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**CONVERSE MEASURABILITY THEOREMS FOR YEH-WIENER  
SPACE**

KUN SOO CHANG

## CONVERSE MEASURABILITY THEOREMS FOR YEH-WIENER SPACE

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**Cameron and Storvick established a theorem for evaluating in terms of a Wiener integral the Yeh-Wiener integral of a functional of  $x$  which depends on the values of  $x$  on a finite number of horizontal lines. Skoug obtained the converse of the theorem in case of one horizontal line. In this paper we extend Skoug's result to the case of a finite number of horizontal lines.**

**1. Introduction.** Let  $C_1[a, b]$  denote the Wiener space of functions of one variable, i.e.,  $C_1[a, b] = \{x(\cdot) \mid x(a) = 0 \text{ and } x(s) \text{ is continuous on } [a, b]\}$ . Let  $R = \{(s, t) \mid a \leq s \leq b, \alpha \leq t \leq \beta\}$  and let  $C_2[R]$  be Yeh-Wiener space (or 2 parameter Wiener space), i.e.,  $C_2[R] = \{x(\cdot, \cdot) \mid x(a, t) = x(s, \alpha) = 0, x(s, t) \text{ is continuous on } R\}$ . Let  $\nu$  be Wiener measure on  $C_1[a, b]$  and let  $m$  be Yeh-Wiener measure on  $C_2[R]$ . For a discussion of Yeh-Wiener measure see [1], [3] and [4].  $\mathbf{R}$  will denote the real numbers and  $\mathbf{C}$  the complex numbers. We shall use the following notation for the Cartesian product of  $n$  Wiener spaces  $\times_n C_1[a, b] = C_1[a, b] \times \cdots \times C_1[a, b]$  and  $\times_n \nu = \nu \times \cdots \times \nu$  will denote the product of  $n$  Wiener measures on  $\times_n C_1[a, b]$ .

Let  $\alpha = t_0 < t_1 < \cdots < t_n = \beta$  be a subdivision of  $[\alpha, \beta]$ . Define  $\varphi: \times_n C_1[a, b] \rightarrow \times_n C_1[a, b]$  by

$$\begin{aligned} \mathcal{P}(y_1, y_2, \dots, y_n) &= \left( \sqrt{\frac{t_1 - t_0}{2}} y_1, \sqrt{\frac{t_1 - t_0}{2}} y_1 + \sqrt{\frac{t_2 - t_1}{2}} y_2, \dots, \sqrt{\frac{t_1 - t_0}{2}} y_1 + \dots \right. \\ &\quad \left. + \sqrt{\frac{t_n - t_{n-1}}{2}} y_n \right). \end{aligned}$$

Then  $\varphi$  is 1-1, onto and continuous with respect to the uniform topology. Let  $G: C_2[R] \rightarrow \times_n C_1[a, b]$  be defined by  $G(x) = (x(\cdot, t_1), x(\cdot, t_2), \dots, x(\cdot, t_n))$ . Then  $G$  is a continuous function from  $C_2[R]$  onto  $\times_n C_1[a, b]$ .

In [1] Cameron and Storvick evaluated certain Yeh-Wiener integrals in terms of Wiener integrals. In particular they obtained the following theorem;

**THEOREM A ( $n$ -parallel lines theorem).** *Let  $f(y_1, y_2, \dots, y_n)$  be a real or complex valued functional defined on  $\times_n C_1[a, b]$  such that*

$f \circ \varphi$  is a Wiener measurable functional of  $(y_1, y_2, \dots, y_n)$  on  $\overset{n}{\times}C_1[a, b]$ . Then  $f \circ G$  is a Yeh-Wiener measurable functional of  $x$  on  $C_2[R]$  and

$$\int_{C_2[R]} f \circ G(x) dx = \int_{\overset{n}{\times}C_1[a, b]} f \circ \varphi(y_1, y_2, \dots, y_n) d(y_1 \times \dots \times y_n)$$

where the existence of either integral implies the existence of the other and their equality.

We note that Theorem A, in the case  $n=1$ , is called the one line theorem. Now we explicitly state and prove the following corollary of Theorem A which plays a key role in the proof of Lemma 3 in §2.

**COROLLARY.** Let  $A$  be any subset of  $\overset{n}{\times}C_1[a, b]$ . If  $\varphi^{-1}A$  is  $\overset{n}{\times}\nu$ -measurable, then  $G^{-1}A$  is Yeh-Wiener measurable and  $\overset{n}{\times}\nu(\varphi^{-1}A) = m(G^{-1}A)$ .

*Proof.* Let  $f(y_1, y_2, \dots, y_n) = \chi_A(y_1, y_2, \dots, y_n)$ . Then

$$f \circ \varphi(y_1, y_2, \dots, y_n) = \chi_A(\varphi(y_1, y_2, \dots, y_n)) = \chi_{\varphi^{-1}A}(y_1, y_2, \dots, y_n).$$

If  $\varphi^{-1}A$  is  $\overset{n}{\times}\nu$ -measurable, then  $f \circ \varphi$  is  $\overset{n}{\times}\nu$ -measurable. Hence by Theorem A,  $f \circ G$  is a Yeh-Wiener measurable functional of  $x$  on  $C_2[R]$ . But  $f \circ G(x) = \chi_A(G(x)) = \chi_{G^{-1}A}(x)$ . Thus  $G^{-1}A$  is Yeh-Wiener measurable.

$$\begin{aligned} \overset{n}{\times}\nu(\varphi^{-1}A) &= \int_{\overset{n}{\times}C_1[a, b]} \chi_{\varphi^{-1}A}(y_1, y_2, \dots, y_n) d(y_1 \times \dots \times y_n) \\ &= \int_{\overset{n}{\times}C_1[a, b]} f \circ \varphi(y_1, y_2, \dots, y_n) d(y_1 \times \dots \times y_n) \\ &= \int_{C_2[R]} f(x(\cdot, t_1), \dots, x(\cdot, t_n)) dx \\ &= \int_{C_2[R]} f \circ G(x) dx = \int_{C_2[R]} \chi_{G^{-1}A}(x) dx = m(G^{-1}A). \end{aligned}$$

It has long been known that measurability questions in Wiener space and Yeh-Wiener space are often rather delicate. In [3] Skoug established some relationships between Yeh-Wiener measurability and Wiener measurability of certain sets and functionals. Furthermore he obtained the converse of the one line theorem. In this paper we extend his result to the  $n$ -parallel lines theorem. In particular we show that if  $A$  is any subset of  $\overset{n}{\times}C_1[a, b]$ , then  $G^{-1}A$  is a Yeh-Wiener measurable subset of  $C_2[R]$  if and only if  $\varphi^{-1}A$  is a Wiener measurable subset of  $\overset{n}{\times}C_1[a, b]$ .

**2. Lemmas.** The converse measurability theorems in §3 will follow quite readily once we establish three lemmas.

DEFINITION. Let  $\delta$  be a fixed constant satisfying  $0 < \delta < 1/2$  and let  $\lambda > 0$  be given. Let

$$A_\lambda \equiv A_\lambda(\delta) \equiv \{x \in C_2[R]: |x(s_2, t_2) - x(s_1, t_1)| \leq \lambda[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{1/2} \\ \text{for all } s_1, s_2 \in [a, b] \text{ and } t_1, t_2 \in [\alpha, \beta]\} .$$

Our first lemma is taken from [3]. We state it without proof.

LEMMA 1. (a) For any  $\varepsilon > 0$ , there exists  $\lambda_0 > 0$  such that  $m(A_\lambda^\delta) < \varepsilon$  for all  $\lambda \geq \lambda_0$ . In fact  $m(\bigcup_{n=1}^\infty A_n) = 1$ . (b) For each  $\lambda > 0$ ,  $A_\lambda$  is compact in the uniform topology in  $C_2[R]$ .

LEMMA 2. Let  $A$  be any subset of  $\overset{n}{\times}C_1[a, b]$  and let  $V$  be any open set in  $C_2[R]$  containing  $G^{-1}A$ . Let  $\lambda > 0$  be given. Then there exists an open set  $U$  in  $\overset{n}{\times}C_1[a, b]$  such that  $A \subseteq U$  and  $A_\lambda \cap G^{-1}U \subseteq V$ .

*Proof.* Case 1. Assume that  $A$  consists of just one point, say,  $(y_1, \dots, y_n)$ . Suppose that Lemma 2 is false. For  $n = 1, 2, 3, \dots$ , let  $U_n$  be open sphere of radius  $1/n$  about  $(y_1, y_2, \dots, y_n)$ . Then there exists a sequence of points  $\{x_n\}_{n=1}^\infty$  in  $(A_\lambda \cap G^{-1}U_n) \setminus V$ . Hence  $\{x_n\}_{n=1}^\infty \subseteq A_\lambda$  and  $\|Gx_n - (y_1, y_2, \dots, y_n)\| < 1/n$  where  $\|\cdot\|$  is a product norm in  $\overset{n}{\times}C_1[a, b]$ . Since  $A_\lambda$  is compact in the uniform topology for  $C_2[R]$ , there exists a subsequence  $\{x_{n_k}\}_{k=1}^\infty$  which converges uniformly on  $R$  to some element, say  $x_0$ , of  $C_2[R]$ . By continuity of  $G$ ,  $(y_1, \dots, y_n) = \lim_{k \rightarrow \infty} Gx_{n_k} = Gx_0$ . Thus  $Gx_0$  is in  $A$  and  $x_0$  is in  $G^{-1}A$ . But  $V^c$  is closed and so  $x_0$  is in  $V^c \subseteq (G^{-1}A)^c$  which is contrary to  $x_0 \in G^{-1}A$ .

Case 2. General case. Let  $A$  be any set in  $\overset{n}{\times}C_1[a, b]$ . By Case 1, we see that for each point  $z$  in  $A$  there exists an open set  $U_z$  in  $\overset{n}{\times}C_1[a, b]$  such that  $z \in U_z$  and  $A_\lambda \cap G^{-1}U_z \subseteq V$ . Then  $U \equiv \bigcup_{z \in A} U_z$  is an open set in  $\overset{n}{\times}C_1[a, b]$  containing  $A$  and

$$A_\lambda \cap G^{-1}U = A_\lambda \cap \left(G^{-1}\left(\bigcup_{z \in A} U_z\right)\right) = \bigcup_{z \in A} (A_\lambda \cap G^{-1}U_z) \subseteq V .$$

LEMMA 3. Let  $A$  be any subset of  $\overset{n}{\times}C_1[a, b]$ . Then  $m^*G^{-1}A = (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$  where  $m^*$  and  $(\overset{n}{\times}\nu)^*$  are outer Yeh-Wiener and product Wiener measures respectively.

*Proof.* First we will show that  $m^*G^{-1}A \leq (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$ . Let  $\tilde{A}$  be a subset of  $\overset{n}{\times}C_1[a, b]$  such that  $A \subseteq \tilde{A}$ ,  $\varphi^{-1}\tilde{A}$  is  $\overset{n}{\times}\nu$ -measurable and  $\overset{n}{\times}\nu(\varphi^{-1}\tilde{A}) = (\overset{n}{\times}\nu)^*(\varphi^{-1}A)$ . Note that such  $\tilde{A}$  exists since there exists a subset  $B$  of  $\overset{n}{\times}C_1[a, b]$  such that  $B$  is  $\overset{n}{\times}\nu$ -measurable and

$\overset{n}{\mathbf{X}}\nu(B) = (\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A)$  and  $\varphi^{-1}A \subseteq B$ . Let  $\tilde{A} = \varphi(B)$ . Then  $A = \varphi(\varphi^{-1}A) \subseteq \varphi(B) = \tilde{A}$  and  $\varphi^{-1}(\tilde{A}) = \varphi^{-1}(\varphi(B)) = B$ . By Corollary of Theorem A,  $G^{-1}\tilde{A}$  is Yeh-Wiener measurable and  $m^*G^{-1}A \subseteq m^*G^{-1}\tilde{A} = mG^{-1}\tilde{A} = \overset{n}{\mathbf{X}}\nu(\varphi^{-1}\tilde{A}) = (\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A)$ .

To show  $(\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A) \subseteq m^*G^{-1}A$ , it suffices to show that for given  $\varepsilon > 0$ ,  $(\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A) \subseteq m^*G^{-1}A + \varepsilon$ . Now choose a Yeh-Wiener measurable set  $H$  such that  $G^{-1}A \subseteq H$  and  $m^*G^{-1}A = mH$ . Next we choose  $n > 0$  so large that  $m(A_n^c) < \varepsilon/2$  [Lemma 1]. Then

$$(1) \quad m(H \cup A_n^c) \leq mH + m(A_n^c) < m^*G^{-1}A + \varepsilon/2.$$

Let  $V$  be an open subset of  $C_2[R]$  such that  $H \cup A_n^c \subseteq V$  and  $m(V \setminus [H \cup A_n^c]) < \varepsilon/2$  [2, Theorem 1.2, p. 27]. Then

$$(2) \quad mV < m(H \cup A_n^c) + \varepsilon/2.$$

By Lemma 2 (note that  $G^{-1}A \subseteq H \subseteq H \cup A_n^c \subseteq V$  and  $V$  is open), there exists an open set  $U \subseteq \overset{n}{\mathbf{X}}C_1[a, b]$  such that  $A \subseteq U$  and  $A_n \cap G^{-1}U \subseteq V$ . But  $(G^{-1}U) \cap A_n^c \subseteq A_n^c \subseteq H \cup A_n^c \subseteq V$ . Hence

$$(3) \quad G^{-1}U = (G^{-1}U \cap A_n) \cup (G^{-1}U \cap A_n^c) \subseteq V.$$

Since  $U$  is open and  $\varphi$  is continuous,  $\varphi^{-1}U$  is open. Hence  $\varphi^{-1}U$  is  $\overset{n}{\mathbf{X}}\nu$ -measurable. By continuity of  $G$ ,  $G^{-1}U$  is Yeh-Wiener measurable and  $m(G^{-1}U) = \overset{n}{\mathbf{X}}\nu(\varphi^{-1}U)$ . By (1), (2) and (3) we obtain  $(\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A) \subseteq \overset{n}{\mathbf{X}}\nu(\varphi^{-1}U) = m(G^{-1}U) \subseteq mV < m(H \cup A_n^c) + \varepsilon/2 < m^*G^{-1}A + \varepsilon$ .

**3. Converse measurability theorems.** Our first theorem in this section establishes a relationship between Yeh-Wiener measurability and product Wiener measurability of certain related sets. In Theorem 2 we obtain the converse of Theorem A.

**THEOREM 1.** *Let  $A$  be any subset of  $\overset{n}{\mathbf{X}}C_1[a, b]$ . Then  $G^{-1}A$  is Yeh-Wiener measurable if and only if  $\varphi^{-1}A$  is  $\overset{n}{\mathbf{X}}\nu$ -measurable. Furthermore  $m(G^{-1}A) = \overset{n}{\mathbf{X}}\nu(\varphi^{-1}A)$ .*

*Proof.* We only need to show that if  $G^{-1}A$  is Yeh-Wiener measurable then  $\varphi^{-1}A$  is  $\overset{n}{\mathbf{X}}\nu$ -measurable. So assume that  $G^{-1}A$  is Yeh-Wiener measurable. By Lemma 3,  $m^*G^{-1}A = (\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A)$ . Another application of Lemma 3 yields  $(\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A)^c = (\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A^c) = m^*G^{-1}A^c = m^*(G^{-1}A)^c = m(G^{-1}A)^c$ . Thus we obtain that

$$\begin{aligned} (\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A)^c + (\overset{n}{\mathbf{X}}\nu)^*(\varphi^{-1}A) &= m^*(G^{-1}A)^c + m^*(G^{-1}A) \\ &= m(G^{-1}A) + m(G^{-1}A)^c = 1 \end{aligned}$$

from which it follows that  $\varphi^{-1}A$  is  $\overset{n}{\mathbf{X}}\nu$ -measurable.

**THEOREM 2.** *Let  $\alpha = t_0 < t_1 < \dots < t_n = \beta$  and let  $f(y_1, y_2, \dots, y_n)$  be a real or complex valued functional defined on  $\overset{n}{\mathbf{X}}C_1[a, b]$ . Then  $f \circ \varphi$  is a Wiener measurable functional of  $(y_1, y_2, \dots, y_n)$  on  $\overset{n}{\mathbf{X}}C_1[a, b]$  if and only if  $f \circ G$  is a Yeh-Wiener measurable functional of  $x$  on  $C_2[R]$ . In this case,*

$$\int_{C_2[R]} f \circ G(x) dx = \int_{\overset{n}{\mathbf{X}}C_1[a, b]} f \circ \varphi(y_1, y_2, \dots, y_n) d(y_1 \times \dots \times y_n)$$

where the existence of either integral implies the existence of the other and their equality.

*Proof.* By Theorem A it suffices to show measurability only. Let  $B$  be any Borel set in  $\mathbf{R}$  or  $\mathbf{C}$ . Suppose that  $f \circ G$  is Yeh-Wiener measurable. Then  $G^{-1}(f^{-1}B) = (f \circ G)^{-1}(B)$  is Yeh-Wiener measurable. By Theorem 1,  $\varphi^{-1}(f^{-1}B) = (f \circ \varphi)^{-1}(B)$  is  $\overset{n}{\mathbf{X}}\nu$ -measurable. Hence  $f \circ \varphi$  is  $\overset{n}{\mathbf{X}}\nu$ -measurable.

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