A NOTE ON REAL ORTHOGONAL MEASURES

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Let $X$ be an open Riemann surface and $K$ a compact subset of $X$ such that $X - K$ has only finitely many connected components. Let $R(K)$ denote the space of meromorphic functions with poles off $K$. In this note, we investigate the space of real measures supported on $\partial K$ and orthogonal to $R(K)$ and connect it with the first homology group of the interior of $K$.

1. Introduction and preliminary notations. Let $X$ be a fixed open connected Riemann surface; $K$ a compact subset of $X$ such that $X - K$ has only finitely many connected components. Let $C(\partial K)$ denote the space of all real valued continuous functions on $\partial K$; $\mathcal{B}(K)$ denote the space of all meromorphic functions on $X$ with poles outside $K$; $\overline{\text{Re} \mathcal{B}(K)}$ denote the closure of the space of real parts of functions in $\mathcal{B}(K)$ under the sup norm on $\partial K$. Let $\mathcal{M}(K)$ denote the space of all measures on $\partial K$ that are orthogonal to $\mathcal{B}(K)$ and $m(K)$ denote those measures of $\mathcal{M}(K)$ that are real.

The sole purpose of this note is to establish the following theorems.

THEOREM 1.1. There exists a natural isomorphism between $m(K)$ and the first cohomology group of $K$ (which we shall denote by $\Omega$ hereafter) with real coefficients.

THEOREM 1.2. One can select a set of functions depending only on a homology basis of $\Omega$ in a natural way so that they form a basis for $C(\partial K)$ modulo $\overline{\text{Re} \mathcal{B}(K)}$.

When $X$ is the complex plane, Theorem 1.1 has already been established by Ahern and Sarason in [2] and Glicksberg in [5]. Walsh [9] already proved in this case that $\log |z - a_i|$, $1 \leq i \leq n$ generate $C(\partial K)$ modulo $\overline{\text{Re} \mathcal{B}(K)}$ where $a_i$ are selected one each from the connected components of $X - K$. He also saw that they need not form a basis as in the case of the crescent moon.

The precise determination of which logarithmic terms are necessary was first given in [2] and later by Glicksberg in [5] by another method. In the case of the plane, we prove these theorems in a separate note without recourse to the techniques of uniform algebras.

2. Topological preliminaries. We need some results that are
purely topological and we give proofs where we can not give a
good reference.

**Theorem 2.1.** Let $U$ be an open subset of an open Riemann
surface $Y$ such that $Y - U$ has only finitely many connected com-
ponents each of which is noncompact. Then the canonical homomor-
phism $i: H_1(U) \to H_1(Y)$ is injective where $H_1$ is the first homology
group functor.

**Proof.** Let $K$ be a triangulation of $Y$ and $K^{(n)}$ denote the $n$th
barycentric subdivision of $K$ and let $L^{(n)}$ denote the subcomplex
made up of all those 2-simplices of $K^{(n)}$ that are contained in $U$.

Let $i_n: H_1(L^{(n)}) \to H_1(Y)$ be the natural homomorphism. It is
enough to prove that $i_n$ is injective for all $n$ since $H_1(U)$ is the
direct limit of $H_1(L^{(n)})$. Writing the homology exact sequence

$$H_1(Y, L^{(n)}) \longrightarrow H_1(L^{(n)}) \longrightarrow H_1(Y)$$

we see that it is enough to prove that $H_2(Y, L^{(n)}) = 0$. Since the
considerations are the same for all $n$, we shall drop the superscript
$n$. Let $z = \sum_{i=1}^k n_i s_i$ be any two cycle made up of simplices not in
$L$ such that $z \in L$. Let $|z|$ denote the set of all points that belong
to at least one of the $s_i$, i.e., the so-called support of $z$. We claim
that the topological boundary of $|z|$ is contained in $|L| = \text{support}
of L$. Let $P$ be a boundary point of $|z|$ and $P \notin L$. But $P \in |\partial s_i|
for some $i$. Let $a$ be the 1-simplex of $s_i$ to which $P$ belongs. By
hypothesis, $a \notin L$ and since $\partial z \subset L$, this $a$ must get cancelled by
another 1-simplex of $s_j$ for some $j \neq i$. Thus if $P$ is not a vertex
of $s_i$, $P \in \text{interior of } |z|$. And if $P$ is a vertex of $s_i$, then star of
$P$ must be part of $|z|$. In either case if $P \notin |L|$, $P \in \text{interior of } |z|.

Also the interior of $|z|$ must contain points of $Y - U$ for other-
wise $|z|$ would be contained in $U$ and hence $z \subset L$. Hence the
interior of $|z|$ must intersect some connected component $C$ of $Y - U.$
Since $C \cap |L| = \emptyset$ and boundary of $|z| \subset |L|$, $C \subset |z|$. But then $C$
is noncompact whereas $|z|$ is compact. A contradiction! \[\square\]

Hence $z = 0$ ie $H_2(Y, L) = 0$.

**Lemma 2.2.** $H_1(\Omega)$ is finitely generated.

**Proof.** We can suitably shrink the ambient Riemann surface
$X$ to $X_0$ so that $K \subset X_0$, $X_0 - K$ has finitely many connected com-
ponents each of which is noncompact and further $H_1(X_0)$ is a free
Abelian group of finite rank.

By the preceding theorem, $H_1(\Omega)$ is a subgroup of $H_1(X_0)$ and
hence is a free Abelian group of finite rank.

For complete details regarding barycentric subdivisions, homology groups etc. one can confer [3], Ch. I.

**Lemma 2.3.** Let $Y$ be a connected open Riemann surface and assume $H_1(Y)$ is finitely generated. Then there exists a subregion $\Omega_0$ relatively compact and bounded by simple closed curves $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that every component of $Y - \Omega_0$ is an annulus.

**Proof.** Canonical form of $Y$ (see [3], p. 94) is (let us say) with $p$ handles and $q$ contours i.e., by cutting out $2p + q$ discs out of the Riemann sphere and then attaching $p$ handles by pairing off $2p$ of the holes, we get a homeomorph of $Y$.

Thus by taking off $q$ ringed domains one around each hole, we get a subregion $\Omega_0$ such that every connected component of $Y - \Omega_0$ is an annulus.

**Definition 2.4.** Let $U$ be an open subset of a Riemann surface $X$. A path at $x$ in $U$ is a Jordan arc entirely lying in $U$ except possibly at one endpoint which is $x$ when $x \in \partial U$.

Two paths at $x$ in $U$ are said to be equivalent if and only if given any neighborhood $N$ of $x$, there exists an arc joining the two paths and lying entirely in $N \cap U$. A point $x$ is said to be a multiple point of $U$ if there exist two inequivalent paths at $x$ in $U$.

**Lemma 2.5.** Let $K$ be a compact subset of an open connected Riemann surface $X$ such that $X - K$ has only finitely many connected components. Let $\Omega = \hat{K}$. The set of multiple points of $\Omega$ is countable and given any multiple point $x$ of $\Omega$, there exists at most countably many inequivalent paths at $x$ in $\Omega$.

**Proof.** Let $x_0 \in \partial \Omega$. Since $X - K$ has only finitely many connected components, there exists a closed parametric disc $\Delta$ with center at $x_0$ such that no connected component of $X - K$ is completely contained in $\Delta$.

Let $\phi: \Delta \to C$ denote the coordinate mapping and $C$, the image of $\Delta \cap K$ by $\phi$. $C$ is compact and the complement of $C$ is connected since any connected component of $X - K$ that intersects $\Delta$ would have points on the rim of $\Delta$. Thus any multiple point of $\Omega$ contained in the interior of $\Delta$ is mapped into a multiple point of $\hat{C}$ and further any two inequivalent paths at $x$ in $\Omega$ are mapped to inequivalent paths at $\phi(x)$ in $\hat{C}$.

Just for this discussion alone, let us make the convention that
capital letters denote paths and small letters their extremeties. Thus \( xPy \) shall denote a path \( P \) with extremeties \( x, y \) and oriented from \( x \) to \( y \).

Now let \( xP_1y_1, xP_2y_2 \) be two paths at \( x \) in \( \hat{C} \) and \( x \) a multiple point of \( \hat{C} \). Assume further that these two paths lie in the same connected component \( U \) of \( \hat{C} \). We join these two paths by a path \( y_1Qy_2 \) completely contained in \( U \). Then \( xP_1y_1Qy_2P_2x \) is a Jordan curve completely contained in \( U \) but for the point \( x \). Certainly the interior of this curve must be completely contained in \( \hat{C} \) for otherwise it would intersect the complement of \( C \) thus trapping a connected component of the complement of \( C \). But complement of \( C \) is connected and unbounded leading to a contradiction. Thus \( xP_1y_1Qy_2P_2x \) is the boundary of a Jordan domain contained in \( U \). But Jordan domains are locally arc-wise connected even at the boundary (see Goluzin [6], p. 46). Hence \( xP_1y_1 \) and \( xP_2y_2 \) are equivalent paths at \( x \) in \( \hat{C} \).

This proves that two paths are inequivalent if and only if they are contained in different connected components of \( \hat{C} \). Thus the number of inequivalent paths at a point \( x \) does not exceed the number of connected components of \( \hat{C} \) and hence they are at most countable.

Now let \( U_1, U_2 \) be two connected components of \( \hat{C} \) and let \( x, u \) belong to \( \partial U_1 \cap \partial U_2 \), \( xP_1y_1, xP_2y_2 \) be paths at \( x \) in \( U_1 \) and \( U_2 \) respectively and \( uQ_u z_1, uQ_2 z_2 \) be paths at \( u \) in \( U_1 \) and \( U_2 \) respectively. Let \( y_1 R_1 z_1, y_2 R_2 z_2 \) be two paths lying entirely in \( U_1 \) and \( U_2 \) respectively. Now interior of the Jordan curve \( xP_1y_1R_1 z_1Q_u z_2R_2 y_2P_2x \) must trap a component of the complement of \( \hat{C} \) for otherwise it would be completely contained in \( C \) and hence in \( \hat{C} \) joining \( U_1 \) and \( U_2 \) which is impossible. This means that given any multiple point \( x \) of \( \hat{C} \), we can associate a pair of connected components of \( \hat{C} \) where the inequivalent paths to \( x \) in \( C \) come from and this association is one-to-one. Since the number of connected components of \( \hat{C} \) is at most countable, we obtain that the set of multiple points of \( \hat{C} \) is also at most countable.

Since \( K \) can be covered by the interiors of a finite number of parametric discs, the lemma is proved.

**Lemma 2.6.** Let \( \Delta \) denote the annulus \( \delta < |z| < 1 \) and \( \phi: \Delta \to U \) be a conformal isomorphism and \( U \) be a relatively compact subset of a connected open Riemann surface \( X \). Assume \( \partial U = C \cup D \) where \( C \) and \( D \) are both compact and disjoint.

Let \( \phi(|z| = \delta) \) denote the set of all points \( \zeta \) in \( X \) for which
there exists a sequence \( z_n \in A \), \( |z_n| \to \delta \) as \( n \to \infty \) and \( \phi(z_n) \to \zeta \) as \( n \to \infty \). By analogy, we can define \( \phi(|z| = 1) \).

Then \( \phi(|z| = 1) \), \( \phi(|z| = \delta) \) are both connected and either \( \phi(|z| = 1) = C \), \( \phi(|z| = \delta) = D \) or \( \phi(|z| = 1) = C \), \( \phi(|z| = \delta) = D \).

**Proof.** Evidently \( \phi(|z| = \delta) \) is a closed set in \( X \). Assume that \( \phi(|z| = \delta) \) is disconnected i.e., \( \phi(|z| = \delta) = A_1 \cup A_2 \) where \( A_1, A_2 \) are mutually disjoint nonempty closed sets in \( X \). Then there exist open sets \( V_1, V_2 \) such that \( V_i \supset A_i \), \( i = 1, 2 \) and \( V_1 \cap V_2 = \phi \). We claim that \( \phi(\delta < |z| < r) \subset V_1 \cup V_2 \) for all \( r \) sufficiently close to \( \delta \). If not, there exists a sequence \( r_n \downarrow \delta \) and \( z_n \) with \( |z_n| = r_n \) and \( \phi(z_n) \notin V_1 \cup V_2 \).

This is impossible since on the one hand all limit points of \( \phi(z_n) \) would belong to \( \phi(|z| = \delta) \) and on the other hand should lie outside \( V_1 \cup V_2 \) which is an open set containing \( \phi(|z| = \delta) \).

Since \( \phi(\delta < |z| < r) \) is connected, the fact that \( \phi(\delta < |z| < r) \subset V_1 \cup V_2 \) implies that \( \phi(\delta < |z| < r) \subset V_1 \) or \( V_2 \) which means that \( \phi(|z| = \delta) \subset V_1 \) or \( V_2 \). Since \( V_1 \cap V_2 = V_2 \cap V_1 = \phi \), \( \phi(|z| = \delta) \cap V_2 = \phi \) or \( \phi(|z| = \delta) \cap V_1 = \phi \). That is impossible. Hence \( \phi(|z| = \delta) \) is connected. Similarly \( \phi(|z| = 1) \) is connected.

Further any boundary point of \( U \) must belong either to \( \phi(|z| = \delta) \) or \( \phi(|z| = 1) \). Let \( \xi_0 \in \partial U \) and \( \{\xi_n\} \) be a sequence of points in \( U \) such that \( \xi_n \to \xi_0 \) as \( n \to \infty \). Then if \( \phi(z_n) = \xi_n \), \( z_n \in A \), any limit point \( z_0 \) of \( \{z_n\} \) must belong to \( \partial A \). For if not, \( z_{n_k} \to z_0 \) as \( k \to \infty \) and \( z_0 \in A \) and \( \phi(z_{n_k}) = \xi_{n_k} \to \xi_0 = \phi(z_0) \) as \( k \to \infty \). But \( \phi(z_0) \) is an interior point of \( U \) and \( \xi_0 \) is a boundary point of \( U \). A contradiction. A similar reasoning would prove that \( \phi(\partial A) \subset \partial U \). Consequently \( \phi(\partial A) = \partial U \).

This proves that \( \partial U \) has at most two connected components. By hypothesis \( \partial U \) has at least two connected components. Hence \( C \) and \( D \) must be connected and \( \phi(|z| = 1) \), \( \phi(|z| = \delta) \) must be disjoint.

Hence \( \phi(|z| = 1) = C \) and \( (|z| = \delta) = D \) or \( \phi(|z| = 1) = D \) and \( \phi(|z| = \delta) = C \).

**Lemma 2.7.** Hypothesis and notation same as in the previous lemma. There exists a Borel set \( E \subset [0, 2\pi] \) of length \( 2\pi \) such that \( \lim_{\theta \to \pi} \phi(re^{i\theta}), \lim_{\theta \to \alpha} \phi(re^{i\theta}) \) exist for all \( \theta \in E \).

**Proof.** Narasimhan [8] proved that any open Riemann surface can be imbedded in \( C^3 \) as a closed sub-manifold. Hence there exist three holomorphic functions \( \psi_i, \ i = 1, 2, 3 \) such that \( \psi(\zeta) = (\psi_1(\zeta), \psi_2(\zeta), \psi_3(\zeta)) \) from \( X \to C^3 \) is a one-one holomorphic map.

Since \( \bar{U} \) is compact, \( \psi/\bar{U} \) is bounded and hence \( \psi_i \circ \phi \) is bounded for \( i = 1, 2, 3 \). By Fatou's theorem (see [10] pp. 99–100) on radial
limits, there exists a Borel set $E \subseteq [0, 2\pi]$ of length $2\pi$ such that $\lim_{r \to 1} \psi_i \circ \phi(re^{i\theta})$, $\lim_{r \to 1} \psi_i \circ \phi(re^{i\theta})$ exist for all $\theta \in E$, $i = 1, 2, 3$. 

Let $\theta \in E$, $r_n \uparrow 1$ and $\phi(r_n e^{i\theta}) \to \zeta_0$ as $n \to \infty$. Then $\lim_{n \to \infty} \psi_i \circ \phi(r_n e^{i\theta}) = \psi_i(\zeta_0) = \lim_{r \to 1} \psi_i \circ \phi(re^{i\theta})$ for $i = 1, 2, 3$. Since $\psi$ is $1-1$, this shows that $\zeta_0$ does not depend on the sequence $\{r_n\}$. Hence $\lim_{r \to 1} \phi(re^{i\theta})$ exists. Similarly $\lim_{r \to 3} \phi(re^{i\theta})$ exists for all $\theta \in E$.

**Lemma 2.8.** Hypothesis same as in Lemma 2.6. Further assume that $X - \bar{U}$ has only finitely many connected components. Then by discarding a countable subset of $E$ ($E$ as in Lemma 2.7), we can assume that $\theta \to \lim_{r \to 1} \phi(re^{i\theta})$ and $\theta \to \lim_{r \to 3} \phi(re^{i\theta})$ are both one-one on $E$.

**Proof.** Let $\theta \in E$, $P_0$ denote the path $\phi(re^{i\theta})$, $1 - \varepsilon < r < 1$, $\varepsilon$ a fixed small positive number; $\zeta_0 = \lim_{r \to 1} \phi(re^{i\theta})$.

Now if $\theta_1 \neq \theta_2$ and $\zeta_{\theta_1} = \zeta_{\theta_2}$, then $\zeta_{\theta_1}$ is a multiple point and $P_{\theta_1}, P_{\theta_2}$ are inequivalent (see [6], pp. 38–39). Thus $\zeta_{\theta_1}$ is a multiple point of $U$. By Lemma 2.5, the set of multiple points is countable and at any given multiple point, there can be at most countably many inequivalent paths.

Thus given a $\theta_0 \in E$, the set of all $\theta \in E$, $\theta \neq \theta_0$, $\zeta_\theta = \zeta_{\theta_0}$ is countable; further the set of all $\theta_0$ for which there exists a $\theta \neq \theta_0$ such that $\zeta_\theta = \zeta_{\theta_0}$ is also countable. Hence by discarding all such $\theta_0$ out of $E$, we obtain a new Borel set $E$ of length $2\pi$ such that $\theta \to \lim_{r \to 1} \phi(re^{i\theta})$ is a $1-1$ map. A similar reasoning applied as $r \to \delta$ would prove the rest of the lemma.

3. Boundary measures and analytic differentials.

**Definition 3.1.** Let $U$ be an open subset of a connected open Riemann surface $X$. An increasing sequence $\{U_n\}$ of open sets is said to be a regular exhaustion of $U$ if $U_n$ is a relatively compact subset of $U_{n+1}$ for all $n$; $\bigcup_{n=1}^{\infty} U_n = U$; $\partial U_n$ consists of finitely many piecewise analytic Jordan curves and $U - \bar{U}_n$ has no relatively compact connected components in $U$.

**Remark.** Existence of regular exhaustions can be proved by triangulations (see [3], pp. 62–63).

**Definition 3.2.** Let $U$ be an open subset of $X$. $\mathcal{H}(U)$ denotes the set of all holomorphic 1-forms $\omega$ for which there exists a regular exhaustion $\{U_n\}$ of $U$ such that $\int_{\partial U_n} |\omega| \leq c$ where $c$ is independent of $n$. 


DEFINITION 3.3. Let $U$ be a relatively compact open subset of an open connected Riemann surface $X$. Let $\omega \in \mathcal{H}(U)$. A finite Borel measure $\mu$ on $\partial U$ is called a boundary measure of $\omega$ if there exists a regular exhaustion $U_n$ of $U$ such that $\int_{\partial U_n} h\omega \to \int_{\partial U} h\mu$ as $n \to \infty$ for any continuous function $h$ on $U$ where $\partial U_n$ is positively oriented with respect to $U_n$.

THEOREM 3.4. (Bishop-Kadama, see [7]). Let $K$ be a compact subset of $X$ such that $X - K$ has only finitely many connected components. Let $K = \Omega$. Given any $\omega \in \mathcal{H}(\Omega)$, there exists one and only one boundary measure $\mu_\omega$ of $\omega$.

The mapping $\omega \mapsto \mu_\omega$ is a linear isomorphism between $\mathcal{H}(\Omega)$ and $\mathcal{M}(K)$ (see § 1 for the definition of $\mathcal{M}(K)$).

DEFINITION 3.5. Let $U$ be an open subset of $X$. A point $x \in \partial U$ is said to be an accessible boundary point of $U$ if and only if there exists a path at $x$ in $U$. Acc $\partial U$ shall denote the set of all accessible boundary points of $U$.

THEOREM 3.6. Let $K$ be a compact subset of $X$ and $X - K$ have only finitely many connected components. Let $K = \Omega$. Let $\{U_i, i \in I\}$ be the family of all connected components of $\Omega$. By Lemma 2.2, $H_i(\Omega)$ is finitely generated and consequently $H_i(U_i)$ is finitely generated for all $i \in I$. By Lemma 2.3, there exists a relatively compact subregion $V_i$ of $U_i$ bounded by finitely many analytic Jordan curves such that each component of $U_i - V_i$ is an annulus. Let $\{A_{ij}, 1 \leq j \leq N(i)\}$ denote the set of all connected components of $U_i - V_i$. Let $\omega \in \mathcal{H}(\Omega)$.

Then $\omega|A_{ij} \in \mathcal{H}(A_{ij})$. Let $\mu_{ij}$ denote the boundary measure of $\omega|A_{ij}$ located on $\partial A_{ij} \cap \partial \Omega$. Then $\mu_{ij}$, $\mu_{i'j'}$ are mutually singular for $(i, j) \neq (i', j')$. Further $\sum_{i \in I} \sum_{1 \leq j \leq N(i)} \|\mu_{ij}\|$ is finite and $\mu_\omega = \sum_i \sum_{1 \leq j \leq N(i)} \mu_{ij}$.

Before proceeding to the proof of the Theorem 3.6, we need two lemmas.

LEMMA 3.7. Let $\Delta$ denote the annulus $\{z; \delta < |z| < 1\}$ and $\omega \in \mathcal{H}(\Delta)$. Let $\omega = f(z)dz$ where $f$ is holomorphic in $\Delta$. Then there exists a Borel measurable function $f$ defined on $\partial \Delta$ such that

$$\lim_{r \to 1^{-}} \int_{0}^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|d\theta = 0 \quad \text{and} \quad \lim_{r \to \delta^{+}} \int_{0}^{2\pi} |f(re^{i\theta}) - f(\delta e^{i\theta})|d\theta = 0.$$ 

Proof. Let $\mu$ denote the boundary measure $\mu_\omega$ of $\omega$. 
Let \( f_1(z) = \int_{|\zeta| = \delta} \frac{d\mu(\zeta)}{\zeta - z} \) and \( f_2(z) = \int_{|\zeta| = \delta} \frac{d\mu(\zeta)}{\zeta - z} \) so that \( f_1 \) is holomorphic in \(|z| < 1\) and \( f_2 \) is holomorphic in \(|z| > \delta\) and \( f = f_1 + f_2 \) in \( \Delta \).

Let \( \nu_1, \nu_2 \) be finite complex Borel measures defined by

\[
d\nu_1(\zeta) = d\mu(\zeta) - \frac{1}{2\pi i} f_1(\zeta) d\zeta \text{ on } |\zeta| = 1
\]

\[
d\nu_2(\zeta) = d\mu(\zeta) + \frac{1}{2\pi i} f_1(\zeta) d\zeta \text{ on } |\zeta| = \delta.
\]

Then for \( \delta < |z| < 1 \),

\[
\int \frac{d\nu_1(\zeta)}{\zeta - z} = \int_{|\zeta| = 1} \frac{d\mu(\zeta)}{\zeta - z} - \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{f_1(\zeta) d\zeta}{\zeta - z}
\]

\[
= f_1(z) - \frac{1}{2\pi i} \int_{|\zeta'| = \delta} \int_{|\zeta| = 1} \frac{d\mu(\zeta') d\zeta}{(\zeta' - \zeta)(\zeta' - z)}
\]

\[
= f_1(z)
\]

since

\[
\int_{|\zeta| = 1} \frac{d\zeta}{(\zeta' - \zeta)(\zeta - z)} = 0 \text{ when } |\zeta'| < 1 \text{ and } |z| < 1.
\]

By analytic continuation, we get that

\[
\int \frac{d\nu_1(\zeta)}{\zeta - z} = f_1(z) \text{ for } |z| < 1.
\]

Further for \(|z| > 1\),

\[
\int \frac{d\nu_1(\zeta)}{\zeta - z} = f_1(z) + f_2(z)
\]

since

\[
\int_{|\zeta| = 1} \frac{d\zeta}{(\zeta' - \zeta)(\zeta - z)} = -2\pi i/(\zeta' - z).
\]

Therefore

\[
\int \frac{d\nu_1(\zeta)}{\zeta - z} = f_1(z) \text{ for } |z| < 1
\]

\[
= 0 \text{ for } |z| > 1.
\]

By F. and M. Riesz theorem ([4], for a very general form), we
obtain that
\[ \int_0^{2\pi} \left| f_1(re^{i\theta}) - f_1(r'e^{i\theta}) \right| d\theta \rightarrow 0 \text{ as } r, r' \rightarrow 1. \]

Now by a similar reasoning, we find that
\[ \int \frac{d\nu_2(\zeta)}{\zeta - z} = f_0(z) \text{ for } |z| > \delta \]
\[ = 0 \text{ for } |z| < \delta. \]

Applying an inversion and F and M. Riesz theorem, we obtain that
\[ \int_0^{2\pi} \left| f_2(re^{i\theta}) - f_2(r'e^{i\theta}) \right| d\theta \rightarrow 0 \text{ as } r, r' \rightarrow \delta. \]

This together with completeness of \( L^1([0, 2\pi]) \) proves our lemma.

**Definition 3.8.** Let \( \phi: X \rightarrow Y \) be a holomorphic map where \( X \) and \( Y \) are Riemann surfaces. Then for any holomorphic 1-form \( \omega \) on \( Y \), \( \phi^*\omega \) denotes the holomorphic 1-form defined as follows: for any \( p \in X \) and a coordinate function \( \zeta \) in a neighborhood \( N \) of \( \phi(P) \), \( \phi^*\omega = f(\zeta \circ \phi) d\zeta \circ \phi \) where \( \omega = f(\zeta) d\zeta \) in a neighborhood of \( \zeta \circ \phi(p) \).

**Definition 3.9.** Let \( X, Y \) be two measurable spaces and \( \phi: X \rightarrow Y \) be a measurable map. For any measure \( \mu \) on \( X \), \( \phi^*\mu \) denotes the measure defined by \( (\phi^*\mu)(S) = \mu(\phi^{-1}(S)) \) for any measurable subset \( S \) of \( Y \).

**Lemma 3.10.** Let \( \Delta_{ij} \) be as introduced in Theorem 3.6 and \( \phi: \Delta \rightarrow \Delta_{ij} \) be a conformal isomorphism where \( \Delta = \{z; \delta < |z| < 1\} \) and \( \delta \) depends on \( i, j \).

Let \( B \) denote the set of all points \( z \) on \( \partial \Delta \) for which \( \lim_{r \rightarrow 1-0} \phi(rz) \) or \( \lim_{r \rightarrow \delta+0} \phi(rz) \) exists and let us extend \( \phi \) to \( B \) by these limits. Let \( \omega \in \mathcal{H}(\Delta_{ij}) \). Then \( \phi^*\omega \in \mathcal{H}(\Delta) \) and if \( \nu \) is the boundary measure of \( \phi^*\omega \), there exists a Borel subset \( B_0 \) of \( B \) on which \( \nu \) is supported and \( \phi^*\nu \) is the boundary measure of \( \omega \).

**Proof.** If \( \{U_n\} \) is a regular exhaustion of \( \Delta_{ij} \), then \( \phi^{-1}(U_n) \) is a regular exhaustion of \( \Delta \) and further
\[ \int_{\phi^{-1}(U_n)} |\phi^*\omega| = \int_{U_n} |\omega|. \]

Consequently by definition, \( \phi^*\omega \in \mathcal{H}(\Delta) \). By Lemma 3.7, if \( \phi^*\omega = \)


\[ f(z)dz; \text{ we can extend } f \text{ as a Borel measurable function to } \Delta \text{ such that} \]

\[
\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})| \, d\theta = 0 \quad \text{and} \\
\lim_{r \to \delta} \int_0^{2\pi} |f(re^{i\theta}) - f(\delta e^{i\theta})| \, d\theta = 0.
\]

In view of Lemma 2.8 there exists a Borel set \( E \subset [0, 2\pi] \) of measure \( 2\pi \) such that \( \lim_{r \to 1} \phi(re^{i\theta}), \lim_{r \to \delta} \phi(re^{i\theta}) \) exist for all \( \theta \in E \). Let \( B_0 \) denote the set \( \{ z; z = e^{i\theta} \text{ or } \delta e^{i\theta} \text{ for some } \theta \in E \} \). Obviously \( B_0 \) is a Borel set and \( \phi \) can be extended by radial limits to \( \Delta \cup B_0 \) as a Borel measurable function.

The above considerations imply that if \( h \) is any continuous function on \( \overline{A}_{ij} \),

\[
\lim_{r \to 1} \int_{|z|=r} h \phi(z)f(z)dz = \lim_{r \to 1} \int_{|z|=r} h\omega \quad \text{and} \\
\lim_{r \to \delta} \int_{|z|=r} h \phi(z)f(z)dz = \lim_{r \to \delta} \int_{|z|=r} h\omega
\]

exist and are respectively equal to

\[
\int_{B_0 \cap |z|=1} h \phi(e^{i\theta})f(e^{i\theta})de^{i\theta} \quad \text{and} \quad \int_{B_0 \cap |z|=\delta} h(\delta e^{i\theta})f(\delta e^{i\theta})d\delta e^{i\theta}
\]

for any continuous function \( h \) on \( \partial \Delta \).

Let us define the boundary measure \( \nu \) on \( \partial \Delta \) as follows: \( d\nu = f(e^{i\theta})de^{i\theta} \) on \( |z| = 1 \) and \( d\nu = -f(\delta e^{i\theta})d\delta e^{i\theta} \) on \( |z| = \delta \). Because of (1), \( \nu \) is the boundary measure of \( \phi^*\omega \) and because of (2),

\[
\int_{\partial \Delta} h\phi \, d\nu = \lim_{n \to \infty} \int_{\partial V_n} h\omega = \int_{\partial A_{ij}} h d\phi^*_n \nu
\]

where \( V_n = \phi((z; \delta + 1/n < |z| < 1 - 1/n)) \). Since \( \{ V_n \} \) is a regular exhaustion of \( A_{ij} \), by the Theorem 3.4 follows that \( \phi^*\nu \) is indeed the boundary measure of \( \omega \) on \( A_{ij} \).

**Remark 3.11.** Boundary measure of \( \omega \) is supported on \( \text{acc} \partial A_{ij} \) and any countable set is a null set for this measure.

Proof of Theorem 3.6. By Remark 3.11, it follows that \( \mu_{ij} \) is supported on a Borel set contained in \( \text{acc} \partial A_{ij} \subset \text{acc} \partial U_i \) and any countable set has measure zero.

Now fixing \( i, \) \( \text{acc} \partial A_{ij} \cap \text{acc} \partial A_{ij'} \) is countable for \( j \neq j' \) thanks to Lemma 2.5. Hence \( \mu_{ij}, \mu_{ij'} \) are mutually singular.

Let us assume \( i \neq j' \). The support of \( \mu_{ij} \) and support of \( \mu_{i'j'} \) are respectively contained in \( \text{acc} \partial U_i \) and \( \text{acc} \partial U_{i'} \). By Lemma 2.5, \( \text{acc} \partial U_i \cap \text{acc} \partial U_{i'} \) is at most countable and by Remark 3.11 follows
that $\mu_{ij}, \mu_{i'j'}$ are mutually singular.

Let $\mu_i$ denote the boundary measure of $\omega$ restricted to $U_i$. We shall now prove that $\mu_i = \sum_{j=1}^{N(i)} \mu_{ij}$. The boundary of $A_{ij}$ falls into two parts, a Jordan curve $\gamma_{ij}$ contained in $U_i$ and $\partial \Omega \cap \partial A_{ij}$ which of course are disjoint closed sets. Thus as in lemma 3.10, $\phi: \{z; \delta < |z| < 1\} \to A_{ij}$ is a conformal isomorphism, by lemma 2.6 the limit sets $\phi(|z| = \delta)$ and $\phi(|z| = 1)$ are disjoint and must coincide with $\gamma_{ij}$ and $\partial \Omega \cap \partial A_{ij}$ is some order. We can assume without loss of generality that $\phi(|z| = 1) = \partial \Omega \cap \partial A_{ij}$. Let $\gamma_{ijm}$ denote the Jordan curve $\phi(|z| = 1 - 1/n)$ oriented positively with respect to $\phi(\delta < |z| < 1 - 1/n)$. For any fixed $n$ and $i$, $\{\gamma_{ijm}\}_{1 \leq j \leq N(i)}$ bound a domain $U_{im}$ contained in $U_i$ and further for any continuous function $h$ on $\bar{\Omega}$, 

$$\lim_{n \to \infty} \int_{\gamma_{ijm}} h \omega = \int h d\mu_{ij} \text{ because of Lemma 3.10.}$$

Hence

$$\lim_{n \to \infty} \int_{\partial U_{im}} h \omega = \sum_{j=1}^{N(i)} \int h d\mu_{ij},$$

i.e., $\mu_i = \sum_{j=1}^{N(i)} \mu_{ij}$. This also proves that $\mu_i, \mu_{i'}$ are mutually singular if $i \neq i'$. Now we shall prove that $\sum_{i \in I} \|\mu_i\| < \infty$.

Since $\omega \in \mathcal{H}^1(\Omega)$, it follows that there exists a regular exhaustion $\{\Omega_n\}$ of $\Omega$ such that

$$\int_{\partial \Omega_n} |\omega| \leq C \text{ where } C \text{ does not depend on } n.$$

Further for any $h$ continuous on $\bar{\Omega}$, $\int_{\partial \Omega_n} h \omega \to \int h d\mu_\omega$ as $n \to \infty$.

Let $F$ be a finite subset of $I$ and let $U_F = \bigcup_{i \in F} U_i$. Now from the above considerations, we obtain that $\int_{\partial (\Omega_n \cap U_F)} |\omega| \leq C$ for all $n$ and by weak compactness of measures follows that by passing to a subsequence if necessary that $\int_{\partial (\Omega_n \cap U_F)} h \omega \to \int h d\mu_F$ as $n \to \infty$ where $\mu_F$ is the boundary measure of $\omega$ restricted to $U_F$. Hence $\|\mu_F\| < C$. But since as $n \to \infty$, $\int_{\partial (\Omega_n \cap U_F)} h \omega = \sum_{i \in F} \sum_{j=1}^{N(i)} \int_{\gamma_{ijm}} h \omega = \sum_{i \in F} \int h d\mu_i$ and $\{\bigcup_{i \in F} U_{im}\}$ is a regular exhaustion of $U_F$, we see that $\sum_{i \in F} \mu_i$ is also a boundary measure of $\omega/U_F$. By Theorem 3.4, $\sum_{i \in F} \mu_i = \mu_F$.

Consequently $\|\sum_{i \in F} \mu_i\| \leq C$ for an arbitrary finite subset $F$ of $I$ and now by the fact that $\mu_i$ are mutually singular, we obtain that $\sum_{i \in I} \|\mu_i\| \leq C$.

Now if $\mu' = \sum_{i \in I} \mu_i$, we can prove that any function $f$ meromorphic on $X$ with poles off $\partial K$, $\int f d\mu' = \int f d\mu_\omega$. It is enough to prove for a function with one pole. If the pole is not in $\Omega$, it is immediate that $\int_{\partial \Omega} f \omega = 0$ and $\int_{U_{im}} f \omega = 0$ for all $i$ and $n$. Hence
\[
\int f d\mu' = \int f d\mu_\omega = 0. \quad \text{Now if the pole is in some } U_i, \text{ then } \int_{\partial \Omega} f \omega = \int_{\partial U_i} f \omega \text{ provided the pole is in } \Omega \cap U_i. \quad \text{Hence by going to the limits, } \int f d\mu_\omega = \int f d\mu, \text{ and of course } \int f d\mu_j = 0 \text{ for } j \neq i.
\]

Thus \( \int (\mu' - \mu_\omega) = 0 \) for all functions meromorphic with poles off \( \partial K \). By a theorem of Kodama (see [7]), we obtain \( \mu' = \mu_\omega \).

Thus \( \mu = \sum_{i \in I} \mu_i = \sum_{i \in I} \sum_{j=1}^{N(i)} \mu_{ij} \).

**Corollary 3.12.** Let \( \bar{U}_i = K_i \). Given \( \mu_i \in \mathcal{M}(K_i) \) such that \( \sum \| \mu_i \| < \infty \), then \( \sum \mu_i \in \mathcal{M}(K) \). Further, \( \mu_i \) are mutually singular. Conversely given any \( \mu \in \mathcal{M}(K) \), \( \mu \) can be uniquely expressed as \( \sum \mu_i \) where \( \mu_i \in \mathcal{M}(K_i) \) and \( \sum \| \mu_i \| < \infty \).

**Proof.** By Theorem 3.6 \( \mu_i \) is supported on a Borel set contained in \( \text{acc} \partial U_i \) and any countable set is a null set modulo \( \mu_i \). By Lemma 2.5, \( \text{acc} \partial U_i \cap \text{acc} \partial U_j \) is a countable set and consequently, \( \mu_i \) and \( \mu_j \) are mutually singular.

Since \( \int f d\mu_i = 0 \) for any \( f \) continuous on \( K_i \) and analytic in \( U_i \), \( \int f d\mu_i = 0 \) for any \( f \) continuous on \( K \) and analytic in \( \Omega \). Therefore \( \mu_i \in \mathcal{M}(K) \forall i \) and \( \sum \mu_i \in \mathcal{M}(K) \).

For the converse, the fact that \( \mu = \sum \mu_i \), \( \mu_i \in \mathcal{M}(K_i) \) is a consequence of Theorem 3.6. Uniqueness follows from mutual singularity.

**Corollary 3.13.** Assume that \( m(K_i) \cong H^1(U_i) \forall i \in I \). Then \( m(K) \cong H^1(\Omega) \).

**Proof.** \( H^1(\Omega) \) is finitely generated by Lemma 2.2. Hence \( H^1(U_i) = 0 \) but for finitely many \( i \). The set of \( i \) for which \( H^1(U_i) \neq 0 \), we shall denote by \( F \).

Then \( H^1(\Omega) \cong \bigoplus_{i \in F} H^1(U_i) \). On the other hand, given any \( \mu \in m(K) \) by Corollary 3.12, \( \mu = \sum_{i \in I} \mu_i \), \( \mu_i \in M(K_i) \), \( \mu_i \), \( \mu_j \) are mutually singular; which implies that \( \mu_i \) is real for all \( i \), i.e., \( \mu_i \in m(K_i) \) for every \( i \) and by our assumption above

\[
\mu_i = 0 \text{ for } i \notin F.
\]

Thus the natural mapping \( m(K) \rightarrow \bigoplus_{i \in F} m(K_i) \) is an isomorphism.

Thus by our hypothesis,

\[
H^1(\Omega) \cong m(K).
\]

4. Harmonic 1-forms, real boundary measures.

**Lemma 4.1.** Let \( \omega \) be a holomorphic 1-form defined on an
annulus \( D = \{ z; \delta < |z| < 1 \} \). Assume that \( \exists \) a real measure \( \mu \) on \( |z| = 1 \) such that for any continuous function \( h \) on \( D \), \( \int_{|z|=r} h\omega \to \int h\mu \) as \( r \to 1 - 0 \). Then \( \int h\omega \to \int h\mu < \infty \) and for any \( C^1 \)-function \( h \) defined on \( D \), vanishing in a neighborhood of \( |z| = \delta \) and \( \int_D dhA*dh < \infty \), \( \int_D dhA \text{Im} \omega = 0 \).

(For the definition of \( * \omega \), \( \text{Im} \omega \) see Ahlfors-Sario [3] p. 271.)

Proof. Since \( \omega \) is a holomorphic 1-form, there exists a holomorphic function \( g(z) \) on \( D \) such that \( \omega = g(z)dz \).

Let \( \bar{D} \) denote the annulus \( \delta < |z| < 1/\delta \), the double of \( D \). Define \( \bar{\omega} \) a holomorphic 1-form on \( \bar{D} \) in the following way. Define \( \bar{\omega} = g(z)dz \) for \( |z| < 1 \) and for \( |z| > 1 \),

\[
\bar{\omega} = -\bar{g}\left(\frac{1}{\bar{z}}\right)\frac{dz}{z^2}.
\]

We note that \( \bar{\omega} \) is not defined on \( |z| = 1 \).

By hypothesis, we obtain that there exists a constant \( C \) such that \( \int_{|z|=r} |\omega| < C \) for \( r \) such that \( (1 + \delta)/2 \leq r < 1 \).

i.e., \( \int_{|z|=r} |g(z)||dz| \leq C \).

Thus if \( g \) is defined as \( g(z) \) on \( |z| < 1 \) and \( -\bar{g}(1/\bar{z})1/\bar{z} \) on \( |z| > 1 \), \( g \) belongs \( L^1,\text{loc} \((\bar{D})\). \) We shall now prove that \( \partial \bar{g}/\partial \bar{z} = 0 \) in the sense of distributions.

Let \( h \) be any \( C^\infty \)-function with compact support in \( \bar{D} \). Then

\[
\int_{\bar{D}} \frac{\partial h}{\partial \bar{z}} \bar{g}(z)d\bar{z} = \lim_{\varepsilon \to 0} \int_{|z|=1-\varepsilon} h\omega \to \int_{|z|=1+\varepsilon} h\bar{g}\left(\frac{1}{\bar{z}}\right)\frac{dz}{z^2}
\]

(by Stoke's formula applied to the annulii \( \delta < |z| < 1 - \varepsilon \), \( 1 + \varepsilon < |z| < 1/\delta \))

\[
= \int h\partial \mu + \lim_{\varepsilon \to 0} \int_{|z|=1+\varepsilon} h\bar{g}\left(\frac{1}{\bar{z}}\right)\frac{dz}{z^2}
\]

\[
= \int h\partial \mu + \lim_{\varepsilon \to 0} \int_{|z|=1/1+\varepsilon} h\left(\frac{1}{\bar{z}}\right)\bar{g}(z)d\bar{z}
\]

\[
= \int h\partial \mu - \int h\bar{\partial} \mu = 0 \text{ since } \mu \text{ is real}.
\]

Therefore we obtain that \( g \) can be defined suitably on \( |z| = 1 \) so that \( g \) is holomorphic in all of \( \bar{D} \). Hence \( \int \omega A*\omega < \infty \) and consequently,
\[
\left\langle \omega A^\ast \omega \right\rangle < \infty .
\]

\((1 + \delta)/2 < |z| < 1.\)

Also, for any real \(h, \mathcal{C}^1\) on \(\bar{D}\) and vanishing in a neighborhood of \(|z| = \delta,\)

\[
\int_D dhA \omega = \int_{|z|=1} h \omega = \int h d\mu
\]

and so

\[
\text{Im} \int_D dhA \omega = \int_D dhA \text{Im} \omega = \int_D dhA \text{Im} \omega = \text{Im} \int h d\mu = 0 .
\]

Now given any \(h, \mathcal{C}^1\) on \(D\) and vanishing in a neighborhood of \(|z| = \delta,\) define \(h_\varepsilon(z) = h(z/(1+\varepsilon)).\) Then \(h_\varepsilon\) is \(\mathcal{C}^1\) on \(\bar{D}\) for every \(\varepsilon > 0\) and vanishes in a neighborhood of \(|z| = \delta\) and furthermore \(\int dh_\varepsilon A^\ast dh_\varepsilon < \infty\) and \(\int (dh_\varepsilon - dh) \omega^\ast (dh - dh_\varepsilon) \to 0\) as \(\varepsilon \to 0.\)

Hence, since we already know that \(\int_D dh_\varepsilon A \text{Im} \omega = 0\) for all \(\varepsilon\) and \(\int \text{Im} \omega A^\ast \text{Im} \omega < \infty,\) we can take the limit under the integral sign and obtain that

\[
\int_D dhA \text{Im} \omega = 0 .
\]

**Lemma 4.2.** Let \(\omega\) be a holomorphic 1-form on \(D = \{z; \delta < |z| < 1\}\) such that \(\int_D \omega A^\ast \omega < \infty.\) Further assume that for any \(h, \mathcal{C}^1\) on \(D\) and vanishing in a neighborhood of \(|z| = \delta\) and \(\int_D dhA^\ast dh < \infty,\) \(\int_D dhA \text{Im} \omega = 0.\)

Then \(\exists\) a real measure \(\mu\) on \(|z| = 1\) such that for any continuous function \(h\) on \(\bar{D},\)

\[
\int_{|z|=r} h \omega \to \int h d\mu\text{ as } r \to 1 - 0.
\]

**Proof.** Let \(\omega = g(z)dz\) for \(\delta < |z| < 1\) and \(\tilde{\omega}\) be defined as \(\omega\) on \(\delta < |z| < 1\) and

\[
=-\tilde{g}\left(\frac{1}{z}\right)\frac{dz}{z^2}\text{ on } 1 < |z| < \frac{1}{\delta}.
\]

By hypothesis, \(\int_{\delta < |z| < 1/\delta} \omega A^\ast \omega < \infty.\) We shall now establish that \(\delta \tilde{\omega} = 0\) in the sense of distributions.

Let \(h\) be any \(\mathcal{C}^1\)-function with compact support in \(\delta < |z| < 1/\delta.\) Then

\[
\text{Im} \int_R dhA \omega = \int_R dhA \text{Im} \omega = \int_R dhA \text{Im} \omega = \text{Im} \int h d\mu = 0 .
\]
\[\oint_{\delta < |z| < 1/\delta} \hat{b} \Lambda \omega = \oint \oint_{\delta < |z| < 1/\delta} dh \Lambda \omega = \lim_{r \to 0} \oint_{\delta < |z| < r} dh \Lambda \omega + \oint_{1/r < |z| < 1/\delta} dh \Lambda \omega \]

\[= (\text{By Stoke's}) \lim_{r \to 0} \left( \oint \omega - \oint_{|z|=1/r} h(z) \left( - \bar{g}\left( \frac{1}{z} \right) \right) dz \right) \]

\[= \lim_{r \to 0} \left( \oint \omega - \oint_{|z|=r} h\left( \frac{1}{z} \right) \overline{\omega} \right) \]

\[= \lim_{r \to 0} \left( \oint \left( h(z) - h\left( \frac{1}{z} \right) \right) \text{Re} \omega + i \oint_{|z|=r} \left( \text{Re} \omega \right) \right) . \]

Since \( h(z) + h(1/z) \) vanishes in a neighborhood of \( |z| = \delta \) and \( \oint_{D} dh \Lambda * dh < \infty \), we have, by hypothesis,

\[\int \int_{D} dh \Lambda \text{Im} \omega = 0 = \oint dh \left( \frac{1}{z} \right) \Lambda \text{Im} \omega \text{ i.e.,} \]

(\text{By Stoke's})

\[\lim_{r \to 0} \oint_{\delta < |z| < r} dh \Lambda \text{Im} \omega = \lim_{r \to 0} \oint_{|z|=r} h \text{Im} \omega = 0 . \]

Hence

\[\oint_{\delta < |z| < 1/\delta} \hat{b} \Lambda \omega = \lim_{r \to 0} \oint_{|z|=r} \left( h(z) - h\left( \frac{1}{z} \right) \right) \text{Re} \omega \]

\[= \lim_{r \to 0} \oint_{\delta < |z| < r} \left( h(z) - h\left( \frac{1}{z} \right) \right) \Lambda \text{Re} \omega \text{ (By Stoke's)} . \]

Since \( h(z) - h(1/z) \in H^{1}(D) \) (here it denotes the Sobolev space) and vanishes on \( \partial D \), we find that

\[h(z) - h(1/z) \in H^{1}(D) \text{ (see Agmon [1], p. 131, Lemma 9.10). But} \]

\[\int_{D} dh \Lambda \text{Re} \omega = 0 \text{ for any } h \text{ that is } C^{1} \text{ and has compact support in} \]

\( D \) and hence for any \( h \in \hat{H}^{1}(D) \).

Therefore \( \hat{b} \omega = 0 \). Hence \( \omega \) is a holomorphic 1-form on \( \delta < |z| < 1/\delta \) which implies that \( \int_{|z|=r} |g(z)||dz| \) is bounded as \( r \to 1-0 \). That means that \( \omega \) defines a real boundary measure on \( |z| = 1 \).

**Theorem 4.3.** Borrowing the notation of Corollary 3.12, \( m(K_{i}) \equiv H^{1}(U_{i}) \) for every \( i \).

**Proof.** Let \( \Gamma(U_{i}) \) denote the set of all holomorphic 1-forms \( \omega \)
such that \[
\int_{U_i} \omega A^* \omega < \infty \quad \text{and for any } C^1\text{-function } h \text{ on } U_i \text{ such that } \int_{U_i} dh A^* dh < \infty, \int_{U_i} e h A \text{ Im } \omega = 0.
\]

The fact that \( H^1(U_i) \equiv \Gamma(U_i) \) is well-known and can be found in Ahlfors-Sario [2], p. 284–288. Thus we need only prove that \( m(K_i) \equiv \Gamma(U_i) \).

Let \( A_{ij} \) be the annulii as introduced in Theorem 3.6. Now if \( \omega \) is a holomorphic 1-form on \( U_i \) whose boundary measure is real, then \( \omega \big|_{A_{ij}} \in \mathcal{H}(A_{ij}) \) and further its boundary measure \( \mu_{ij} \) on \( \partial U_i \cap \partial A_{ij} \) is real. We can apply now Lemma 4.1 to \( \omega \big|_{A_{ij}} \) and obtain \[
\int_{A_{ij}} \omega A^* \omega < \infty \quad \text{and} \quad \int_{A_{ij}} dh A \text{ Im } \omega = 0 \quad \text{provided } h \text{ is a } C^1\text{-function vanishing in a neighborhood of } \partial A_{ij} - \partial U_i.
\]
Thus using partition of unity, we obtain that \( \int_{U_i} \omega A^* \omega < \infty \) and \( \int_{U_i} dh A \text{ Im } \omega = 0 \) for any \( h, C^1 \) on \( U_i \) and \( \int_{U_i} dh A^* dh < \infty \).

Now assume that \( \omega \in \Gamma(U_i) \). Now \( \omega \big|_{A_{ij}} \) satisfies the following conditions: \[
\int_{A_{ij}} \omega A^* \omega < \infty \quad \text{and any } C^1\text{-function } h \text{ vanishing in a neighborhood of } \partial A_{ij} - \partial U_i \text{ and } \int_{A_{ij}} dh A^* dh < \infty, \int_{A_{ij}} dh A \text{ Im } \omega = 0.
\]
This is easily obtained by defining \( h = 0 \) on \( U_i - A_{ij} \). Now we can apply Lemma 4.2 to obtain that the boundary measure \( \mu_{ij} \) of \( \omega \) on \( \partial A_{ij} \cap \partial U_i \) is real. Since boundary measure \( \mu_i \) of \( \omega \) is \( \sum_{j=1}^{N(i)} \mu_{ij} \) by Theorem 3.6, \( \mu_i \) is real.

**Theorem 4.4.** \( m(K) \equiv H^1(\Omega) \).

**Proof.** It is immediate from Corollary 3.13 and Theorem 4.3.

5. A natural basis for \( C(\partial K)/\text{Re } \mathcal{R}(K) \) (Theorem 1.2). We may assume without loss of generality that \( X \) is a noncompact surface with analytic boundary and \( K \) a compact subset of \( X \) such that \( X - K \) has only finitely many connected components none of which is relatively compact. By Theorem 2.1, the canonical homomorphism \( H_i(K) \to H_i(X) \) is injective.

Let \( \gamma_i(1 \leq i \leq k) \) be a homology basis for \( \hat{K} \) and \( \gamma_i(1 \leq i \leq k + l) \) be a homology basis for \( X \). Let \( \Theta \) denote the space of all harmonic functions \( h \) on \( X \) such that \( \int dh A^* dh < \infty \).

We contend that given any \( \sum a_i \gamma_i \neq 0, a_i \) real, there exists \( h \in \Theta \) such that \( \int_{\Sigma a_i \gamma_i}^* dh \neq 0 \). Assume the contrary .

Then there exists a harmonic differential \( \sigma \) with compact support (see Ahlfors-Sario [3], p. 288) such that
\[ \int_{\Sigma_{a_i \gamma_j}} \ast \, dh = \int \sigma A \ast \, dh \]
and so
\[ \int \sigma A \ast \, dh = 0 \forall h \in \mathbb{C}, \]
i.e. \( \ast \sigma \) also has compact support. But \( \sigma - i \ast \sigma \) is a holomorphic 1-form and it cannot have compact support unless \( \sigma = i \ast \sigma = 0 \) which implies \( \sum a_i \gamma_i \) is homologous to zero.

This proves that the mapping \( \psi: \mathbb{C} \to R^{k+1} \) given by \( \psi(h) = (\int_{\gamma_1} \ast \, dh, \ldots, \int_{\gamma_{k+1}} \ast \, dh) \) is a surjection. Now let us pick \( h_i \in \mathbb{C} \) such that \( \int_{\gamma_i} \ast \, dh_i = 1 \) and \( \int_{\gamma_j} \ast \, dh_i = 0 \) for \( j \neq i \).

We claim now that \( h_1, h_2, \ldots, h_k \) form a basis of \( \mathbb{C}(\partial K) \) modulo \( \text{Re} \mathbb{B}(K) \). Assume \( \sum a_i h_i \in \text{Re} \mathbb{B}(K) \). Then there exists a function \( f \) holomorphic in a neighborhood of \( K \) such that \( |\sum a_i h_i - \text{Re} f| < \varepsilon \) on \( \partial K \).

Since \( \gamma_i \) lie in \( K \) for \( 1 \leq i \leq k \), and \( \int_{\gamma_j} |\sum a_i \ast \, dk_i - \text{Im} \, df| < C \varepsilon \) where \( C \) depends only on \( \gamma_j \).

Since \( \int_{\gamma_j} df = 0 \) and \( \int_{\gamma_i} \ast \, dh_i = \delta_{ij} \) (Kronecker \( \delta \)), we obtain that \( |a_i| < C \varepsilon \) for \( 1 \leq i \leq k \). Since this is true for all \( \varepsilon > 0 \), \( a_i = 0 \forall i \).

Thus \( \{h_i\}_{1 \leq i \leq k} \) are linearly independent modulo \( \text{Re} \mathbb{B}(K) \) and because \( \dim \mathbb{C}(\partial K)/\text{Re} \mathbb{B}(K) = k \), we have that \( \{h_i\}_{1 \leq i \leq k} \) is a basis for \( \mathbb{C}(\partial K)/\text{Re} \mathbb{B}(K) \).


REFERENCES


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