

Pacific Journal of Mathematics

TORSION DIVISORS ON ALGEBRAIC CURVES

GEORGE KEMPF

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Let D be a divisor on a smooth complete algebraic curve C such that the multiple dD is rationally equivalent to zero for some positive integer d . We may write $D = D_0 - D_\infty$ where D_0 and D_∞ are distinct effective divisors.

The purpose of this note is to prove the

THEOREM. *Assume that C is a curve with general moduli in characteristic zero. For such a divisor D , the cohomology $H^1(C, \mathcal{O}_C(D_0 + D_\infty))$ must be zero (equivalently $|K - D_0 - D_\infty|$ is empty).*

This theorem answers a question which arose in Sevin Recillas' unpublished work which motivated this note. This paper contains simple computational techniques for handling infinitesimal deformations of curves. I hope that these techniques may be useful for solving similar problems.

The theorem is well-known for the case $d = 1$. For instance see Farkas' paper [2]. The recent work [1] contains further bilogographic material and related material concerning deformations of mappings of curves into projective spaces.

1. First order deformations defined by principal parts. Let \mathcal{L} be an invertible sheaf on a smooth curve C . For any point c of C , $\text{Prin}_c(\mathcal{L}) \equiv \text{Rat}(\mathcal{L})/\mathcal{L}_c$ measures the principal (polar) part of the rational sections of \mathcal{L} . Let $\text{Prin}(\mathcal{L})$ be the space of principal parts, $\bigoplus \text{Prin}_c(\mathcal{L})$, where c runs through C . For a given principal part $p = (p_c)$ in $\text{Prin}(\mathcal{L})$, the support, $\text{supp}(p)$, is the set of c in C with $p_c \neq 0$.

As in [3], we have a natural exact sequence

$$0 \longrightarrow \Gamma(C, \mathcal{L}) \longrightarrow \text{Rat}(\mathcal{L}) \longrightarrow \text{Prin}(\mathcal{L}) \longrightarrow H^1(C, \mathcal{L}) \longrightarrow 0.$$

Let p_c be a principal part in $\text{Prin}_c(\mathcal{L})$ with a pole of order one. By duality the image of p_c in $H^1(C, \mathcal{L})$ is zero if and only if each section of $\Omega_c \otimes \mathcal{L}^{\otimes -1}$ vanishes at c . Consequently, if $H^1(C, \mathcal{L}) \neq 0$ (i.e., there is a nonzero such section), the cohomology class of p_c is nonzero for all but a finite number of points c .

Let $\theta_c = \Omega_c^{\otimes -1}$ be the sheaf of regular vector fields on C . For any principal part X in $\text{Prin}(\theta_c)$, we want to define a deformation C_x of C over $T = \text{Spec}(k[\delta])$, where $k[\delta]$ is the ring of dual numbers $k[t]/(t^2)$. First note that, if f is a rational function which is regular at each point of $\text{supp}(X)$, then $X(f)$ makes sense in an obvious way

as a principal part in $\text{Prin}(\mathcal{O}_c)$. Secondly define the structure sheaf of C_x to be the subsheaf \mathcal{A}_x of $\mathcal{O}_c \oplus \delta \mathcal{A}t(\mathcal{O}_c)$, whose sections $f_0 + f_1 \delta$ locally satisfy the conditions: f_0 is regular and $X(f_0)$ is the principal part of f_1 (symbolically $(1 - X\delta)(f_0 + f_1 \delta)$ is regular).

Locally on C , X is the principal part of a rational vector field \underline{X} . For regular functions g_0 and g_1 , the expression

$$\phi_{\underline{X}}(g_0 + \delta g_1) \equiv (1 + \underline{X}\delta)(g_0 + g_1 \delta) = g_0 + (\underline{X}g_0 + g_1)\delta$$

defines an $k[\delta]$ -algebra isomorphism between \mathcal{A}_x and $\mathcal{O}_c \oplus \delta \mathcal{O}_c$, which preserves the augmentations of the two algebras onto \mathcal{O}_c . Therefore $C_x \rightarrow T$ is a (locally trivial) deformation of C over T , which has been given an obvious trivialization over the open subset $C\text{-supp}(X)$.

EXAMPLE. If t is a parameter function at a point c and $t(c) = 0$, let X be the principal part of the vector field $(1/t)(d/dt)$ at c . This principal part is determined by c upto constant. In this case, C_x is called a (first-order) Schiffer variation of the curve C . (If one replaces $(1 + X\delta)$ by $\exp(Xt) \bmod t^n$, as $n \rightarrow \infty$ one gets the formal version of a Schiffer variation in the compact Riemann surface case.)

REMARK. The deformation $C_x \rightarrow T$ depends only on the cohomology class of X in $H^1(C, \theta_c)$. This class is a form of the Kodaira-Spencer class of the deformation.

2. Obstructions to a particular deformation problem. Let D_0 and D_∞ be two effective divisors on our smooth curve C . Assume that D_0 and D_∞ have disjoint support. We are interested in the following special situation. For some positive integer d , dD_0 is linearly equivalent to dD_∞ . In other words, there is a nonconstant morphism $f: C \rightarrow \mathbf{P}^1$ such that $f^{-1}(0) = dD_0$ and $f^{-1}(\infty) = dD_\infty$. We want to examine whether this special situation may be extended to a deformation of C .

Let $g: \tilde{C} \rightarrow M$ be a deformation of \tilde{C} . Thus g is a smooth morphism and C is a closed subscheme of \tilde{C} , which has the form $g^{-1}(m)$ for some point m of M . Then we may ask if we can find a morphism $\tilde{f}: \tilde{C} \rightarrow \mathbf{P}^1$ such that $\tilde{f}|_C = f$, $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(\infty)$ are defined and we have $\tilde{f}^{-1}(0) = d\tilde{D}_0$ and $\tilde{f}^{-1}(\infty) = d\tilde{D}_\infty$ for some effective relative Cartier divisors \tilde{D}_0 and \tilde{D}_∞ on \tilde{C} with $\tilde{D}_i|_C = D_i$ for $i = 0$ or ∞ .

Next we will analyze this concept for first order deformations. Let $C_x \rightarrow T$ be the deformation defined in the first section for a given principal part X in $\text{Prin}(\theta_c)$. Here C and C_x are the same as topological spaces. Let C_i denote the open subscheme $C\text{-supp}(D_i)$ of

C where $\{0, \infty\} = \{i, j\}$. Our original morphism f is given by a pair (f_0, f_∞) where each f_i is a regular function on C_i and $f_0 \cdot f_\infty = 1$ on $C_0 \cap C_\infty$. Similarly, our extension \tilde{f} would be given by a pair $(f_0 + f'_0\delta, f_\infty + f'_\infty\delta)$, where $f_i + f'_i\delta$ is a section of \mathcal{A}_X over C_i , such that

(*) $f_0 f'_0 + f'_\infty f_\infty = 0$ on $C_0 \cap C_\infty$. As the $f_i + f'_i\delta$ are then reciprocals on $C_0 \cap C_\infty$, the conditions on the zeros and poles of \tilde{f} are just that, near $\text{supp}(D_i)$,

(**) there exist sections α and β of \mathcal{A}_X with $f_i + f'_i\delta = \alpha\beta^d$ and α is a unit.

In the simplest case, the question of the existence of such an \tilde{f} is easily interpreted.

PROPOSITION. *Assume that $\text{supp}(X)$ is contained $C_0 \cap C_\infty$ and $\text{char}(k) \nmid d$. Then the required extension \tilde{f} exists if and only if the principal part $(1/f)X(f)$ interpreted in $\text{Prin}(\mathcal{O}_c(D_0 + D_\infty))$ is cohomologous to zero.*

Proof. By the assumption on the $\text{supp}(X)$, any section of \mathcal{A}_X near $\text{supp}(D_i)$ can be written as $\alpha = \alpha_0 + \alpha_1\delta$, where α_0 and α_1 are regular functions. Furthermore, this section $\alpha_0 + \alpha_1\delta$ is a unit near $\text{supp}(D_i)$ if and only if α_0 is.

Claim. Given a rational function f'_i on C , then $f_i + f'_i\delta$ is a section of \mathcal{A}_X over C_i - $\text{supp}(X)$ and satisfies the condition (**) $\Leftrightarrow \text{div}(f'_i/f_i) + D_i \geq 0$ on C_i - $\text{supp}(X)$.

We will first see how this claim implies the proposition. Assume that $(f_1 + f'_1\delta, f_\infty + f'_\infty\delta)$ gives a required \tilde{f} . By the identity (*), we must have $f'_0/f_0 = -f'_\infty/f_\infty$. Thus the claim shows that the desired conditions on the zeros and poles of \tilde{f} imply that f'_i/f_i is a regular section of $\mathcal{O}_c(D_0 + D_\infty)$ over C - $\text{supp}(X)$. As $f_i + f'_i\delta$ is a section of \mathcal{A}_X over C_i , $X(f_i)$ is the principal part of the rational function f'_i except at $\text{supp}(D_0 + D_\infty)$. So f'_i/f_0 is a rational section of $\mathcal{O}_c(D_0 + D_\infty)$ with principal part $1/f_0 X(f_0)$. Therefore, if our \tilde{f} exists, $(1/f) X(f)$ is cohomologous to zero in $\text{Prin}(\mathcal{O}_c(D_0 + D_\infty))$.

Conversely, if $1/f X(f)$ is cohomologous to zero, we may have a rational section f'_0/f_0 of $\mathcal{O}_c(D_0 + D_\infty)$ with principal part $1/f_0 X(f_0)$. Define f'_∞ by the identity (*). Reversing the above argument, one can readily check that the required extension \tilde{f} is given by $(f_0 + f'_0\delta, f_\infty + f'_\infty\delta)$. Thus the claim implies the proposition.

To prove the claim, assume that $f_i + f'_i\delta$ satisfies condition (**). Then near $\text{supp}(D_i)$ we have $f_i + f'_i\delta = (\alpha_0 + \alpha_1\delta)(\beta_0 + \beta_1\delta)^d = \alpha_0\beta_0^d +$

$(\alpha_1\beta_0^d + d\beta_0^{d-1}\beta_1\alpha_0)\delta$, where the α 's and β 's are regular and α_0 is a unit. So $\beta_0 f'_i/f_i = (\alpha_1\beta_0 + d\beta_1\alpha_0)/\alpha_0$ is regular. As $D_i = (\beta_0 = 0)$, this means that $\text{div}(f'_i/f_i) + D_i \geq 0$ along D_i . Conversely, assume that this is true. By the assumption on f , we have $f_i = \alpha_0\beta_0^d$, where α_0 is unit and $D_i = (\beta_0 = 0)$ near D_i . Take $\alpha_1 = 0$ and $\beta_1 = 1/d(\beta_0 f'_i/f_i)$. Thus β_1 is regular and condition (***) is satisfied for $\alpha = \alpha_0 + \alpha_1\delta$ and $\beta = \beta_0 + \beta_1\delta$ near D_i .

Away from D_i on $C_i\text{-supp}(X)$, $\text{div}(f'_i/f_i) + D_i \geq 0 \Leftrightarrow f'_i$ is regular because f_i is a unit there. On the other hand, away from $\text{supp}(X)$, $f_i + f'_i\delta$ is a section of $\mathcal{A}_X \Leftrightarrow f'_i$ is regular. This completes the proof of the claim. \square

Immediately we get

COROLLARY. *Assume that $H^1(C, \mathcal{O}_C(D_0 + D_\infty))$ is not zero and $\text{char}(k) \nmid \deg(f)$. Then, for all but a finite number of points c of C , the required extension \tilde{f} does not exist if X is the Schiffer variation $(1/t)(d/dt)$ at the point c .*

Proof. The rational differential df is not zero. In characteristic zero, this is true because f is not constant. In characteristic p , it follows because p does not divide the degree of f .

Thus for all but a finite number of points c of C , $X(f) = (1/t)(df/dt)$ is a principal part at c of order one. As $H^1(C, \mathcal{O}_C(D_0 + D_\infty)) \neq 0$, by the previous remark, we know that almost of these principal parts are not zero. Hence our corollary follows from the proposition. \square

Now we will be done as soon as we finish the

Proof of the theorem. Let $g: \tilde{C} \rightarrow M$ be a family of smooth complete curves where M is an irreducible variety. We will also assume that, for each point m of M , g gives a universal deformation of the fiber $g^{-1}(m)$.

For some positive integer e , let $\text{Div} \rightarrow M$ be the variety of effective relative divisors on \tilde{C}/M of degree e . Let X be the subset $\{(D_0, D_\infty) \mid dD_0 \sim dD_\infty\}$ of $\text{Div} \times_M \text{Div}$.

By the see-saw principle, X is a closed subset. Consider the conditions $D_0 \neq D_\infty$ and $H^1(C, \mathcal{O}_C(D_0 + D_\infty)) \neq 0$. The first condition defines an open subset of X and the second condition defines a closed subset of X by upper-semicontinuity. Let Y be the subvariety of X defined by these two conditions. We have a tautological morphism $\sigma: Y \rightarrow M$. The precise statement of the theorem is that σ omits an open dense subset of M .

Assume otherwise. Then by Sard's theorem ($\text{char } k = 0$) we may find a locally closed smooth subvariety M' of Y , which is etale over

M . Replacing M by M' and \tilde{C} by $\tilde{C} \times_M M'$, we may assume that we have effective relative divisors \bar{D}_0 and \bar{D}_∞ on C , whose fibers over M satisfy the above two conditions. We may find effective relative divisors \bar{E} , \bar{D}_0 and \bar{D}_∞ on \tilde{C} such that \bar{D}_0 and \bar{D}_∞ have no common component and $\bar{E} + \bar{D}_i = \bar{D}_i$. Further replacing M by an open subset we may assume that \bar{D}_0 and \bar{D}_∞ are disjoint.

By construction we have a rational function \tilde{f} on \tilde{C} with zero divisor $d\bar{D}_0$ and polar divisor $d\bar{D}_\infty$. By the first condition \bar{D}_0 and \bar{D}_∞ are relative divisors of positive degree and consequently \tilde{f} is not constant. We may now specialize our attention to a particular fiber C of g . By the corollary and the universality of the deformation, $H^1(C, \mathcal{O}_{\tilde{C}}(\bar{D}_0 + \bar{D}_\infty)|_C) = 0$. Hence its quotient $H^1(C, \mathcal{O}_{\tilde{C}}(\bar{D}_0 + \bar{D}_\infty)|_C) = 0$. As this violates the second condition, the theorem follows by contradiction. \square

REFERENCES

1. E. Arbarello and M. Cornalba, *Su una congettura di Petri*, to appear.
2. H. M. Farkas, *Special divisors and analytic subloci of the Teichmüller space*, Amer. J. Math., **88** (1966), 881-901.
3. G. Kempf, *On algebraic curves*, J. für reine und and. Math., **295** (1977), 40-48.

Received January 30, 1980. Research supported by NSF Grant #7900965.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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Manufactured and first issued in Japan

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