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STRONG COMPLETENESS IN PROFINITE GROUPS

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A profinite group is strongly complete if every subgroup of finite index is open. In this paper it is shown that a profinite group with finitely generated p -Sylow subgroups is strongly complete and that if G is a finitely generated strongly complete profinite group and A is a finitely generated pseudocompact G -module then any extension of A by G is strongly complete.

The purpose of this paper is to extend some results of Anderson [1] in the theory of strong completeness of profinite groups. A *profinite group* is a topological group whose topology is Hausdorff, compact and has neighborhood base of the identity consisting of certain subgroups of finite index. A profinite group is *strongly complete* if every subgroups of finite index is open. Since all open subgroups are also closed, a strongly complete profinite group has no dense subgroups of finite index except itself.

Our first result is:

THEOREM 1. *Let G be a profinite group, G_p a p -Sylow subgroup, $U \trianglelefteq G$ with $(G:U) = n < \infty$. U is open in G if and only if $U \cap G_p$ is open in G_p for every prime p which divides n .*

COROLLARY 1. *Let G be a profinite group all of whose p -Sylow subgroups are finitely generated. Then G is strongly complete.*

Our second result is:

THEOREM 2. *Let $A \twoheadrightarrow E \twoheadrightarrow G$ be a short exact sequence of profinite groups. If G is a finitely generated strongly complete profinite group and A is a finitely generated pseudocompact $\hat{Z}[[G]]$ -module then E is strongly complete.*

COROLLARY 1. *Let $A \twoheadrightarrow E \twoheadrightarrow G$ be a short exact sequence of profinite groups where G is as in the theorem and A contains a finite sequence of subgroups which are normal in E : $A = A_0 \supseteq A_1 \supseteq \dots \supseteq A_n = (e)$ such that A_i/A_{i+1} is a finitely generated pseudocompact $\hat{Z}[[G]]$ -module for $i = 0, \dots, n-1$. Then E is strongly complete.*

In this paper all groups are profinite, all subgroups are closed, and all homomorphisms are continuous unless otherwise stated. We

will call a proper subgroup of finite index *large*.

1. For any group, G , $x \in G$, the closed subgroup generated by x , $\overline{\langle x \rangle}$, is cyclic and so there is a continuous homomorphism $\rho: \hat{Z} \rightarrow \overline{\langle x \rangle}$ defined by $\rho(\lambda) = x^\lambda$. Writing \hat{Z} as $\prod_p \hat{Z}_p$, the product over all primes p of p -adic integers, and then as $\hat{Z}_p \times \prod_{q \neq p} \hat{Z}_q$ and allowing the generator of $\hat{Z}_p \times (0)$ to be $(1, 0)$ and the generator of $(0) \times \prod_{q \neq p} \hat{Z}_q$ to be $(0, 1)$ one sees that $\overline{\langle x^{(1,0)} \rangle}$ is the p -Sylow subgroup of $\overline{\langle x \rangle}$ and $\overline{\langle x^{(0,1)} \rangle}$ its p -complement. Finitely generated pro-abelian groups are known to be strongly complete. Hence any homomorphism from $\overline{\langle x \rangle}$ to a finite group is continuous. With this we prove:

PROPOSITION 1. *Let U be a large normal subgroup of G , U not necessarily open, $x \in G$ such that $\bar{x} \in (G/U)_p$, p -Sylow subgroup of G/U . Then $\overline{x^{(1,0)}} = \bar{x}$ in G/U .*

Proof. The morphism $\overline{\langle x \rangle}$ to $\langle \bar{x} \rangle \leq G/U$ is continuous as we have noted. $\langle \bar{x} \rangle$ is a finite cyclic p -group. Since $x = x^{(1,0)} \cdot x^{(0,1)}$ and $x^{(0,1)}$ is an element of G whose order is prime to p , its image $\langle \bar{x} \rangle$ is the identity. Hence

$$\bar{x} = \overline{x^{(1,0)} \cdot x^{(0,1)}} = \overline{x^{(1,0)} \cdot x^{(0,1)}} = \overline{x^{(1,0)}}. \quad \square$$

We call an element of G a p -element if it belongs to some p -Sylow subgroup of G . For all x in G , $x^{(1,0)}$ is a p -element and x is a p -element if and only if $x = x^{(1,0)}$ (see [4]).

A net of elements $\{x_\alpha\}$ of a profinite group G converges to an element x if for all open normal subgroups V of G , $x_\alpha V = xV$ for almost all α .

PROPOSITION 2. *Let $\{x_\alpha\}$ be a net in G converging to a p -element x . Then $\{x_\alpha^{(1,0)}\}$ is a net of p -elements which also converges to x .*

Proof. If x is a p -element then for any open normal subgroup V of G , xV is a p -element in G/V . By Proposition 1, $x_\alpha V = x_\alpha^{(1,0)} V$ if $x_\alpha V = xV$. The set $\{x_\alpha^{(1,0)}\}$ is clearly a net and hence the result.

Before proving Theorem 1 we need the following lemma.

LEMMA 1. *Let $U \trianglelefteq G$, U not necessarily closed, such that for some p -Sylow subgroup G_p of G , $U \cap G_p$ is closed in G_p . The set of all p -elements in U is closed in G .*

Proof. Let $U_p = U \cap G_p$. The set of all p -elements in U is

$$\bigcup_{x \in G} U \cap G_p^x = \bigcup_{x \in G_p} U_p^x$$

since U is normal. Consider the function $U_p \times G \rightarrow G$ defined by $(u, g) \rightarrow g^{-1}ug$. Since U_p is closed in G_p it is compact and hence the function, which is easily continuous, is a closed function. Its image, which is precisely the set of p -elements of U , is therefore closed in G . □

Proof of Theorem 1. Let $U \trianglelefteq G$ of finite index. If U is open then $U \cap G_p$ is open in G_p for all G_p . Conversely suppose there exists large U not open, the quotient group \bar{U}/U has a nontrivial p -Sylow subgroup for some prime p . Hence there exists $x \notin U$ such that $\bar{x} \neq \bar{x} \in \bar{U}/U$ is a nontrivial p -element. By Proposition 1 we may assume x is a p -element of G . Since $x \in \bar{U}$ there is a net $\{x_\alpha\}$ of elements of U which converges to x . By Proposition 2, the net $\{x_\alpha^{(1,0)}\}$ also converges to x . Clearly, $x_\alpha \in U$ then $x_\alpha^{(1,0)} \in U$ by the strong completeness of $\overline{\langle x_\alpha \rangle}$. Hence the net $\{x_\alpha^{(1,0)}\}$ is a net of p -elements in U which converge to a p -element x not in U . By hypothesis and Lemma 1, the set of p -element of U is closed in G . Hence x must be a p -element of U , contradiction. □

Proof of Corollary 1 to Theorem 1. Finitely generated pro- p -groups are strongly complete, [1], [6]. Hence if $U \trianglelefteq G$, U large then $U \cap G_p$ is large in G_p and so open. Therefore the theorem applies.

The above corollary is another proof of the result due to Oltikar and Ribes, [5], that finitely generated prosupersolvable groups are strongly complete since in the same paper they prove that such groups have finitely generated p -Sylow subgroups.

2. In this section we first show that the completed group algebra $\hat{Z}[[G]]$ (which we denote by Δ) for a finitely generated profinite group, G , is in some sense strongly complete. Let $\text{Mod}(G)$ be the category of G -modules, G considered as an abstract group.

PROPOSITION 3. *Let G be a finitely generated profinite group, $A \leq \Delta$ such that Δ/A is finite and $A \in \text{Mod}(G)$. Then A is open in the topology of Δ .*

Before proving Proposition 3 we first review the topological structure of Δ .

$$\Delta \simeq \varinjlim_{n, U \text{ open}} \mathbf{Z}/n\mathbf{Z}(G/U).$$

A neighborhood base of (0) consists of the kernels, $\pi_{n,U}$ of the continuous morphisms $\Delta \rightarrow \mathbf{Z}/n\mathbf{Z}(G/U)$. In [2], Brummer notes that $\pi_{n,U}$ is the closed ideal generated by $\{(u-1) \mid u \in U\}$. In fact, as a pseudocompact Δ -module, $\pi_{n,U}$ is precisely $n\Delta + \sum \Delta(u_i - 1)$ where $\{u_i\}$ is a set of topological generators of U . Therefore if G and hence U is finitely generated $I_{n,U}$ is a finitely generated pseudocompact Δ -module.

Proof of Proposition 3. Since Δ/A is finite, there exists n such that $n\Delta \leq A$. As well, Δ/A is trivial U -action for some large but not necessarily open subgroup U of G . However U contains the topological generators $\{u_1, \dots, u_s\}$ of \bar{U} , its closure in G . In this case $I_{n,\bar{U}} = n\Delta + \sum_{i=1}^s \Delta(u_i - 1) = n\Delta + \sum_{i=1}^s \Delta(u_i - 1)$ and since clearly $B = \sum_{i=1}^s \Delta(u_i - 1) \leq A$ one has $I_{n,\bar{U}} \leq A$ which implies A is open as well. □

The category of pseudocompact Δ -modules, PC_Δ^p , is studied by Brummer, [2], and in the thesis of Gabriel. These modules are inverse limits of finite discrete G -modules with the corresponding profinite topology. If $M \in PC_\Delta^p$ and M is (topologically) finitely generated then M is the continuous homomorphic image of $\bigoplus^m \Delta$, for some finite m .

COROLLARY 1. *Let G be a finitely generated profinite group, $M \in PC_\Delta^p$, M finitely generated. If $A \leq M$ such that M/A is finite and $A \in \text{Mod}(G)$, then A is open in M .*

Proof. If $\pi: \bigoplus^m \Delta \rightarrow M$ is defined, which is the case for M finitely generated by at most m elements, then one easily shows $\pi^{-1}(A)$ open in $\bigoplus^m \Delta$ and hence A is open in M . □

We now prove Theorem 2.

Proof of Theorem 2. If U is a large normal subgroup of E but not necessarily open, its image in G is open since G is strongly complete and $U \cap A$ is open in A by Corollary 1 to Proposition 3 since $U \cap A$ is preserved under the action of G and hence belongs to $\text{Mod}(G)$.

Consider the following commutative diagram of profinite groups

$$\begin{array}{ccccc} A & \twoheadrightarrow & E & \xrightarrow{\pi} & G \\ \downarrow & & \rho \downarrow & & \downarrow \\ A/U \cap A & \longrightarrow & E/U \cap A & \xrightarrow{\pi_1} & G. \end{array}$$

Clearly $\rho^{-1}(\rho(U)) = U$ and ρ is continuous so it suffices to show $\rho(U)$ is open or closed in $E/U \cap A$.

However, π_1 is a monomorphism when restricted to $\rho(U)$ and $\pi_1 \circ \rho(U)$ is open in G . Therefore, restricted to $\pi_1 \circ \rho(U)$, π_1 has an inverse π_1^{-1} , such that $\pi_1 \circ \pi_1^{-1} = 1_{\pi_1 \circ \rho(U)}$ and $\pi_1^{-1} \circ \pi_1 = 1_{\rho(U)}$. Hence there is a topology which we can place on $\rho(U)$ to make it a profinite group. Namely, $V \leq \rho(U)$ is open iff $\pi_1(V)$ is open in G . But this is clearly the original relative topology on $\rho(U)$. We argue as follows: Let $V \leq E/A \cap U$ be open in $E/A \cap U$. Then $\pi_1(V \cap \rho(U))$ is open in G . Hence $V \cap \rho(U)$ is open in $\rho(U)$ equipped with its profinite topology.

Hence the profinite topology of $\rho(U)$ is finer than its relative topology. Conversely, if the profinite topology is properly finer then we extend this topology to a profinite topology on $E/A \cap U$. Hence $E/A \cap U$ can be equipped with two profinite topologies, one coarser than the other and this is impossible. Hence the two topologies on $\rho(U)$ are identical so that $\rho(U)$ is closed in $E/A \cap U$ since it is compact. Hence the result. \square

Proof of Corollary 1 to Theorem 2. The profinite group, E , of Theorem 2 is finitely generated. By the Theorem, E/A_1 is strongly complete. By induction, if E/A_i is strongly complete then the short exact sequence $A_i/A_{i+1} \twoheadrightarrow E/A_{i+1} \rightarrow E/A_i$ shows E/A_{i+1} to be strongly complete. Hence, by induction, the corollary holds. \square

Finally we notice that in the case $A \twoheadrightarrow E \rightarrow G$ verifies the hypothesis of Corollary 1 to Theorem 2 then E is finitely generated.

PROPOSITION 4. *Let $A \twoheadrightarrow E \rightarrow G$ be a split short exact sequence of profinite groups where E is generated by n elements and A is abelian. Then A is a pseudocompact Δ -module generated by n elements.*

Proof. A similar results is proved by Hartley, [3, Lemma 5] for finite groups and easily carries over to profinite groups. \square

COROLLARY 2 TO THEOREM 2. *If $A \twoheadrightarrow E \rightarrow G$ is a split short exact sequence of profinite groups such that E is finitely generated, A is abelian and G is strongly complete, then E is strongly complete.*

Proof. Proposition 4 allows us to say A is a finitely generated pseudocompact G -module and so we may apply the theorem. \square

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