STRONG COMPLETENESS IN PROFINITE GROUPS

Andrew Pletch
A profinite group is strongly complete if every subgroup of finite index is open. In this paper it is shown that a profinite group with finitely generated p-Sylow subgroups is strongly complete and that if $G$ is a finitely generated strongly complete profinite group and $A$ is a finitely generated pseudocompact $G$-module then any extension of $A$ by $G$ is strongly complete.

The purpose of this paper is to extend some results of Anderson [1] in the theory of strong completeness of profinite groups. A profinite group is a topological group whose topology is Hausdorff, compact and has neighborhood base of the identity consisting of certain subgroups of finite index. A profinite group is strongly complete if every subgroups of finite index is open. Since all open subgroups are also closed, a strongly complete profinite group has no dense subgroups of finite index except itself.

Our first result is:

**Theorem 1.** Let $G$ be a profinite group, $G_p$ a p-Sylow subgroup, $U \leq G$ with $(G: U) = n < \infty$. $U$ is open in $G$ if and only if $U \cap G_p$ is open in $G_p$ for every prime $p$ which divides $n$.

**Corollary 1.** Let $G$ be a profinite group all of whose p-Sylow subgroups are finitely generated. Then $G$ is strongly complete.

Our second result is:

**Theorem 2.** Let $A \rightarrow E \rightarrow G$ be a short exact sequence of profinite groups. If $G$ is a finitely generated strongly complete profinite group and $A$ is a finitely generated pseudocompact $\hat{Z}[[G]]$-module then $E$ is strongly complete.

**Corollary 1.** Let $A \rightarrow E \rightarrow G$ be a short exact sequence of profinite groups where $G$ is as in the theorem and $A$ contains a finite sequence of subgroups which are normal in $E$: $A = A_0 \geq A_1 \geq \cdots \geq A_n = (e)$ such that $A_i/A_{i+1}$ is a finitely generated pseudocompact $\hat{Z}[[G]]$-module for $i = 0, \cdots, n - 1$. Then $E$ is strongly complete.

In this paper all groups are profinite, all subgroups are closed, and all homomorphisms are continuous unless otherwise stated. We
1. For any group, $G$, $x \in G$, the closed subgroup generated by $x$, $\langle x \rangle$, is cyclic and so there is a continuous homomorphism $\rho: \hat{Z} \to \langle x \rangle$ defined by $\rho(\lambda) = x^i$. Writing $\hat{Z}$ as $\prod_p \hat{\mathbb{Z}}_p$, the product over all primes $p$ of $p$-adic integers, and then as $\hat{\mathbb{Z}}_p \times \prod_{q \neq p} \hat{\mathbb{Z}}_q$ and allowing the generator of $\hat{\mathbb{Z}}_p \times \langle 0 \rangle$ to be $(1, 0)$ and the generator of $\langle 0 \rangle \times \prod_{q \neq p} \hat{\mathbb{Z}}_q$ to be $(0, 1)$ one sees that $\langle x^{(1,0)} \rangle$ is the $p$-Sylow subgroup of $\langle x \rangle$ and $\langle x^{(0,1)} \rangle$ its $p$-complement. Finitely generated pro-abelian groups are known to be strongly complete. Hence any homomorphism from $\langle x \rangle$ to a finite group is continuous. With this we prove:

**Proposition 1.** Let $U$ be a large normal subgroup of $G$, $U$ not necessarily open, $x \in G$ such that $\overline{x} \in (G/U)_p$, $p$-Sylow subgroup of $G/U$. Then $x^{(1,0)} = \overline{x}$ in $G/U$.

**Proof.** The morphism $\langle x \rangle$ to $\langle \overline{x} \rangle \leq G/U$ is continuous as we have noted. $\langle \overline{x} \rangle$ is a finite cyclic $p$-group. Since $x = x^{(1,0)} \cdot x^{(0,1)}$ and $x^{(0,1)}$ is an element of $G$ whose order is prime to $p$, its image $\langle \overline{x} \rangle$ is the identity. Hence $\overline{x} = x^{(1,0)} \cdot x^{(0,1)} = x^{(1,0)} \cdot x^{(0,1)} = x^{(1,0)}$.

We call an element of $G$ a $p$-element if it belongs to some $p$-Sylow subgroup of $G$. For all $x$ in $G$, $x^{(1,0)}$ is a $p$-element and $x$ is a $p$-element if and only if $x = x^{(1,0)}$ (see [4]).

A net of elements $\{x_\alpha\}$ of a profinite group $G$ converges to an element $x$ if for all open normal subgroups $V$ of $G$, $x_\alpha V = x V$ for almost all $\alpha$.

**Proposition 2.** Let $\{x_\alpha\}$ be a net in $G$ converging to a $p$-element $x$. Then $\{x_\alpha^{(1,0)}\}$ is a net of $p$-elements which also converges to $x$.

**Proof.** If $x$ is a $p$-element then for any open normal subgroup $V$ of $G$, $x V$ is a $p$-element in $G/V$. By Proposition 1, $x_\alpha V = x_\alpha^{(1,0)} V$ if $x_\alpha V = x V$. The set $\{x^{(1,0)}_\alpha\}$ is clearly a net and hence the result.

Before proving Theorem 1 we need the following lemma.

**Lemma 1.** Let $U \trianglelefteq G$, $U$ not necessarily closed, such that for some $p$-Sylow subgroup $G_\circ$, $G \cap G_\circ$ is closed in $G_\circ$. The set of all $p$-elements in $U$ is closed in $G$. 
Proof. Let $U_p = U \cap G_p$. The set of all $p$-elements in $U$ is

$$\bigcup_{x \in G} U \cap G_p^x = \bigcup_{x \in G_p} U_p^x$$

since $U$ is normal. Consider the function $U_p \times G \to G$ defined by $(u, g) \to g^{-1}ug$. Since $U_p$ is closed in $G_p$, it is compact and hence the function, which is easily continuous, is a closed function. Its image, which is precisely the set of $p$-elements of $U$, is therefore closed in $G$.

Proof of Theorem 1. Let $U \leq G$ of finite index. If $U$ is open then $U \cap G_p$ is open in $G_p$ for all $G_p$. Conversely suppose there exists large $U$ not open, the quotient group $\bar{U}/U$ has a nontrivial $p$-Sylow subgroup for some prime $p$. Hence there exists $x \in U$ such that $\bar{e} \neq \bar{x} \in \bar{U}/U$ is a nontrivial $p$-element. By Proposition 1 we may assume $x$ is a $p$-element of $G$. Since $x \in \bar{U}$ there is a net $\{x_a\}$ of elements of $U$ which converges to $x$. By Proposition 2, the net $\{x_a^{(1,0)}\}$ also converges to $x$. Clearly, $x_a \in U$ then $x_a^{(1,0)} \in U$ by the strong completeness of $\langle x_a \rangle$. Hence the net $\{x_a^{(1,0)}\}$ is a net of $p$-elements in $U$ which converge to a $p$-element $x$ not in $U$. By hypothesis and Lemma 1, the set of $p$-element of $U$ is closed in $G$. Hence $x$ must be a $p$-element of $U$, contradiction.

Proof of Corollary 1 to Theorem 1. Finitely generated pro-$p$-groups are strongly complete, [1], [6]. Hence if $U \leq G$, $U$ large then $U \cap G_p$ is large in $G_p$ and so open. Therefore the theorem applies.

The above corollary is another proof of the result due to Oltikar and Ribes, [5], that finitely generated prosupersolvable groups are strongly complete since in the same paper they prove that such groups have finitely generated $p$-Sylow subgroups.

2. In this section we first show that the completed group algebra $\hat{Z}[[G]]$ (which we denote by $\Delta$) for a finitely generated profinite group, $G$, is in some sense strongly complete. Let $\text{Mod}(G)$ be the category of $G$-modules, $G$ considered as an abstract group.

**Proposition 3.** Let $G$ be a finitely generated profinite group, $A \leq \Delta$ such that $\Delta/A$ is finite and $A \in \text{Mod}(G)$. Then $A$ is open in the topology of $\Delta$.

Before proving Proposition 3 we first review the topological structure of $\Delta$. 

\[ \Delta \simeq \lim_{n,U \text{ open}} Z/nZ(G/U) . \]

A neighborhood base of \( (0) \) consists of the kernals, \( \pi_{n,U} \) of the continuous morphisms \( \Delta \to Z/nZ(G/U) \). In [2], Brummer notes that \( \pi_{n,U} \) is the closed ideal generated by \( \{ (u - 1) \mid u \in U \} \). In fact, as a pseudocompact \( \Delta \)-module, \( \pi_{n,U} \) is precisely \( n \Delta + \sum_{i=1}^{r} \Delta(u_i - 1) \) where \( \{u_i\} \) is a set of topological generators of \( U \). Therefore if \( G \) and hence \( U \) is finitely generated \( I_{n,U} \) is a finitely generated pseudocompact \( \Delta \)-module.

**Proof of Proposition 3.** Since \( \Delta/A \) is finite, there exists \( n \) such that \( n \Delta \leq A \). As well, \( \Delta/A \) is trivial \( U \)-action for some large but not necessarily open subgroup \( U \) of \( G \). However \( U \) contains the topological generators \( \{ u_1, \ldots, u_r \} \) of \( \bar{U} \), its closure in \( G \). In this case \( I_{n,U} = n \Delta + \sum_{i=1}^{r} \Delta(u_i - 1) = n \Delta + \sum_{i=1}^{r} \Delta(u_i - 1) \) and since clearly \( B = \sum_{i=1}^{r} \Delta(u_i - 1) \leq A \) one has \( I_{n,U} \leq A \) which implies \( A \) is open as well. \( \square \)

The category of pseudocompact \( \Delta \)-modules, \( PC_{\Delta} \), is studied by Brummer, [2], and in the thesis of Gabriel. These modules are inverse limits of finite discrete \( G \)-modules with the corresponding profinite topology. If \( M \in PC_{\Delta} \) and \( M \) is (topologically) finitely generated then \( M \) is the continuous homomorphic image of \( \bigoplus^n \Delta \), for some finite \( m \).

**Corollary 1.** Let \( G \) be a finitely generated profinite group, \( M \in PC_{\Delta} \), \( M \) finitely generated. If \( A \leq M \) such that \( M/A \) is finite and \( A \in \text{Mod}(G) \), then \( A \) is open in \( M \).

**Proof.** If \( \pi: \bigoplus^n \Delta \to M \) is defined, which is the case for \( M \) finitely generated by at most \( m \) elements, then one easily shows \( \pi^{-1}(A) \) open in \( \bigoplus^n \Delta \) and hence \( A \) is open in \( M \). \( \square \)

We now prove Theorem 2.

**Proof of Theorem 2.** If \( U \) is a large normal subgroup of \( E \) but not necessarily open, its image in \( G \) is open since \( G \) is strongly complete and \( U \cap A \) is open in \( A \) by Corollary 1 to Proposition 3 since \( U \cap A \) is preserved under the action of \( G \) and hence belongs to \( \text{Mod}(G) \).

Consider the following commutative diagram of profinite groups

\[
\begin{array}{ccc}
A & \to & E \\
\downarrow & & \downarrow \rho \\
A/U \cap A & \to & E/U \cap A
\end{array}
\]

\[
\begin{array}{ccc}
& & \pi \\
& & \downarrow \pi_1 \\
& & G
\end{array}
\]

\[
\begin{array}{ccc}
A & \to & E \\
\downarrow & & \downarrow \rho \\
A/U \cap A & \to & E/U \cap A
\end{array}
\]

\[
\begin{array}{ccc}
& & \pi \\
& & \downarrow \pi_1 \\
& & G
\end{array}
\]
Clearly $\rho^{-1}(\rho(U)) = U$ and $\rho$ is continuous so it suffices to show $\rho(U)$ is open or closed in $E/U \cap A$.

However, $\pi_1$ is a monomorphism when restricted to $\rho(U)$ and $\pi_1 \circ \rho(U)$ is open in $G$. Therefore, restricted to $\pi_1 \circ \rho(U)$, $\pi_1$ has an inverse $\pi_1^{-1}$, such that $\pi_1 \circ \pi_1^{-1} = 1_{\pi_1 \circ \rho(U)}$ and $\pi_1^{-1} \circ \pi_1 = 1_{\rho(U)}$. Hence there is a topology which we can place on $\rho(U)$ to make it a profinite group. Namely, $V \leq \rho(U)$ is open iff $\pi_1(V)$ is open in $G$. But this is clearly the original relative topology on $\rho(U)$. We argue as follows: Let $V \leq E/A \cap U$ be open in $E/A \cap U$. Then $\pi_1(V \cap \rho(U))$ is open in $G$. Hence $V \cap \rho(U)$ is open in $\rho(U)$ equipped with its profinite topology.

Hence the profinite topology of $\rho(U)$ is finer than its relative topology. Conversely, if the profinite topology is properly finer then we extend this topology to a profinite topology on $E/A \cap U$. Hence $E/A \cap U$ can be equipped with two profinite topologies, one coarser than the other and this is impossible. Hence the two topologies on $\rho(U)$ are identical so that $\rho(U)$ is closed in $E/A \cap U$ since it is compact. Hence the result.

**Proof of Corollary 1 to Theorem 2.** The profinite group, $E$, of Theorem 2 is finitely generated. By the Theorem, $E/A_i$ is strongly complete. By induction, if $E/A_i$ is strongly complete then the short exact sequence $A_i/A_{i+1} \rightarrow E/A_{i+1} \rightarrow E/A_i$ shows $E/A_{i+1}$ to be strongly complete. Hence, by induction, the corollary holds.

Finally we notice that in the case $A \rightarrow E \rightarrow G$ verifies the hypothesis of Corollary 1 to Theorem 2 then $E$ is finitely generated.

**Proposition 4.** Let $A \rightarrow E \rightarrow G$ be a split short exact sequence of profinite groups where $E$ is generated by $n$ elements and $A$ is abelian. Then $A$ is a pseudocompact $A$-module generated by $n$ elements.

**Proof.** A similar results is proved by Hartley, [3, Lemma 5] for finite groups and easily carries over to profinite groups.

**Corollary 2 to Theorem 2.** If $A \rightarrow E \rightarrow G$ is a split short exact sequence of profinite groups such that $E$ is finitely generated, $A$ is abelian and $G$ is strongly complete, then $E$ is strongly complete.

**Proof.** Proposition 4 allows us to say $A$ is a finitely generated pseudocompact $G$-module and so we may apply the theorem.
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Instituto de Matematica e Estatistica
Universidade de Sao Paulo
Sao Paulo, Brasil

Current address: Depto Mathematicas
Universidad de Puerto Rico
Mayaguez, PR 00708
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