) \hat{L}(z_1, z_2)|^2 |dz_1| |dz_2| \end{aligned}$$

holds. Furthermore, for any fixed nonconstant $h(z)$, we show that the function $\hat{L}(z_1, z_2)$ on $G \times G$ is characterized by that equation in some class.

1. Introduction and statement of result. Let S denote an arbitrary compact bordered Riemann surface. Let $W(z, t)$ be a meromorphic function whose real part is the Green's function $g(z, t)$ with pole at $t \in S$. The differential $\text{id } W(z, t)$ is positive along ∂S . For simplicity, we do not distinguish between points $z \in S \cup \partial S$ and local parameters z . For an arbitrary integer q and for any positive continuous function $\rho(z)$ on ∂S , let $H_{p, \rho}^q(S) [p \geq 1]$ be the Banach space of analytic differentials $f(z)(dz)^q$ on S of order q with finite norms

$$\left\{ \frac{1}{2\pi} \int_{\partial S} |f(z)(dz)^q|^p \rho(z) [\text{id } W(z, t)]^{1-2q} \right\}^{1/p} < \infty,$$

where $f(z)$ means the Fatou boundary value of f at $z \in \partial S$. Let $K_{q, t, \rho}(z, \bar{u})(dz)^q$ be the reproducing kernel for $H_{2, \rho}^q(S)$ which is characterized by the reproducing property

$$f(u) = \frac{1}{2\pi} \int_{\partial S} f(z)(dz)^q \overline{K_{q, t, \rho}(z, \bar{u})(dz)^q} \rho(z) [\text{id } W(z, t)]^{1-2q}$$

for all $f(z)(dz)^q \in H_{2, \rho}^q(S)$.

See [9]. Let $L_{q, t, \rho}(z, u)(dz)^{1-q}$ denote the adjoint L -kernel of $K_{q, t, \rho}(z, \bar{u})(dz)^q$. The function $L_{q, t, \rho}(z, u)(dz)^{1-q}$ is a meromorphic differential on S of order $1 - q$ with a simple pole at u having residue 1. Moreover,

$$\begin{aligned} (1.1) \quad & \overline{K_{q, t, \rho}(z, \bar{u})(dz)^q} \rho(z) [\text{id } W(z, t)]^{1-2q} \\ & = \frac{1}{i} L_{q, t, \rho}(z, u)(dz)^{1-q} \text{ along } \partial S. \end{aligned}$$

We note that $|K_{q,t,\rho}(z, \bar{u})|$ and $|L_{q,t,\rho}(z, u)|$ can be extended continuously on ∂S . In addition, $K_{q,t,\rho}(z, \bar{u}) = \overline{K_{q,t,\rho}(u, \bar{z})}$ and $L_{q,t,\rho}(z, u) = -L_{1-q,t,\rho^{-1}}(u, z)$ on S .

If S is a bounded regular region in the plane, then we can define the kernels for arbitrary real values of q . In this case, for $q = 1/2$ and $\rho(z) \equiv 1$, we have the classical Szegő kernels $\hat{K}(z, \bar{u}) = K_{1/2,t,1}(z, \bar{u})/2\pi$ and $\hat{L}(z, u) = L_{1/2,t,1}(z, u)/2\pi$. Cf. [8] and [9].

A classical characterization of $L_{q,t,\rho}(z, u)(dz)^{1-q}$ can be now stated as follows:

PROPOSITION (P. R. Garabedian [3, 4], Z. Nehari [6, 7] and S. Saitoh [8, 9]). *The adjoint L-kernel $L_{q,t,\rho}(z, u)(dz)^{1-q}$ is characterized by the following extremal property*

$$\begin{aligned} K_{q,t,\rho}(u, \bar{u}) &= \frac{1}{2\pi} \int_{\partial S} |L_{q,t,\rho}(z, u)(dz)^{1-q}|^2 (\rho(z))^{-1} [\text{id } W(z, t)]^{2q-1} \\ &= \min \left\{ \frac{1}{2\pi} \int_{\partial S} |F(z, u)(dz)^{1-q}|^2 (\rho(z))^{-1} [\text{id } W(z, t)]^{2q-1} \right\}. \end{aligned}$$

The minimum is taken here over all meromorphic differentials $F(z, u)(dz)^{1-q}$ on S of order $1 - q$ with a simple pole at u having residue 1 and with finite integral

$$\int_{\partial S} |F(z, u)(dz)^{1-q}|^2 [\text{id } W(z, t)]^{2q-1} < \infty.$$

In this paper, we establish the following theorem:

THEOREM 1.1. *For any analytic function $h(z)$ on S with a finite Dirichlet integral, we have the equation*

$$\begin{aligned} (1.2) \quad & \frac{1}{\pi} \iint_S |h'(z)|^2 dx dy \\ &= \frac{1}{4\pi^2} \int_{\partial S} \int_{\partial S} |(h(v) - h(u))L_{q,t,\rho}(v, u)(dv)^{1-q}(du)^q|^2 \\ & \quad \times (\rho(v))^{-1} [\text{id } W(v, t)]^{2q-1} \rho(u) [\text{id } W(u, t)]^{1-2q}, \quad z = x + iy. \end{aligned}$$

Furthermore, for any fixed nonconstant $h(z)$, the adjoint L-kernel $L_{q,t,\rho}(v, u) (dv)^{1-q}(du)^q$ is characterized by the following extremal property:

$$\begin{aligned} (1.3) \quad & \int_{\partial S} \int_{\partial S} |(h(v) - h(u))L_{q,t,\rho}(v, u)(dv)^{1-q}(du)^q|^2 \\ & \quad \times (\rho(v))^{-1} [\text{id } W(v, t)]^{2q-1} \rho(u) [\text{id } W(u, t)]^{1-2q} \end{aligned}$$

$$= \min \left\{ \int_{\partial S} \int_{\partial S} |(h(v) - h(u))F(v, u)(dv)^{1-q}(du)^q|^2 \right. \\ \left. \times (\rho(v))^{-1}[\text{id } W(v, t)]^{2q-1}\rho(u)[\text{id } W(u, t)]^{1-2q} \right\} .$$

The minimum is taken here over all meromorphic differentials $F(v, u)(dv)^{1-q}(du)^q$ on $S \times S$ such that

$$(1.4) \quad F(v, u) = \frac{f(u, v)}{h(v) - h(u)}$$

for an analytic differential $f(u, v)(du)^q(dv)^{1-q}$ on $S \times S$ satisfying

$$(1.5) \quad f(z, z) = h'(z) \text{ on } S$$

and

$$(1.6) \quad \int_{\partial S} \int_{\partial S} |f(u, v)(du)^q(dv)^{1-q}|^2[\text{id } W(u, t)]^{1-2q}[\text{id } W(v, t)]^{2q-1} < \infty .$$

In particular, we note that when $q = 1/2$ and $\rho(z) \equiv 1$, we can define the adjoint L -kernels of the Szegő kernels of S with characteristics. Cf. D. A. Hejhal [5] and J. D. Fay [2]. Then, the adjoint L -kernels are, in general, multiplicative functions, but our proof of Theorem 1.1 will show that Theorem 1.1 is still valid for these adjoint L -kernels in a modified form.

2. Preliminaries. Let $\{\Phi_j(z)(dz)^q\}_{j=1}^\infty$ and $\{\Psi_j(z)(dz)^{1-q}\}_{j=1}^\infty$ be complete orthonormal systems for $H_{2,\rho}^q(S)$ and $H_{2,\rho^{-1}}^{1-q}(S)$, respectively. Let $H = H_{2,\rho}^q(S) \otimes H_{2,\rho^{-1}}^{1-q}(S)$ denote the direct product of $H_{2,\rho}^q(S)$ and $H_{2,\rho^{-1}}^{1-q}(S)$. The space H is composed of all differentials $f(z_1, z_2)(dz_1)^q(dz_2)^{1-q}$ on $S \times S$ such that

$$(2.1) \quad f(z_1, z_2) = \sum_{j=1}^\infty \sum_{k=1}^\infty A_{j,k} \Phi_j(z_1) \Psi_k(z_2) , \quad \sum_{j=1}^\infty \sum_{k=1}^\infty |A_{j,k}|^2 < \infty .$$

The scalar product $(,)_H$ is given as follows:

$$(2.2) \quad (f(z_1, z_2), h(z_1, z_2))_H = \sum_{j=1}^\infty \sum_{k=1}^\infty A_{j,k} \overline{B_{j,k}}$$

where $h(z_1, z_2) = \sum_{j=1}^\infty \sum_{k=1}^\infty B_{j,k} \Phi_j(z_1) \Psi_k(z_2)$ and $\sum_{j=1}^\infty \sum_{k=1}^\infty |B_{j,k}|^2 < \infty$. Cf. [1, § 8].

We let $H_{D(0)}$ denote the subspace in H composed of all differentials which vanish along the diagonal set $D = \{(z, z) | z \in S\}$ and $(H_{D(0)})^\perp$ the orthocomplement of $H_{D(0)}$ in H .

3. Proof of theorem. For $h(z) \in H_{1,1}^0(S)$, we set

$$(3.1) \quad f_h(u, v) = \int_{\partial S} h(z) \overline{K_{q,t,\rho}(z, \bar{u})} K_{1-q,t,\rho^{-1}}(z, \bar{v}) dz.$$

From (1.1) and the residue theorem, we have

$$(3.2) \quad f_h(u, v) = -2\pi i L_{q,t,\rho}(v, u)(h(v) - h(u))$$

and so

$$(3.3) \quad f_h(z, z) = -2\pi i h'(z) \text{ on } S.$$

When $h(z)$ has a finite Dirichlet integral, from [12, Theorem 4.1] and [11, Corollary 3.2], we see that $f_h(u, v)(du)^q(dv)^{1-q}$ belongs to $(H_{D(0)})^\perp$. From [12, Corollary 2.1] and [10, Equation (3.2)], we thus obtain (1.2).

Next, suppose that $F^*(v, u)$ attains the minimum in (1.3). Then, in the case such that $h(z)$ is not constant, we set

$$(3.4) \quad f_h^*(u, v) = F^*(v, u)(h(v) - h(u))$$

and so

$$(3.5) \quad f_h^*(z, z) = h'(z) \text{ on } S.$$

We note that any $f(u, v)(du)^q(dv)^{1-q} \in H$ satisfying $f(z, z) = h'(z)$ on S is expressible in the form

$$f(u, v) = F(v, u)(h(v) - h(u))$$

for an $F(v, u)$ stated in the theorem. From the extremal property of $f_h^*(u, v)(du)^q(dv)^{1-q}$ in the subspace H satisfying $f(z, z) = h'(z)$ on S , we see that $f_h^*(u, v)(du)^q(dv)^{1-q} \in (H_{D(0)})^\perp$. Cf. [10, Equation (3.2)]. Therefore, by [12, Theorem 4.2], $f_h^*(u, v)$ is expressible in the form

$$(3.6) \quad f_h^*(z_1, z_2) = \frac{1}{2\pi} \int_{\partial S} \frac{h^*(\zeta) d\zeta \overline{K_{q,t,\rho}(\zeta, \bar{z}_1)} K_{1-q,t,\rho^{-1}}(\zeta, \bar{z}_2) d\bar{\zeta}}{\text{id } W(\zeta, t)}$$

for a uniquely determined $h^*(z)dz$ in $H_{1,1}^1(S)$. Furthermore, from [12, Equations (4.11) and (4.12)], $h^*(z)$ can be determined as follows:

$$(3.7) \quad h^*(z) = -W'(z, t)(h(z) - h(t)).$$

From (3.6) and (1.1), we have

$$(3.8) \quad f_h^*(u, v) = L_{q,t,\rho}(v, u)(h(v) - h(u)).$$

We thus have the desired result $F^*(v, u) = L_{q,t,\rho}(v, u)$.

4. Corollary. *In particular, from the proof of Theorem 1.1, we obtain*

COROLLARY 4.1. *For any fixed nonconstant analytic function $h(z)$ on S with a finite Dirichlet integral, the unique extremal function which minimizes*

$$\|f(z_1, z_2)\|_{H_{2, \rho^{-1}}^{1-q}(S) \otimes H_{2, \rho}^q(S)}$$

in the subspace in $H_{2, \rho^{-1}}^{1-q}(S) \otimes H_{2, \rho}^q(S)$ satisfying $f(z, z) = h'(z)$ on S is given by $(h(z_1) - h(z_2))L_{q, t, \rho}(z_1, z_2)(dz_1)^{1-q}(dz_2)^q$.

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