RIGHT CHAIN RINGS AND THE GENERALIZED SEMIGROUP OF DIVISIBILITY

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Let $R$ be a ring with unit element and without zero-divisors and let $\tilde{H}(R) = \{\tilde{x} | 0 \neq x \in R\}$ where $\tilde{x}$ is the mapping from the set of all nonzero principal right ideals of $R$ into itself defined by $\tilde{x}(aR) = xaR$. $\tilde{H}(R)$ is a partially ordered semigroup that can be considered as a generalization of the group of divisibility of a commutative integral domain. We study those rings $R$ for which $\tilde{H}(R)$ is totally ordered.

1. Introduction. Associated with any commutative integral domain $A$ is the partially ordered group $G(A)$ of nonzero fractional principal ideals of $A$ with $aA \leq bA$ if and only if $aA$ contains $bA$. It is well known (see [4], [5], [8]) that $G(A)$, the group of divisibility, reflects certain properties of $A$, like $A$ being a unique factorization domain, the fact that any two elements in $A$ have a greatest common divisor or $A$ being a valuation ring. This concept of a group of divisibility cannot be extended directly to a not necessarily commutative integral domain $R$.

In this paper we associate with any ring $R$ with unit element and without zero-divisors a partially ordered semigroup $\tilde{H}(R)$ which is isomorphic to the semigroup $H(A) \subseteq G(A)$ of nonzero principal ideals $aA$ in $A$ if $A$ is a commutative domain.

After observing some basic facts about $\tilde{H}(R)$ we characterize in §3 those rings $R$ with $\tilde{H}(R)$ totally ordered as right chain rings $R$ with $Ja \subseteq aR$ for all $a$ in $R$ and $J = J(R)$ the Jacobson radical of $R$. These rings are localizations of right invariant right chain rings. The main result of §4 is the theorem that a ring with $\tilde{H}(R)$ totally ordered and d.c.c. for prime ideals is right invariant. In a final §5 we show by examples that for every totally ordered group $G$ there exists a ring $R$ with $\tilde{H}(R)$ totally ordered and $G$ (not only the positive cone of $G$) can be embedded into $\tilde{H}(R)$. The value group $G(A)$ is particularly useful in case $A$ is a commutative valuation ring. The nonzero principal right ideals in a right chain ring $R$ form a semigroup $H(R)$ under ideal multiplication only if $R$ is right invariant. In the general case it is the semigroup $\tilde{H}(R)$ which takes the place of $H(R)$. Mathiak in [6] studies right and left chain domains with the help of a group that could be considered a generalization of $G(A)$. We found that in the case of one-sided conditions a generalization of $H(A)$, which will be a semigroup only, will be more natural.

2. Definition and preliminary results. We consider only rings
with unit element and without zero-divisors. We call a ring $R$ right invariant if $Ra \subseteq aR$ (if and only if $RaR = aR$) holds for all elements $a$ in $R$ and $R$ is a right chain ring (sometimes called a right valuation ring) if for $a, b$ in $R$ either $aR \subseteq bR$ or $bR \subseteq aR$ holds. Here $I \subseteq L$ always means that the set $I$ is contained in $L$; $J = J(R)$ is the Jacobson radical and $U = U(R)$ the group of units of $R$.

Let $W = \{aR | 0 \neq a \in R\}$ be the set of nonzero principal right ideals of $R$. Every element $0 \neq x$ in $R$ induces a mapping $\bar{x}$ on $W$ with $\bar{x}(aR) = xaR$; and $\bar{xy} = \bar{x}\bar{y}$ follows. With $\bar{x} \geq \bar{y}$ defined as $xaR \subseteq yaR$ for all $a$ in $R$ we can consider $\bar{H}(R) = \{\bar{x} | 0 \neq x \in R\}$ as a partially ordered semigroup. Further, $x + y \geq \inf (\bar{x}, \bar{y})$; i.e., $\bar{z} \leq \bar{x}$, $\bar{z} \leq \bar{y}$ implies $\bar{z} \leq \bar{x} + \bar{y}$. The mapping $\sim$ from $R^* (= R^0)$ to $\bar{H}(R)$ is called the regular right valuation of $R$ with the value-semigroup $\bar{H}(R)$. This semigroup satisfies the following conditions:

1. $\bar{H}(R)$ is a partially ordered semigroup with unit element $\bar{1}$.

2. $\bar{a} \leq \bar{b}$ if and only if there exists a $\bar{t}$ in $\bar{H}(R)$ with $\bar{a}\bar{t} = \bar{y}$ and $\bar{1} \leq \bar{t}$.

3. $\bar{a}\bar{y} = \bar{a}\bar{z}$ implies $\bar{y} = \bar{z}$ for $\bar{a}$, $\bar{y}$, $\bar{z}$ in $\bar{H}(R)$.

This means that the order in $\bar{H}$ is a right natural order and $H$ is left cancellative.

We draw a few immediate conclusions from these properties:

(i) $\bar{x} \leq \bar{1}$ implies that $\bar{x}$ is a unit in $\bar{H}$, i.e., there exists $\bar{y}$ with $\bar{xy} = \bar{y}\bar{x} = \bar{1}$.

(ii) $\bar{1} \leq \bar{x}$ implies $\bar{a}\bar{a} = \bar{a}\bar{a}'$ for some $\bar{a}'$ in $\bar{H}$.

To prove (i) we have by (2) an element $\bar{t}$ with $\bar{x}\bar{t} = \bar{1}$. This implies $\bar{x}\bar{t}\bar{x} = \bar{x}$ and $\bar{t}\bar{x} = \bar{1}$ using (3). For $\bar{1} \leq \bar{x}$ and $\bar{a}$ in $\bar{H}$ we have $\bar{a} \leq \bar{x}\bar{a}$ and $\bar{x}\bar{a} = \bar{a}\bar{a}'$ for some $\bar{a}'$ using (2) again. Let $\bar{U} = \bar{U}(R)$ be the subgroup of units of $\bar{H}(R)$. The following condition is satisfied by $\bar{H}(R)$:

(4) Let $\bar{U}'$ be a subgroup of $\bar{U}$ with $\bar{U}'\bar{x} \subseteq \bar{x}\bar{U}$ for all $\bar{x}$ in $\bar{H}(R)$. Then $\bar{U}' = \{\bar{1}\}$. In particular $\bar{U} = \{\bar{1}\}$ for $R$ commutative. The following is an easy example of a semigroup $S$ satisfying conditions (1)-(3), but not (4).

Let $S = \{(n, a); n, a \in \mathbb{Z}; n \geq 0\}$ considered as a subsemigroup of $G = \mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}$ the integers. We write $(n, a) > (m, b)$ if either $n > m$ or $n = m$ and $a > b$. Conditions (1), (2), (3) hold for $S$, but $U = \{(0, a); a \in \mathbb{Z}\}$ is a subgroup $\neq \{e\}$ of $S$, violating (4).

Two obvious problems arise: What is the structure of semigroups with (1), (2), (3), (4)? Given a semigroup $S$ satisfying (1), (2), (3), (4) is $S = \bar{H}(R)$ for some $R$? We are not able to answer these questions in general.

**Definition.** Let $R$ be a ring. Then
\[ \hat{R} = \{ r \in R \mid r \geq 1 \} \cup \{0\} = \{ r \in R \mid raR \subseteq aR \ \text{for all} \ a \ \text{in} \ R \} . \]

It is obvious that \( \hat{R} \) is a subring of \( R \).

**Lemma 1.** (1) \( \hat{R}a \subseteq a\hat{R} \) for all \( a \) in \( R \); in particular \( \hat{R} \) is a right invariant subring of \( R \).

(2) The mapping \( a\hat{R} \) to \( a \) for \( a \neq 0 \) in \( R \) defines an isomorphism between the semigroup \( C(R) \) of \( R \)-modules \( a\hat{R} \) with \( a \) in \( R \) onto \( \hat{H}(R) \). In \( C(R) \) we have \( a\hat{R}b\hat{R} = ab\hat{R} \) as operation and \( a\hat{R} \leq b\hat{R} \) if and only if \( a\hat{R} \supseteq b\hat{R} \).

(3) \( \hat{H}(R) \cong R^* / U(\hat{R}) \) where \( U(\hat{R}) \) is the group of units of \( \hat{R} \) and \( r_1 \equiv r_2 \) if and only if \( r_1 = r_2u \) with \( u \) in \( U(\hat{R}) \) defines a congruence relation on \( R^* \), the multiplicative semigroup of nonzero elements in \( R \).

**Proof.** (1) \( \hat{R}a \subseteq a\hat{R} \) by definition. If \( r \) is in \( \hat{R} \) then \( ra = ar \) and \( rab = abr = arb \) for any \( a, b \) in \( R \) with \( r, r \) in \( R \). But \( r,b = br \) implies \( r \) in \( \hat{R} \) and \( \hat{R}a \subseteq a\hat{R} \) for \( a \neq 0 \) in \( R \).

(2) Using (1) it follows that \( a\hat{R}b\hat{R} = ab\hat{R} \) for \( a, b \) in \( R \). If \( \hat{a} \geq \hat{b} \) then \( axR \subseteq bxR \) for all \( x \) in \( R \) and \( a = bs \) and \( s \) in \( \hat{R} \), hence \( a\hat{R} \subseteq b\hat{R} \) follows. Reversing these arguments yields the converse and \( \hat{H}(R) \cong [a\hat{R} \mid 0 \neq a \ \text{in} \ R] \) as a partially ordered semigroup.

(3) is just a different version of (2). \( \square \)

**Remark.** If \( R \) is embeddable into some skew field then \( \hat{R} = \bigcap_{0 \neq a \in R} aRa^{-1} \).

If \( R \) is a ring such that the product of any two nonzero principal right ideals is again a nonzero principal right ideal we write \( H(R) \) for the semigroup of the nonzero principal right ideals of \( R \); \( H(R) \) is a partially ordered semigroup with \( aR \leq bR \) if and only if \( a\hat{R} \subseteq b\hat{R} \).

If \( H(R) \) exists and is isomorphic to \( \hat{H}(R) \) under the mapping that assigns \( \bar{x} \) to \( xR \) then \( R \) is right invariant. On the other hand \( H(R) \) does exist for some rings that are not right invariant; simple rings or not right invariant principal ideal domains are obvious examples.

The following lemma shows that \( H(R) \) exists for a local ring \( R \) if and only if \( R \) is right invariant.

**Lemma 2.** Assume \( H(R) \) exists and let \( 0 \neq a \) be in \( R \). Then \( RaR = bR \) for some \( b \) and if \( a = bc \) then \( c \) is not contained in \( J(R) \).

**Proof.** It only remains to show that \( c \) is not in \( J(R) \). We have \( b = \sum r_i a s_i \) for some \( r_i, s_i \) in \( R \); \( b = \sum r_i b c s_i = \sum br_i c s_i = b \sum r_i c s_i \), where
Corollary. If $R$ is local then $H(R)$ exists if and only if $R$ is right invariant.

3. $\hat{H}(R)$ totally ordered. If $A$ is a commutative integral domain its group of divisibility $G(A)$ is totally ordered only if $A$ is a valuation ring. We will discuss the corresponding question for $\hat{H}(R)$ and characterize the rings with $\hat{H}(R)$ totally ordered. If $x$ and $y$ are nonzero elements in $R$ then $x \leq y$ or $y < x$ and $xR \supseteq yR$ or $yR \supseteq xR$ follows. Therefore, $R$ is a right chain ring if $\hat{H}(R)$ is totally ordered. Examples (see §5) show that for $R$ a right chain ring $\hat{H}(R)$ is not necessarily totally ordered.

Theorem 1. For an integral domain $R$ the following conditions are equivalent:

1. $\hat{H}(R)$ is totally ordered.
2. $R$ is a right chain ring such that $r$ in $R$, not in $\hat{R}$ implies $r^{-1}$ in $\hat{R}$.
3. $R = R_p$, the localization of a right invariant right chain ring $R'$ at a prime ideal $P$ of $R'$.
4. $R$ is a right chain ring such that $Ja \subseteq aR$ for all $a$ in $R$.
5. $R$ is a right chain ring and if $Ra \not\subseteq aR$ then $Ja \subseteq aJ$ for any $a$ in $R$.
6. The submodules of the right $\hat{R}$-module $R$ are totally ordered.

Proof. (1) $\Rightarrow$ (2) We observed that $R$ is a right chain ring if $\hat{H}(R)$ is totally ordered. For an element $r$, not in $\hat{R}$, we have $\tilde{r} < \hat{1}$, hence $raR \supseteq aR$ for all $a \in R$ and $r$ in $U(R)$, $r^{-1}$ in $\hat{R}$ follows. (2) $\Rightarrow$ (3) It follows from (2) that $\hat{R}$ is a right chain ring and from Lemma 1 that $\hat{R}$ is right invariant. The set $S = \hat{R} \cap U(R)$ is multiplicatively closed and $P = \hat{R} \setminus S$ is a prime ideal in $\hat{R}$. Finally, $R = \hat{R}_p = \hat{R}S^{-1}$ is the localization of $\hat{R}$ at $P$.

To prove that (3) implies (1) we need a few lemmas.

Let $R$ be a right invariant right chain ring. We write $\bar{T} = \{\overline{t} \in \bar{H}(R) | t \in T\}$ for a subset $T \subseteq R^*$ and we say $\bar{T}$ ($\neq \emptyset$) is $R$-convex if for $tR \subseteq sR \subseteq R$, $t$ in $T$, the element $\overline{s}$ is contained in $\bar{T}$. One can check the following two statements.

Lemma 3. There is a one-to-one correspondence between the set of $R$-convex subsets of $\bar{H}(R)$ and the right ideals $\neq R$ given by

$$\bar{S} \rightarrow \bar{S}' = \{x \in R | x \in \bar{S}\} \cup \{0\}$$

$$I \rightarrow I' = \{\overline{x} \in \bar{H}(R) | xR \supset I\}$$
where \( \tilde{S} \) is \( R \)-convex and \( I \) is a right ideal \( \neq R \).

**Lemma 4.** The \( R \)-convex subset \( \tilde{S} \) is a subsemigroup of \( \tilde{H}(R) \) if and only if \( \tilde{S}' = P \) is a completely prime ideal of \( R \).

We consider the situation as described in the last lemma. Then \( S = \{ x \in R \mid \tilde{x} \in \tilde{S} \} \) is a multiplicatively closed saturated (i.e., \( ab \) in \( S \) implies \( a, b \) in \( S \)) right Ore system in \( R \). The corresponding prime ideal is \( P = R \backslash S \) and \( R_p = RS^{-1} \) is the corresponding localization. Set \( N = N(S) = \{ r \in R \mid ra = sa, s \in S \text{ for all } a \neq 0 \text{ in } R \} \). \( N \) is an \( R \)-convex subsemigroup of \( S \) maximal with the property that \( a^{-1}Na \subseteq N \) for all nonzero \( a \) in \( R \). To see this, one observes that with \( n \) in \( N \), \( nR \subseteq mR \subseteq R \), we have \( n = mr \) for some \( r \) and \( na = as = am'r' \) for \( m', r' \) in \( R \) with \( ma = am' \), \( ra = ar' \). Therefore \( m'r' = s_a \) is in \( S \) and \( m' \) in \( S \), and \( m \) in \( N \). Further, \( n \) in \( N \) and \( na = as \) implies \( s_a \) in \( N \).

To \( N \) there corresponds a prime ideal \( Q = R \backslash N \) with \( P \subseteq Q \subseteq J \). We want to describe \( \tilde{H}(R_p) \) and we will get the result by considering two special cases:

1. \( N(S) = S \), i.e., \( Q = P \) (Lemma 5) and
2. \( N(S) = U(R) \), i.e., \( Q = J \) (Lemma 6).

**Lemma 5.** Let \( R \) be a right invariant right chain ring, \( P \) a prime ideal in \( R \), \( S = R \backslash P \). Assume \( N(S) = N = S \). Then \( R_p \) is again right invariant and \( \tilde{H}(R_p) \simeq \tilde{H}(R) \backslash \tilde{N} = \tilde{H} \).

**Proof.** That \( R_p \) is again right invariant follows from the fact that every principal right ideal in \( R_p \) has the form \( aR_p \) with \( a \) in \( R \) and that \( sa = as \) for all \( s \) in \( S \) if \( s \) is in \( S = N \). Hence \( rs^{-1}aR_p = raR_p = ar'R_p \) with \( ra = ar' \), \( r, a \) in \( R \). If one defines \( \tilde{r}_1 \equiv \tilde{r}_2, r_1, r_2 \) nonzero elements in \( R \), if and only if \( r_1 = r_2n \) or \( r_1n = r_2 \) for some \( n \) in \( N \), then “\( \equiv \)” is a congruence relation defined on \( \tilde{H} \), and we write \( H = \tilde{H}(R) \backslash \tilde{N} \) for the factor semigroup modulo this congruence. Further, \( \tilde{r}_1 > \tilde{r}_2 \) in \( \tilde{H} \) if and only if \( r_1 > r_2 \) in \( \tilde{H}(R) \) and \( \tilde{r}_1 \neq \tilde{r}_2 \). It follows that \( \tilde{H} \simeq \tilde{H}(R_p) \) as totally ordered semigroups.

**Lemma 6.** Let \( R \) be a right invariant, right chain ring, \( P \) a prime ideal in \( R \), \( S = R \backslash P \). Assume \( N(S) = U(R) \). Then \( R_p \) is not right invariant if \( P \subseteq J \) and \( \tilde{H}(R_p) \simeq \tilde{H}(R)\tilde{S}^{-1} \).

**Proof.** \( \tilde{H}(R) \) contains the subsemigroup \( \tilde{S} \). We will prove that under the above assumption \( \tilde{H}(R) \) can be embedded into the semigroup \( \tilde{H}(R)\tilde{S}^{-1} = \{ \tilde{r}\tilde{s}^{-1} \mid r \in R^*, s \in \tilde{S} \} \) of fractions for \( \tilde{H}(R) \).
The semigroup \( H(R) \) is totally ordered and \( \alpha \beta = \alpha \gamma \) for \( \alpha, \beta, \gamma \) in \( H(R) \) implies \( \beta = \gamma \). Since the other cancellation law does not hold in general, \( H(R) \) itself may not be embeddable into a group. But for every \( \tilde{r} \) in \( \tilde{H}(R) \) and \( \tilde{s} \) in \( \tilde{S} \) there exists an element \( \tilde{a} \) in \( \tilde{H}(R) \) with \( \tilde{r}\tilde{a} = \tilde{s} \) or \( \tilde{r} = \tilde{s}\tilde{a} \) and \( \tilde{H}(R)\tilde{S}^{-1} \) exists ([3], Prop. 5.1; page 21) if we can show that \( \tilde{r}\tilde{s} = \tilde{s}\tilde{s} \) implies \( \tilde{r} = \tilde{r}_1 \) for \( \tilde{r}_1, \tilde{r}_2 \) in \( \tilde{H}(R), \tilde{s} \) in \( \tilde{S} \).

We can assume \( r_1 = r_2c \) for some \( c \) in \( R \) and we are done if we can show that \( c \) is in \( N \). But, \( \tilde{r}\tilde{s} = \tilde{s}\tilde{s} \) implies \( r_2cs = r_2se \) for some \( e \) in \( U(R) \). Therefore \( cs = se \) and \( c \) is an element of \( S \). Let \( a \) be in \( R \). If \( a \) is in \( S \) then \( ca = ac' \) with \( c' \) in \( S \). If \( a \) is not in \( S \) then \( a = sa_i \) for some \( a_i \) in \( R \) and \( ca = csa_i = sa_i = sa_i' = ae' \) with \( e' \) in \( U(R) \). Hence, \( c \) is in \( N = U(R) \) and \( K = \tilde{H}(R)\tilde{S}^{-1} = \{ \tilde{r}\tilde{s}^{-1} | r \in R^*, s \in S \} \) exists.

This semigroup is totally ordered if we define \( \tilde{r}\tilde{s}^{-1} \geq \tilde{r}_2\tilde{s}^{-1} \) if and only if for all \( \tilde{s}, \tilde{s}' \), with \( \tilde{s}\tilde{s} = \tilde{s}\tilde{s}' \) we get \( \tilde{r}\tilde{s} \geq \tilde{r}\tilde{s}' \).

This last condition is equivalent to \( \tilde{r}_1 \geq \tilde{r}_2 \tilde{s}^{-1} \) if \( s_1 = s_2 \) and \( \tilde{r}\tilde{s} \geq \tilde{r}_2 \) if \( s_1 = s_2 \) where \( s \) is some element in \( S \). For the necessary computations it is the easiest to write any finite number of elements in \( K \) in the form \( \tilde{r}\tilde{s}^{-1}, i = 1, \ldots, n \).

It is a bit tedious to check that \( K \) is a totally ordered semigroup with unit element such that

(i) \( \alpha \geq \beta \) in \( K \) implies that there exists \( \gamma \) in \( K \) with \( \alpha = \beta \gamma \)

(ii) \( \gamma \alpha = \gamma \beta \) implies \( \alpha = \beta \) where \( \alpha, \beta, \gamma \) are in \( K \).

Further, it follows from these conditions that all elements \( \gamma \leq \tilde{1} \) in \( K \) have an inverse in \( K \).

It remains to show that \( K \simeq \tilde{H}(R_P) \) as ordered semigroups where the isomorphism is given by \( \tilde{r}\tilde{s}^{-1} \leftrightarrow \tilde{r}\tilde{s}^{-1} \). (Here \( \tilde{r}, \tilde{s} \) are elements in \( \tilde{H}(R), \tilde{s}^{-1} \) is an element in \( \tilde{H}(R_P) \).) We shall show here that the given correspondence is one-to-one and omit the rest.

Let \( \tilde{r}\tilde{s}^{-1} = \tilde{r}_1\tilde{s}^{-1} \) i.e., \( r_1s^{-1}aR_P = r_2s^{-1}aR_P \) for all \( a \) in \( R_P \); in particular \( r_1s^{-1}sbR_P = r_2s^{-1}sbR_P \) for all \( b \) in \( R \) and \( r_1bR_P = r_2bR_P \), \( r_1b = r_2b' \) or \( r_1b' = r_2b \) for some \( b' \) in \( S \) follows. Comparing \( r_1 \) and \( r_2 \) yields \( r_1 = r_2c \) or \( r_1 = r_2c' \) for some \( c \) in \( N \) and \( \tilde{r}_1 = \tilde{r}_2 \) in \( \tilde{H}(R) \).

If conversely \( \tilde{r}_1\tilde{s}^{-1} = \tilde{r}_2\tilde{s}^{-1} \) in \( K \) we get \( \tilde{r}_1 = \tilde{r}_2 \) in \( \tilde{H}(R) \) and therefore \( r_1s^{-1}aR_P = r_2s^{-1}aR_P \) for all \( a \) in \( R \). If \( a \) is in \( S \) this is obvious, otherwise \( a = sb \) and \( r_1bR = r_2bR \) implies \( r_1s^{-1}aR_P = r_2s^{-1}aR_P \) in that case. Finally let \( s \) be in \( S \setminus U(R) \). Then there exists \( a \) in \( R \) with \( sa = ag \) and \( g \) not in \( S \) since \( s \) is not in \( N \). This shows that \( s^{-1}aR_P \supset aR_P \) and \( R_P \) is not right invariant.

If we combine Lemma 5 and Lemma 6 we get the following result:

**Theorem 2.** Let \( R \) be a right invariant right chain ring, \( P a \)
prime ideal in $R$, $S = R\setminus P$; $N = \{x \in R \mid xa = as, s \in S \text{ for all } a \in R\}$. Then:

1. $\bar{\mathcal{H}}(R_p) \simeq \bar{\mathcal{S}}^{-1}$ is a totally ordered semigroup with $\bar{\mathcal{H}} = \bar{\mathcal{H}}(R_Q) \simeq \bar{\mathcal{H}}(R)/\bar{N}$ and $\bar{S} \simeq \bar{S}/\bar{N}$; $Q = R\setminus N$ is a prime ideal and $R_Q$ is right invariant.

2. $R_p$ is right invariant if and only if $N = S$.

With Theorem 2 the equivalence of (1), (2), (3) in Theorem 1 is proven.

We prove the equivalence of (1) and (4). If $\bar{\mathcal{H}}(R)$ is totally ordered and $j$ in $J(R)$, then $\tilde{j} \leq \tilde{1}$ is impossible, since this implies $jR = R$, $j$ a unit. Hence $jaR \subseteq aR$ for all $a$ in $R$. Conversely if $R$ is a right chain ring with $Ja \subseteq aR$ for all $a$ in $R$ we must show that for any nonzero elements $x$, $y$ in $R$ either $\bar{x} \leq \bar{y}$ or $\bar{y} \leq \bar{x}$. If we assume on the contrary that there exist $a$, $b$ in $R$ with $xaR \subseteq yaR$ and $ybR \subseteq xbR$ we obtain $xa = yav_1$, $yb = xbv_2$ and say $a = bs$ for $v_1$, $v_2$, $s$ in $J$ (the case $b = as$ is similar). Then $ya = ybs = xbv_2s = xbsv_2 = xav_2 = yav_1v_2$ and $ya = 0$ where $v_2s = sv_2'$ for some $v_2'$ in $R$, using (4).

The implication $(5) \Rightarrow (4)$ is obvious. To prove $(4) \Rightarrow (5)$ assume there is an $a$ in $R$ with $Ra \nsubseteq aR$ and $Ja \nsubseteq aJ$, but $Ja \subseteq aR$. Then there exist elements $u$ in $U(R)$, $n$ in $J$ with $uaR \supseteq aR$ and $uan = a$; and elements $n'$ in $J$, $u'$ in $U(R)$ with $n'a$ in $aR$, but not in $aJ$, hence $n'au' = a$. This leads to $un'au' = a$ and with $Ja \subseteq aR$ to $a = 0$, a contradiction. The equivalence of (1) and (6) follows from Lemma 1(2) and with this Theorem 1 is proved completely.

**Definition.** A right chain ring $R$ that satisfies the equivalent conditions of Theorem 1 is called **semi-invariant**.

Since $\bar{\mathcal{H}}(R)$ is not known even if $R$ is right invariant unless $R$ is also right noetherian or satisfies some other extra condition (see [1]) we cannot describe the structure of $\bar{\mathcal{H}}(R)$ for a semi-invariant ring $R$. It follows from Theorem 2 that this semigroup is a group of fractions of a semigroup $H = \bar{\mathcal{H}}(R')$ where $R'$ is a right invariant right chain ring with respect to a subsemigroup $T$ of $H$ which satisfies

1. If $t$ is in $T$, $h$ in $H$ and $e$ the unit element in $H$ with $e \leq h \leq t$, then $h$ is in $T$.
2. For every $e \neq t$ in $T$ there exist $h$ and $k$ in $H$ with $th = eh$ and $k$ not in $T$.
3. $h,t = h_2t$ for $t$ in $T$, $h_1$, $h_2$ in $H$ implies $h_1 = h_2$.

One sees that $\bar{\mathcal{H}}(R)$, $R$ semi-invariant, not a division ring, is not a group, but we will show that for every totally ordered group $G$
there exists a semi-invariant ring $R$ such that $G$ can be embedded into $\bar{H}(R)$.

4. Semi-invariant right chain rings with d.c.c. for prime ideals. Investigating the condition $\bar{H}(R)$ totally ordered, we were led to semi-invariant right chain rings. The valuation semigroup can then be described using Theorem 2. In many cases we actually have $H(R) \cong \bar{H}(R)$. The reason for this is the result we will prove in this section: Semi-invariant right chain rings with d.c.c. for prime ideals are right invariant. We recall that an ideal $P$ in $R$ is called completely prime if $ab$ in $P$ implies $a$ or $b$ in $P$ and $P$ is called prime if $aRb$ in $P$ implies $a$ or $b$ in $P$ where $a$, $b$ are elements in $R$. It follows from a result of Thierrin ([10]) that a prime ideal $P$ is completely prime if $a^2$ in $P$ implies $a$ in $P$.

**Lemma 7.** Every prime ideal $P$ in the semi-invariant ring $R$ is completely prime.

**Proof.** Assume $a^2$ in $P$ and $a$ not in $P$. Then there exists $t_1$ in $R$ with $at_1a$ not in $P$ and $t_2$ in $R$ with $at_2(at_1a)$ not in $P$. We can assume $R \neq P$ and $a$ in $J$. Hence $a(t_2at_1)a = a^2r$ for some $r$ in $R$ using (4) of Theorem 1. This contradiction proves the lemma. 

The next result shows how to produce certain prime ideals.

**Lemma 8.** Let $z$ be an element in $R$, a semi-invariant ring. Then $D = \cap z^nR$ is a prime ideal.

**Proof.** We can assume that $z$ is in $J$. Then $D$ is a right ideal and we will first show that $a^2$ in $D$ implies $a$ in $D$ for $a$ in $R$. Assume $a$ is not in $D$, then $a$ is in $J$ and $aj = z^n$ for some natural number $n$ and $j$ in $J$. But then $ajaj = a^2j'j = z^{2n}$ is not in $D$ contradicting $a^2$ in $D$. It remains to prove that $D$ is a left ideal. Let $x$ be in $D$ and $x = z^nq_n$, $q_n$ in $J$ follows. For $r$ in $R$ we get $rxrx = rxrz^2q_n = z^nvq_n$ for some $v$ in $R$. This shows that $(rx)^2$ is in $D$ and hence $rx$ in $D$.

The next theorem will be proved in three steps, Lemmas 9-11.

**Theorem 3.** A semi-invariant right chain ring with d.c.c. for ideals is right invariant.

Let $a$ be an element in the semi-invariant right chain ring $R$. By (5) Theorem 1 we have either $Ra \subseteq aR$ or $Ja \subseteq aJ$. In the first
case we are done and in the second we define a mapping $\phi$ from the set of prime ideals $P \neq R$ into itself by defining $P^\phi$ as the smallest prime ideal with $Pa \subseteq aP^\phi$. We will show that either $J^\phi = J$ which implies $Ra \subseteq aR$ or $J^\phi \subset J$ and $\{J^\phi\}$ is a strictly decreasing chain of prime ideals of $R$.

**Lemma 9.** Let $J = J^\phi$ and $J = mR$, then $Ra \subseteq aR$.

**Proof.** We have $ma = am^kv$ for some unit $v$ in $R$, some integer $k$, some generator $m$ of $J$, since as a right ideal $J^\phi = J$ using Lemma 8.

If $Ra \not\subseteq aR$ there exists a unit $u$ in $R$ and an element $q$ in $J$ with $ua = aq$. Since $q$ is in $J$ and $u^{k+1}a = aq^{k+1}$ we obtain $q^{k+1}R \subseteq m^kR$ and we can assume $qR \subseteq m^kR$ and $q = m^kvt$ with $t$ in $J$. With $us = m$, $ma = am^kv$, $mat = am^kvt = aq = ua$ we obtain $sat = a$, $s$, $t$ in $J$ and $a = 0$ follows.

**Lemma 10.** Let $R$ be semi-invariant, $J$ not finitely generated as a right ideal and $0 \neq a$ an element in $R$ with $Ja \subseteq aJ^\phi$, $J^\phi = J$. Then $Ra \subseteq aR$.

**Proof.** Assume $j \neq 0$ in $J$. We want to find $r$, $s$ in $J$ with $ra = as$ and $sR \supseteq jR$. Let $P = \bigcap j^R$. By Lemma 8, $P$ is a prime ideal and $P \subset J$. Since $J^\phi = J$ there exist elements $r_1$, $s_1$ in $J$ with $s_1$ not in $P$ such that $r_1a = as_1$. Either $s_1R \supseteq jR$ and we are done or there exists an $n$ with $jR \supset j^{n+1}R \supset s_1R \supset j^R$. Hence $s_1q = j^n$ for some $q$ in $R$. We choose an element $z$ in $J$ with $r_1 = z^nv$ with $v$ in $J$ and some $m > n$. This is possible, since $J$ is not finitely generated: Let $r_iR \subset xR \neq R$. We obtain $r_1 = xy$ for $x$, $y$ in $J$. Choose $z_i$ in $J$ with $z_iR \supset xR$ and $z_iR \supset yR$ and $r_1 = z_iu_i$ follows with $u_i$ in $J$. Repeating this process yields an element $z$ with $r_1 = z^nv$, $z$, $v$ in $J$, $m > n$. Consider $za = az'$, $z$, $z'$ in $J$. We claim $z'R \supset jR$. Otherwise $jw = z'$ for some $w$ in $J$. But $r_1a = z^nv = az^nv' = as_1$ for some element $v'$ in $J$ with $va = av'$. Hence $s_1 = z^nv' = (jw)^nv' = j^nbv'$ for some element $b$ in $R$. This implies $j^n = s_1q = j^nbv'q$, a contradiction, since $m > n$. We conclude that we have found an element $r = z$, $s = z'$ with $sR \supset jR$ and $ra = as$ for the given element $j$ in $J$.

If $Ra \not\subseteq aR$ there exist a unit $u$ in $R$ and an element $t$ in $J$ with $ua = at$. By the above argument we have $s$, $r$ in $J$ with $ra = as$ and $sR \supset tR$. Hence, $sv = t$ for some $v$ in $J$ and $rav = asv = at = ua$. We obtain $a = u^{-1}rav = ak$, $k$ in $J$ and $a = 0$, a contradiction.
REMARK. Under the hypothesis of Lemma 10 we have proved that \( J^\phi = J \) is even the smallest two-sided ideal \( I \) satisfying \( Ja \subseteq aI \).

**Lemma 11.** Let \( R \) be semi-invariant, \( a \) in \( R \) with \( Ja \subseteq aJ^{\phi} \) and \( J^{\phi} \subseteq J \). Then \( J^{\phi n+1} \subseteq J^{\phi} \) for all \( n \).

**Proof.** We will write \( J^{(n)} \) instead of \( J^*_{\phi n} \). Then \( J^{(n+1)} \subseteq J^{(n)} \) and we assume \( n \) minimal with \( J^{(n)} = J^{(n+1)} \). Let \( r \) be in \( J^{(n-1)} \setminus J^{(n)} \), \( ra = as \) with \( s \) in \( J^{(n)} \). Then there exists \( a \) in \( J^{(n)} \) with \( qa = aq' \) and \( q'R \supset s^kR \) for some \( k \), since otherwise \( J^{(n+1)} = J^{(n)} \subseteq \cap s^kR \subseteq J^{(n)} \).

After replacing \( r \) by \( r^k \) if \( k > 1 \) we can assume that there is an \( r \) in \( J^{(n-1)} \setminus J^{(n)} \) with \( ra = as \) and an element \( q \) in \( J^{(n)} \) with \( qa = aq' \) and \( q'R \supset sR \). Hence \( q't = s \) for some \( t \) in \( J \) and \( rv = q \) for some \( v \) in \( J^{(n)} \). This yields \( ra = as = aq't = qat = rvat = rav't \) with \( v' \) in \( J \) and the contradiction \( ra = 0 \) proves the lemma. \( \square \)

5. Examples, problems and comments. We begin with an example of a semi-invariant right chain ring \( R \) such that \( \hat{H}(R) \) contains \( G \) where \( G \) is a given totally ordered group.

**Example 1.** For very totally ordered group \( G \) there exists a semi-invariant right chain ring \( R \) such that \( \hat{H}(R) \) contains \( G \).

Let \( K = \bigoplus_{i \in \mathbb{Z}} G_i \) where \( G_i \cong G \) for all \( i \in \mathbb{Z} \). \( K \) is an ordered group with the lexicographic ordering. Next, let \( L = \{ t^n k | n \in \mathbb{Z}, k \in K \} \) with \( t^n k_1 \cdot t^m k_2 = t^{n+m}(k_1^{[m]} k_2) \) be the ordered group where \( k = (g_i) \) and \( k^{[m]} = (g_i^m) \) with \( g_i^m = g_{i+m} \). Further \( t^n k_1 > t^m k_2 \) if and only if \( n > m \) or \( n = m \) and \( k_1 > k_2 \) in \( K \).

Let \( H = \{ t^n k \in L | t^n k \geq e, k = (g_i) \) with \( n \geq 0 \) and \( g_i = 1_{a_i} \) for \( i > 0 \} \). Then \( H \) is a totally ordered semigroup with unit element and both cancellation laws. Further, \( H \) is naturally ordered in the sense that \( h_1 \geq h_2 \) for \( h_i \) in \( H \) holds if and only if there exists an element \( h \geq e \) in \( H \) with \( h_1 = h_2 h \). Therefore it is possible to construct the generalized power series ring.

\[
R' = \{ \alpha = \sum x_h a_h | h \in H, a_h \in R \text{ and } T(\alpha) = \{ h | a_h \neq 0 \} \text{ well ordered in } H \}.
\]

\( R' \) is a right invariant right chain ring with \( \hat{H}(R') \cong H ([7]) \).

To the subsemigroup \( M = \{ t^0 g_i | g_i = 1_{a_i} \) for \( i \neq 0 \} \) there corresponds an \( R' \)-convex subsemigroup in \( \hat{H}(R') \) and a prime ideal \( P \) in \( R' \). We put \( R'_P = R \). Since for \( h \) in \( M \) we have \( ht = th' \) with \( h' \) not in \( M \) unless \( h = 1 \), we conclude that \( \hat{H}(R) \cong HM^{-1} = H \cup M^{-1} \).

It follows that \( G \) can be embedded into \( \hat{H}(R) \) where \( R \) is a semi-invariant right chain ring. We observe that the right ideal \( x_i R \) is
not a left ideal and \( R_x \) is not a right ideal. On the other hand we know ([2]) that for every \( a \) in \( a \) semi-invariant right and left chain ring either \( aR \) or \( Ra \) is a two-sided ideal.

**Example 2.** In our next example we construct a right chain ring \( R \) such that \( \overline{H}(R) \) is not totally ordered, but that the subgroup \( \overline{U}(R) = \{ \bar{u} | u \in U(R) \} \) of \( \overline{H}(R) \) is totally ordered with respect to the order as defined in \( \overline{H}(R) \). This condition

\[(U) \quad \overline{U}(R) \text{ is totally ordered is therefore weaker than the condition } \overline{H}(R) \text{ totally ordered and implies among other things that for a right chain ring } R \text{ with } (U), a \text{ in } R, \text{ there exists a unit } \varepsilon \text{ in } U(R) \text{ with } a\varepsilon \text{ in } R \text{ (see Lemma 12 (ii) below). } \]

The basic idea of this construction has been used in [9], [2] and [6]: Let \( R_1 \) be a right and left chain ring, \( D = Q(R_1) \) the division ring of quotients of \( R_1 \), \( H \) a totally ordered semigroup with unit element that satisfies both cancellation laws. Further, let \( h_1 \geq h_2 \) hold for elements \( h_1, h_2 \) in \( H \) if and only if \( h_1 = h_2h \) for some \( h \) in \( H \). Finally, let \( \tau \) be a mapping from \( H \) into the semigroup \( M(D) \) of monomorphism from \( D \) to \( D \) with \( \tau(h_1h_2) = \tau(h_1)\tau(h_2) \). One then can form the generalized power series ring \( D\{[H]\} = \{ \sum x_h d_h = \alpha \mid h \in H, d_h \in D, T(\alpha) = \{ h \mid d_h \neq 0 \} \} \) well ordered in \( H \) where multiplication is defined by \( x_h x_{h_2} = x_{h_1h_2} \) and \( dx_h = x_h d^{\tau(h)} \). The subring \( R \) of \( D\{[H]\} \) consisting of those elements \( \alpha \) with \( d_\alpha \) in \( R_1 \) is a right chain ring where \( e \) is the unit element in \( H \). It does not seem to be easy to determine \( \overline{H}(R) \) in general.

To consider a special case let \( F = Q(x, y) \), the field of rational functions in the two indeterminates \( x \) and \( y \) over the field \( Q \) of rational numbers. Then \( F \) contains \( R_1 = Q[x, y]_{(x)} \), a chain ring one obtains by localizing the polynomial ring \( Q[x, y] \) at the prime ideal \( (x) \). We form the skew power series ring \( F[[t, \tau]] \), where \( \tau \) is the automorphism of \( F \) exchanging \( x \) and \( y \). Finally, \( R \) consists of all those power series \( \sum t^j f_j(x, y) \) with \( f_j(x, y) \) in \( R_1 \). The principal right ideals of \( R \) are of the form \( t^m x^m R \) with \( n = 0, 1, 2, \cdots \) and \( m \) in \( Z \), but \( m \geq 0 \) if \( n = 0 \). The semigroup \( \overline{H}(R) = \{ t^m x^m y^k \mid n = 0, 1, 2, \cdots; m, k \in Z \text{ and } m \geq 0 \text{ if } n = 0 \} \). It is \( t^m x^m ; y^k ; t^n x^n y^k \) if \( n_1 > n_2 \) or \( n_1 = n_2 \) and \( m_1 > m_2 \) with \( k_1 \geq k_2 \) or \( n_1 = n_2, \text{ and } m_1 = m_2 \) and \( k_1 > k_2 \). Finally, we have \( \overline{U}(R) = \{ \bar{y}^k, k \in Z \} \cong Z \) as ordered groups. Therefore, \( \overline{H}(R) \) satisfies condition \( (U) \), but is not totally ordered: \( \bar{x}\bar{y}^{-1} \) and \( \bar{1} \) for example cannot be compared.

We conclude this paper with some observation for right chain rings that satisfy condition \( (U) \).

**Lemma 12.** Let \( R \) be a ring satisfying condition \( U \).
(i) Let \( a, b \in R \) with \( aR = bR \). Then either \( \bar{a} \leq \bar{b} \) or \( \bar{b} < \bar{a} \).

(ii) For any \( a \) in \( R \), \( R \) local, exists \( x \) in \( \hat{R} \) with \( aR = xR \).

(iii) Let \( R \) be a local ring and \( aR \supset bR \). Then there exists for every \( x \) with \( xR = aR \) a \( y \) in \( R \) with \( \bar{x} < \bar{y} \) and \( yR = bR \). Similarly for every \( y \) in \( R \) with \( yR = bR \) exists \( x \) with \( xR = aR \) and \( \bar{x} < \bar{y} \).

Proof. (i) is obvious, using condition (U). Statement (ii) is correct if \( a \) is a unit. We can therefore assume \( a \) in \( J \), \( a \) not in \( \hat{R} \). Hence \( 1 + a \) is in \( U(R) \backslash \hat{R} \) and \( (1 + a)(1 + x) = (1 + x)(1 + a) = 1 \) for some \( x \) in \( R \). But \( 1 + x \) and \( x \) are in \( \hat{R} \) and \( a(1 + x) = (1 + x)a = -x \) is in \( \hat{R} \). Since \( aR = xR \), (ii) follows.

To prove (iii) assume \( b = xp \). Using (ii) there exists a unit \( u \) in \( R \) with \( pu \) in \( \hat{R} \) and \( bu = xpu \) implies \( \bar{x} < \bar{b}u \). If \( y = ap \) the second part of (iii) is correct for \( p \) in \( \hat{R} \). Otherwise we obtain with (ii): \( (1 + p)^{-1}p \) is in \( \hat{R} \), \( y = a(1 + p)(1 + p)^{-1}p \) and \( x = a(1 + p) \). \( \Box \)

PROBLEMS.

(1) Describe all rings \( R \) for which \( \hat{H}(R) \) satisfies (U). (This class of rings contains all right invariant, in particular all commutative rings.)

(2) Which conditions characterize the semigroups \( S \) with \( S \cong \hat{H}(R) \), \( R \) a ring or additionally: \( R \) a right chain ring.

(3) Find the class of rings \( R \) with \( \hat{H}(R) \) lattice ordered.

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