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INVERTING DOUBLE KNOTS

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We disprove J. Montesinos's conjecture that every invertible knot in S^3 is strongly invertible.

Let K denote a tame, oriented knot in S^3 , and fix an orientation of S^3 . If there exists an orientation-preserving, PL involution of S^3 that inverts K , then K is *strongly invertible*. J. Montesinos proposed this definition [5], and he has conjectured [3; Problem 1.6, p. 277] that *every* invertible knot is strongly invertible. In this paper, we disprove this conjecture; our results are as follows.

THEOREM 1. *A knot K is strongly invertible if and only if each double of K is strongly invertible.*

COROLLARY. *No double of a noninvertible knot is strongly invertible; hence, there exist invertible knots that are not strongly invertible.*

Proof of Corollary. Any double knot is invertible [6; Theorem 1, p. 235].

THEOREM 2. *If L is a strongly invertible knot with exactly one maximal companion C_L , then C_L is also strongly invertible.*

Section 1 contains a preliminary lemma. We prove Theorems 1 and 2, in §2. In §3, we give a counterexample to the converse of Theorem 2; in §4, we discuss surgery on invertible knots, give several examples, and formulate a conjecture.

I wish to thank C. Gordon and K. Murasugi for discussing this work with me.

1. Preliminaries. In this paper, all spaces are polyhedrons; the three-sphere has a fixed orientation; all maps are piecewise linear; all submanifolds, polyhedral; and all knots, oriented. We shall need the following lemma.

POSITIONING LEMMA. *If K is a strongly invertible knot, then there exists an orientation-preserving, K -inverting, PL involution $\rho: S^3 \rightarrow S^3$ with nonempty, fixed point set A and there exists a polyhedral, ρ -invariant 2-sphere S such that (a) the "axis" A of ρ belongs to S , (b) the set $K \cap A$ contains exactly two points, (c) the knot K is transverse with respect to the 2-sphere S , and (d) the set $K \cap S$ contains only a finite number of points.*

Proof. Because K is strongly invertible, there exists an orientation-preserving, K -inverting, PL involution $\rho': S^3 \rightarrow S^3$. The Lefschetz fixed-point theorem implies that $\rho'|K$ has exactly two fixed points. Because the fixed-point set (or *axis*) A' of ρ' is, therefore, nonempty, the axis A' must be a knot [7; Theorem, p. 162]. If we set $S^3 = R^3 \cup \{\infty\}$, then, because ρ' is piecewise linear and orientation-preserving, there exists a PL autohomeomorphism φ of S^3 such that $\varphi\rho'\varphi^{-1}(=\rho_0)$ is a "standard" (orthogonal) 180° -rotation and such that $\varphi(A') = (y\text{-axis}) \cup \{\infty\}(=A_0)$ [10].

Because $\varphi(K)$ is polyhedral, because the rotation ρ_0 takes $\varphi(K)$ onto itself, and because $\varphi(K)$ meets A_0 in only two points, x_1 and x_2 , we can (if necessary) find a small angle $\alpha^\circ(\alpha > 0)$ such that an α° -rotation ρ_α about A_0 takes $\varphi(K)$ to a knot $\rho_\alpha\varphi(K)$ that is transverse to the 2-sphere $S_0(=(yz\text{-plane}) \cup \{\infty\})$ at each of the points, x_1 and x_2 . We shall find a knot K' (ambient isotopic to $\rho_\alpha\varphi(K)$) such that K' , the involution ρ_0 (of S^3) with axis A_0 , and the 2-sphere S_0 satisfy the hypothesis and the conclusion of the lemma. The lemma's proof will easily follow.

Choose $\varepsilon(>0)$ so that (closed) ε -neighborhood V_ε of $\rho_\alpha\varphi(K)$ is a solid torus; such a choice is possible, because $\rho_\alpha\varphi(K)$ is polyhedral in S^3 . Because $\rho_\alpha\varphi(K)$ is transverse to S_0 at x_1 and at x_2 , we can restrict ε so that $V_\varepsilon \cap S_0$ contains (among other things) two disjoint meridional disks, E_1 and E_2 , of V_ε , with $E_i \cap \rho_\alpha\varphi(K) = \{x_i\}(i = 1, 2)$. By a final restriction of ε , we can assume that $V_\varepsilon \cap A_0 = (E_1 \cup E_2) \cap A_0(=\text{two, disjoint arcs})$. (The constructions involved in our restrictions of ε are standard, and we shall omit them.) Finally, note that $\rho_0(V_\varepsilon) = V_\varepsilon$.

The points x_1 and x_2 divide $\rho_\alpha\varphi(K)$ into two (closed) arcs, k_1 and k_2 ; the disks E_1 and E_2 divide V_ε into (closed) 3-cells, B_1 and B_2 , with k_i unknotted in $B_i(i = 1, 2)$ (see [4; p. 134]). We note that $\rho_0(B_1) = B_2$ and that $B_i \cap A_0 = (E_1 \cup E_2) \cap A_0$.

Keeping x_1 and x_2 fixed, we now put k_1 in general position with respect to S_0 by an orientation-preserving autohomeomorphism $h_1: S^3 \rightarrow S^3$ moving each point of k_1 less than ε . We can evidently assume that $h_1|(S^3 - \text{Int } B_1)$ is the identity map.

The arc $\rho_0 h_1(k_1)$ is clearly unknotted in B_2 . Hence, there exists an orientation-preserving autohomeomorphism $h_2: S^3 \rightarrow S^3$ taking k_2 onto $\rho_0 h_1(k_1)$ and leaving each point of $S^3 - \text{Int } B_2$ fixed. The autohomeomorphism h of S^3 given by

$$h(x) = \begin{cases} h_i(x), & \text{if } x \in \text{Int } B_i(i = 1, 2) \\ x, & \text{otherwise,} \end{cases}$$

preserves the orientation of S^3 and takes $\rho_\alpha\varphi(K)$ onto a knot $h_1(k_1) \cup$

$\rho_0 h_1(k_1)$ that is in general position with respect to S_0 and that is strongly inverted by ρ_0 . We set $K' = h\rho_\alpha\varphi(K)(=h_1(k_1) \cup \rho_0 h_1(k_1))$ and note that the knot K' , the involution ρ_0 with axis A_0 , and the 2-sphere S_0 satisfy the hypothesis and conclusion of the lemma. The proof of the lemma now follows by taking $\rho = (h\rho_\alpha\varphi)^{-1}\rho_0(h\rho_\alpha\varphi)$, taking $A = A' (= \varphi^{-1}(A_0) = (h\rho_\alpha\varphi)^{-1}(A_0))$, and taking $S = (h\rho_\alpha\varphi)^{-1}(S_0)$.

2. Proofs.

Proof of Theorem 1. We shall assume that K is not trivial, for otherwise, the theorem is evidently true.

(1) *Necessity.* We assume that K is strongly invertible. Let ρ , and A , and S denote the objects our Positioning Lemma guarantees, and let $K \cap A = \{x_1, x_2\}$. By the Positioning Lemma's proof, we can assume (without loss of generality) that ρ is the 180° -rotation about $A(=(y\text{-axis}) \cup \{\infty\})$ and that $S = (yz\text{-plane}) \cup \{\infty\}$. Moreover, we can choose $\varepsilon(>0)$ and V_ε exactly as in the lemma's proof. We have $K = k_1 \cup k_2$ (with $\rho(k_1) = k_2$) and $V_\varepsilon = B_1 \cup B_2$; moreover, $E_i \cap A(i = 1, 2)$ is a properly imbedded arc in E_i .

Let C denote a cylindrical 3-cell with core k and with two disks, D_1 and D_2 , meeting in an arc and imbedded in C , as shown in Figure 1. Let v be a (closed) arc in $\text{Int}(E_2 \cap A)$ such that $x_2 \in \text{Int}(v)$ (see Figure 2(a)). It is easy to find an arc $v_1 \subset \text{Int} E_1$ such that $v_1 \cap A = \{x_1\} = v_1 \cap \rho(v_1) = A \cap \text{Int} v_1$; note that $\rho(v_1) \subset \text{Int} E_1$ (see Figure 2(b)).

Now, let $g: C \rightarrow B_1$ be a homeomorphism such that $g(E_i) = E_i(i = 1, 2)$, such that $g(v'_1) = v_1$ and $g(v'_2) = \rho(v_1)$, such that $g(v') = v$, and such that $g(k) = k_1$. Then $[g(D_1 \cup D_2)] \cup [\rho g(D_1 \cup D_2)]$ is a singular disk Σ with one clasping singularity, and the $\partial\Sigma$ is a double of K with twisting number σ (an integer depending on the homeomorphism $g: C \rightarrow B_1$) and with self-intersection number $\eta(= \pm 2)$. (By changing g (to change σ) and by replacing C with its mirror image (to change the sign of η), we can assume that $\partial\Sigma$ is any double of K that we desire.) Evidently, $\partial\Sigma$ is strongly invertible (by the involution ρ). This completes the proof of the necessity.

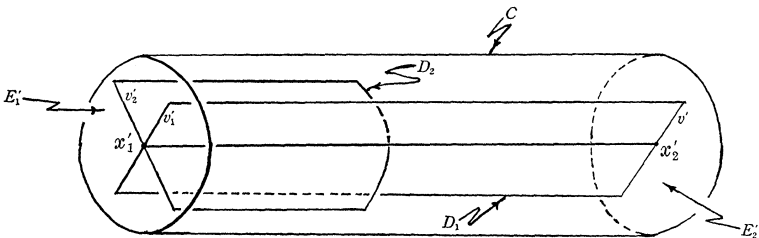


FIGURE 1

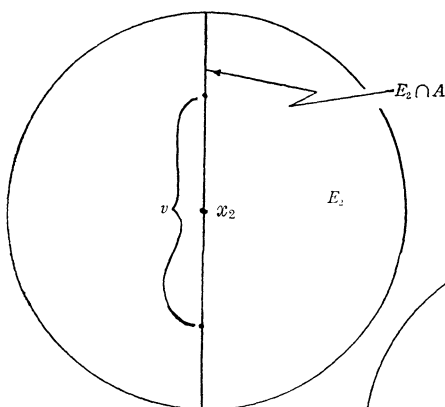


FIGURE 2(a)

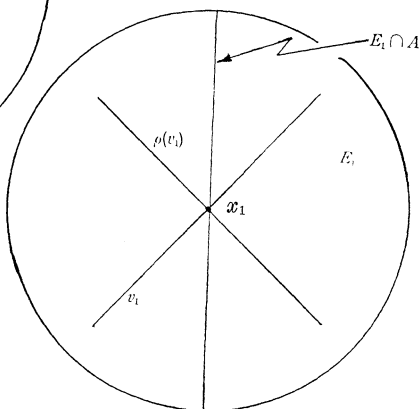


FIGURE 2(b)

(2) *Sufficiency.* We assume that some double, D_K , of K is strongly invertible. Replace K by D_K in the Positioning Lemma; we can assume that ρ is the standard rotation (of period 2) about $A = (y\text{-axis}) \cup \{\infty\}$, that $S = (yz\text{-plane}) \cup \{\infty\}$, and that $D_K \cap A = \{x_1, x_2\}$.

Let V^* denote a (closed) regular neighborhood of a clasping disk whose boundary is D_K ; note that K is equivalent to a core of V^* [6; p. 238]. Now K is a unique maximal companion D_K [6; p. 242]; that is, any companion of D_K , other than K , is also a companion of K . Hence, the torus $\rho(\partial V^*)$ is ambient isotopic to ∂V^* in $S^3 - D_K$. So, by [9; Theorem 1, p. 223], the ∂V^* is ambient isotopic (in $S^3 - D_K$) to a torus T in general position with respect to A , and either $\rho(T) \cap T = \emptyset$ or $\rho(T) = T$. If $\rho(T) \cap T = \emptyset$, then T and $\rho(T)$ are parallel. Because $\rho^2(T) = T$ and because each of $\rho(T)$ and T separates $S^3 - D_K$, it easily follows that ρ moves fixed points of itself, which is absurd. Thus, $\rho(T) = T$.

Now T splits S^3 into a solid torus V (containing D_K in its interior) and a K -knot manifold. If $A \cap T = \emptyset$, then $A \subset \text{Int} V$, because $A \cap D_K \neq \emptyset$. Because K is knotted and A is unknotted, A belongs to a polyhedral 3-cell $\subset \text{Int} V$; otherwise, A would have a companion, which it does not [6]. Applying Tollefson's lemma [8; Lemma 1, p. 141], we can find a 2-sphere $S' \subset \text{Int}(V - A)$ such that S' bounds no 3-cell in $V - A$ and such that either $\rho(S') \cap S' = \emptyset$ or $\rho(S') = S'$. As with the tori T and $\rho(T)$ in the preceding paragraph, we cannot

have $\rho(S') \cap S' = \emptyset$. If $\rho(S') = S'$, then take the 3-cell $B^3(\subset S^3)$ that does not contain A and that S' bounds (in S^3), and consider the homeomorphism $\rho|B: B \rightarrow B$. By the Brouwer fixed-point theorem, $\rho|B$ has a fixed point, and so ρ has a fixed point not on A (which it does not). Hence, $A \cap T \neq \emptyset$.

Because T is in general position with respect to A , the cardinality b of $A \cap T$ is finite. Let T_0 denote the orbit space of $\rho|T$. The projection $p: T \rightarrow T_0$ is a branched covering, and the two Euler characteristics, $\chi(T)$ and $\chi(T_0)$, are related by the Riemann-Hurwitz branch-point formula,

$$\chi(T) = 2\chi(T_0) - b;$$

see [1; p, 93]. But $\chi(T) = 0$ and $b > 0$. Hence, $\chi(T_0) = 2$, and so T_0 is a 2-sphere and $b = 4$. (Because the orbit space of ρ is S^3 and because S^3 contains no projective planes, we cannot have $\chi(T_0) = 1$.)

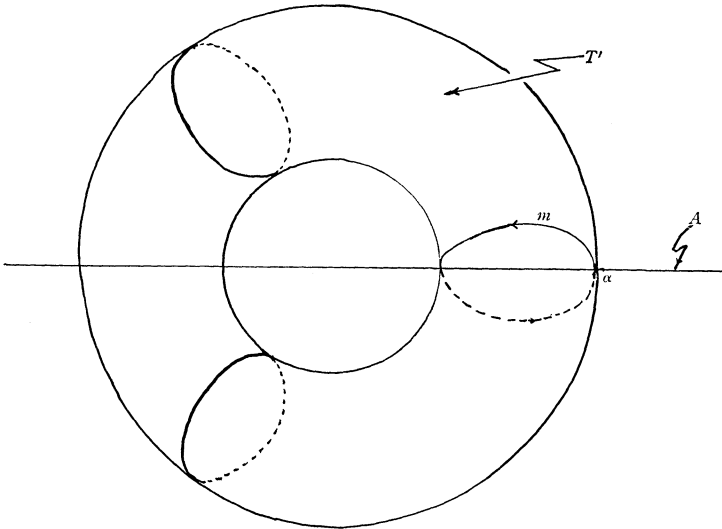


FIGURE 3

Now let T' denote the torus $(r - 2)^2 + z^2 = 1$ (see Figure 3), let m denote the curve $\{(r, z) | \theta = 0 \text{ and } (r - 2)^2 + z^2 = 1\}$ (which we shall take as one of the two components of $T' \cap S$), and let $K_{(\bar{p}, \bar{q})}((\bar{p}, \bar{q}) = 1)$ denote the torus knot $\{(r, z) | r = 2 + \cos(\bar{p}\theta/\bar{q}), z = \sin(\bar{p}\theta/\bar{q})\}$ on T' (cf. [2; p. 92]). To fix the (r, θ, z) -coordinate system on T' , let the point α shown in Figure 3 have (r, θ, z) -coordinates $(3, 0, 0)$. Note that $\rho(T') = T'$ and that $\rho(m) = m^{-1}$ (after we have oriented m). If T'_0 denotes the orbit space of $\rho|T'$, then the projection $p': T' \rightarrow T'_0$ is a branched covering. As with $p: T \rightarrow T_0$, the covering p' has four branch points, and T'_0 is a 2-sphere.

According to [1; Theorem 3.4, p. 94], the coverings p and p' are equivalent; that is, there exist homeomorphisms $\psi: T \rightarrow T'$ and $\gamma: S^2 \rightarrow S^2$ such that $p'\psi = \gamma\rho$. It follows easily that φ preserves covering fibers.

Thus, if $\{x, \rho(x)\}$ is a fiber of p , then $\{\psi(x), \psi\rho(x)\}$ is a fiber of p' , and so $(\rho|T')\psi(\rho|T)(x) = \psi(x)$; that is, $\psi = (\rho|T')\psi(\rho|T)$. Because $\rho^2 = id.$, we have $(\rho|T')\psi = \psi(\rho|T)$. Notice that $\rho(K_{(\bar{p}, \bar{q})}) = K_{(\bar{p}^{-1}, \bar{q})}$; thus, for any (\bar{p}, \bar{q}) -torus knot, there exists a representative, $K_{(\bar{p}, \bar{q})}$, of it on T' that ρ inverts (and, hence, *strongly* inverts).

If λ is an (oriented) longitude of K on T , then $\psi(\lambda)$ is isotopic on T' to m or to one of the torus knots $K_{(\bar{p}_1, \bar{q}_1)}$, for some pair (\bar{p}_1, \bar{q}_1) . Thus, either $\psi^{-1}(m)$ or $\psi^{-1}(K_{(\bar{p}_1, \bar{q}_1)})$ is a longitude of T meeting the axis A of p in exactly two points, because ψ maps branch points of p to branch points of p' . Because $(\rho|T')\psi = \psi(\rho|T)$, we have either $\rho(\psi^{-1}(m)) = \psi^{-1}(\rho|T')(m) = \psi^{-1}(m^{-1}) = [\psi^{-1}(m)]^{-1}$ or, similarly, $\rho(\psi^{-1}(K_{(\bar{p}_1, \bar{q}_1)})) = [\psi^{-1}(K_{(\bar{p}_1, \bar{q}_1)})]^{-1}$. Therefore, ρ strongly inverts a longitude of K , and it follows that K itself is strongly invertible.

Proof of Theorem 2. We need only note that, in the proof of Theorem 1, the sufficiency portion depends on the uniqueness of the maximal companion K of D_K and not on the knot type of D_K .

3. A counterexample. The noninvertible knot \mathcal{K} in [11; Figure 3, p. 1275] is a counterexample to the converse of Theorem 2. Because the knots 3_1 and 5_1 (of the Alexander-Briggs table) are simple, one can apply Schubert's theorem [6; p. 216] to show that \mathcal{K} has exactly one maximal companion, which is a trefoil knot and, hence, strongly invertible; details of the application are routine, and we shall omit them.

4. A conjecture. A link L in S^3 is *strongly invertible*, if there exists an orientation-preserving PL involution of S^3 that inverts each component of L . In [5, Theorem 1, p. 231], Montesinos proved that any 3-manifold derived from surgery on a strongly invertible link is a 2-fold cyclic covering space of S^3 branched over a link and, conversely, that one can produce any particular 2-fold branched cyclic covering space of S^3 by surgery on a suitable, strongly invertible link. I do not know whether nontrivial surgery on a *knot that is not strongly invertible* will produce a 2-fold branched cyclic covering space of S^3 . It is, however, a different story for links. Here are some examples.

F. González-Acuña and J. Montesinos gave the first such examples (unpublished). Assign any rational coefficient to the component K_1 of the unsplitable and noninvertible Borromean rings, $K_1 \cup K_2 \cup K_3$

[5]. Take nonzero integers, a and b , and assign the coefficient $1/a$ to K_2 and the coefficient $1/b$ to K_3 . We now have a surgical description of a closed, connected, orientable 3-manifold, M . By applying an appropriate twist across a disk spanning each of K_2 and K_3 , we can replace our original surgical description on M by one involving only a knot, K , which (with a little adjusting) is easily seen to be strongly invertible. Hence, M is a 2-fold branched cyclic covering space of S^3 . Some of the various knots that K might be are 8_3 , 10_3 , and any twist knot.

For the second group of examples, let K_1 denote a double of a noninvertible knot and let K_2 denote a trivial knot in $S^3 - K_1$ placed near the "critical" part of K_1 so that exactly one (suitable) twist, t , across a disk spanning K_2 , will unknot K_1 . Now assign any rational coefficient to K_1 and assign either $+1$ or -1 to K_2 so that the coefficient of K_2 becomes ∞ after the twist t . The link $K_1 \cup K_2 (= L)$ is invertible, but not strongly invertible. Furthermore, with the two coefficients attached, L provides a surgical description of a manifold N . After twisting by t about a disk spanning K_2 , we can replace our first surgical description of N by one involving only a trivial knot. Hence, N is a 2-fold branched cyclic covering space of S^3 ; in fact, N is a lens space.

CONJECTURE. *No manifold obtained from nontrivial surgery on a double of a noninvertible knot is a 2-fold branched cyclic covering space of S^3 .*

We conclude with two remarks, added in October, 1980, just before the paper went to press.

REMARK 1. Let K be a knot nontrivially imbedded in the interior of an unknotted solid torus V in S^3 , and suppose that one can invert K inside V (without disturbing $S^3 - \text{Int}(V)$). Let W be a solid torus in S^3 whose core is *not* strongly invertible, and let $f: V \rightarrow W$ be a faithful homeomorphism. With only minor technical restrictions on K , we can conclude that $f(K)$ is invertible but not strongly (see Theorem 2 of [12]). One can easily construct examples (each with genus > 1) that are not double knots (see [12]).

REMARK 2. Richard Hartley has independently constructed counterexamples to Montesinos's conjecture (that every invertible knot is strongly invertible); see Hartley's paper [*Knots and involutions*, Math. Zeit., 171 (1980), 175-185].

REFERENCES

1. I. Bernstein and A. L. Edmonds, *On the construction of branched coverings of low-dimensional manifolds*, Trans Amer. Math. Soc., **247** (1979), 81-124.
2. R. H. Crowell and R. H. Fox, *Introduction to Knot Theory*, Graduate Texts in Mathematics **57**, Springer-Verlag, New York, 1977.
3. R. Kirby, *Problems in low dimensional manifold topology*, Proceeding of Synposia in Pure Mathematics **XXXII**, Amer. Math. Soc., Providence, R. I., 1978.
4. E. E. Moise, *Geometric Topology in Dimensions 2 and 3*, Graduate Texts in Mathematics **47**, Springer-Verlag, New York, 1977.
5. J. Montesinos, *Surgery on links and double branched covers of S^3 ; Knots, Groups, and 3-Manifolds*, Annals of Mathematics Studies **84**, 227-259, Princeton University Press, Princeton, N. J., 1975.
6. H. Schubert, *Knoten und Vollringe*, Acta Math., **90** (1953), 131-286.
7. P. A. Smith, *Transformations of finite period*, Ann. of Math., **39** (1938), 127-164.
8. J. L. Tollefson, *Involutions on $S^1 \times S^2$ and other 3-manifolds*, Trans. Amer. Math. Soc., **183** (1973), 139-152.
9. ———, *Periodic homeomorphisms of 3-manifolds fibered over S^1* , Trans. Amer. Math. Soc., **223** (1976), 223-234.
10. F. Waldhausen, *Über Involutionsen der 3-sphäre*, Topology, **8** (1969), 81-91.
11. W. Whitten, *Surgically transforming links into noninvertible knots*, Amer. J. Math., **44** (1972), 1269-1281.
12. ———, *Switching and inverting knots*, preprint.

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