TORSION DIVISORS ON ALGEBRAIC CURVES

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Let $D$ be a divisor on a smooth complete algebraic curve $C$ such that the multiple $dD$ is rationally equivalent to zero for some positive integer $d$. We may write $D = D_0 - D_\infty$ where $D_0$ and $D_\infty$ are distinct effective divisors.

The purpose of this note is to prove the

THEOREM. Assume that $C$ is a curve with general moduli in characteristic zero. For such a divisor $D$, the cohomology $H^1(C, \mathcal{O}_C(D_0 + D_\infty))$ must be zero (equivalently $|K - D_0 - D_\infty|$ is empty).

This theorem answers a question which arose in Sevin Recillas' unpublished work which motivated this note. This paper contains simple computational techniques for handling infinitesimal deformations of curves. I hope that these techniques may be useful for solving similar problems.

The theorem is well-known for the case $d = 1$. For instance see Farkas' paper [2]. The recent work [1] contains further bibliographic material and related material concerning deformations of mappings of curves into projective spaces.

1. First order deformations defined by principal parts. Let $\mathcal{L}$ be an invertible sheaf on a smooth curve $C$. For any point $c$ of $C$, $\text{Prin}_c(\mathcal{L}) = \text{Rat}(\mathcal{L})/\mathcal{L}_c$ measures the principal (polar) part of the rational sections of $\mathcal{L}$. Let $\text{Prin}(\mathcal{L})$ be the space of principal parts, $\bigoplus \text{Prin}_c(\mathcal{L})$, where $c$ runs through $C$. For a given principal part $p = (p_c)$ in $\text{Prin}(\mathcal{L})$, the support, supp($p$), is the set of $c$ in $C$ with $p_c \neq 0$.

As in [3], we have a natural exact sequence

$$0 \rightarrow \Gamma(C, \mathcal{L}) \rightarrow \text{Rat}(\mathcal{L}) \rightarrow \text{Prin}(\mathcal{L}) \rightarrow H^1(C, \mathcal{L}) \rightarrow 0.$$ 

Let $p_c$ be a principal part in $\text{Prin}_c(\mathcal{L})$ with a pole of order one. By duality the image of $p_c$ in $H^1(C, \mathcal{L})$ is zero if and only if each section of $\Omega_c \otimes \mathcal{L}^{\otimes -1}$ vanishes at $c$. Consequently, if $H^1(C, \mathcal{L}) \neq 0$ (i.e., there is a nonzero such section), the cohomology class of $p_c$ is nonzero for all but a finite number of points $c$.

Let $\Theta_c = \Omega_c^{\otimes -1}$ be the sheaf of regular vector fields on $C$. For any principal part $X$ in $\text{Prin}(\Theta_C)$, we want to define a deformation $C_X$ of $C$ over $T = \text{Spec}(k[\delta])$, where $k[\delta]$ is the ring of dual numbers $k[\epsilon]/(\epsilon^2)$. First note that, if $f$ is a rational function which is regular at each point of supp($X$), then $X(f)$ makes sense in an obvious way.
as a principal part in $\text{Prin}(\mathcal{C})$. Secondly define the structure sheaf of $C_x$ to be the subsheaf $\mathcal{X}$ of $\mathcal{O}_C \oplus \delta \text{Mat}(\mathcal{O}_C)$, whose sections $f_0 + f_1\delta$ locally satisfy the conditions: $f_0$ is regular and $X(f_0)$ is the principal part of $f_1$ (symbolically $(1 - X\delta)(f_0 + f_1\delta)$ is regular).

Locally on $C$, $X$ is the principal part of a rational vector field $X$. For regular functions $g_0$ and $g_1$, the expression

$$\phi_\times(g_0 + \delta g_1) = (1 + X\delta)(g_0 + g_1\delta) = g_0 + (Xg_0 + g_1)\delta$$

defines an $k[\delta]$-algebra isomorphism between $\mathcal{X}$ and $\mathcal{O}_C \oplus \delta \mathcal{O}_C$, which preserves the augmentations of the two algebras onto $\mathcal{O}_C$. Therefore $C_x \to T$ is a (locally trivial) deformation of $C$ over $T$, which has been given an obvious trivialization over the open subset $C$-supp($X$).

**Example.** If $t$ is a parameter function at a point $c$ and $t(c) = 0$, let $X$ be the principal part of the vector field $(1/t)(d/dt)$ at $c$. This principal part is determined by $c$ up to constant. In this case, $C_x$ is called a (first-order) Schiffer variation of the curve $C$. (If one replaces $(1 + X\delta)$ by $\exp(X\delta) \mod t^n$, as $n \to \infty$ one gets the formal version of a Schiffer variation in the compact Riemann surface case.)

**Remark.** The deformation $C_x \to T$ depends only on the cohomology class of $X$ in $H^1(C, \theta_C)$. This class is a form of the Kodaira-Spencer class of the deformation.

2. Obstructions to a particular deformation problem. Let $D_0$ and $D_\infty$ be two effective divisors on our smooth curve $C$. Assume that $D_0$ and $D_\infty$ have disjoint support. We are interested in the following special situation. For some positive integer $d$, $dD_0$ is linearly equivalent to $dD_\infty$. In other words, there is a nonconstant morphism $f: C \to \mathbb{P}^1$ such that $f^{-1}(0) = dD_0$ and $f^{-1}(\infty) = dD_\infty$. We want to examine whether this special situation may be extended to a deformation of $C$.

Let $g: \bar{C} \to M$ be a deformation of $\bar{C}$. Thus $g$ is a smooth morphism and $C$ is a closed subscheme of $\bar{C}$, which has the form $g^{-1}(m)$ for some point $m$ of $M$. Then we may ask if we can find a morphism $\tilde{f}: \bar{C} \to \mathbb{P}^1$ such that $\tilde{f}|_C = f$, $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(\infty)$ are defined and we have $\tilde{f}^{-1}(0) = d\bar{D}_0$ and $\tilde{f}^{-1}(\infty) = d\bar{D}_\infty$ for some effective relative Cartier divisors $\bar{D}_0$ and $\bar{D}_\infty$ on $\bar{C}$ with $\bar{D}_i|_C = D_i$ for $i = 0$ or $\infty$.

Next we will analyze this concept for first order deformations. Let $C_x \to T$ be the deformation defined in the first section for a given principal part $X$ in $\text{Prin}(\theta_C)$. Here $C$ and $C_x$ are the same as topological spaces. Let $C_i$ denote the open subscheme $C$-supp($D_i$) of
Our original morphism $f$ is given by a pair $(f_0, f_\infty)$ where each $f_i$ is a regular function on $C_i$ and $f_0 \cdot f_\infty = 1$ on $C_0 \cap C_\infty$. Similarly, our extension $\tilde{f}$ would be given by a pair $(f_0 + f'_0 \delta, f_\infty + f'_\infty \delta)$, where $f_i + f'_i \delta$ is a section of $\mathcal{A}_x$ over $C_i$, such that

\[(*) \quad f_0 f'_0 + f'_\infty f_\infty = 0 \text{ on } C_0 \cap C_\infty.\]

As the $f_i + f'_i \delta$ are then reciprocals on $C_0 \cap C_\infty$, the conditions on the zeros and poles of $\tilde{f}$ are just that, near supp($D_i$),

\[(**) \quad \text{there exist sections } \alpha \text{ and } \beta \text{ of } \mathcal{A}_x \text{ with } f_i + f'_i \delta = \alpha \beta^d \text{ and } \alpha \text{ is a unit.}\]

In the simplest case, the question of the existence of such an $\tilde{f}$ is easily interpreted.

**Proposition.** Assume that supp($X$) is contained $C_0 \cap C_\infty$ and $\text{char}(k) \not| d$. Then the required extension $\tilde{f}$ exists if and only if the principal part $(1/f)X(f)$ interpreted in Prin($\mathcal{O}_c(D_0 + D_\infty)$) is cohomologous to zero.

**Proof.** By the assumption on the supp($X$), any section of $\mathcal{A}_x$ near supp($D_i$) can be written as $\alpha = \alpha_0 + \alpha_1 \delta$, where $\alpha_0$ and $\alpha_1$ are regular functions. Furthermore, this section $\alpha_0 + \alpha_1 \delta$ is a unit near supp($D_i$) if and only if $\alpha_0$ is.

**Claim.** Given a rational function $f'_i$ on $C_i$, then $f_i + f'_i \delta$ is a section of $\mathcal{A}_x$ over $C_i$-supp($X$) and satisfies the condition $(**) \iff \text{div}(f'_i/f_0) + D_i \geq 0 \text{ on } C_i$-supp($X$).

We will first see how this claim implies the proposition. Assume that $(f_i + f'_i \delta, f_\infty + f'_\infty \delta)$ gives a required $\tilde{f}$. By the identity $(*), we must have $f'_i/f_0 = -f'_\infty/f_\infty$. Thus the claim shows that the desired conditions on the zeros and poles of $\tilde{f}$ imply that $f'_i/f_0$ is a regular section of $\mathcal{O}_c(D_0 + D_\infty)$ over $C$-supp($X$). As $f_i + f'_i \delta$ is a section of $\mathcal{A}_x$ over $C_i$, $X(f_i)$ is the principal part of the rational function $f'_i$ except at supp($D_0 + D_\infty$). So $f'_i/f_0$ is a rational section of $\mathcal{O}_c(D_0 + D_\infty)$ with principal part $1/f_0 X(f_0)$. Therefore, if our $\tilde{f}$ exists, $(1/f) X(f)$ is cohomologous to zero in Prin($\mathcal{O}_c(D_0 + D_\infty)$).

Conversely, if $1/f X(f)$ is cohomologous to zero, we may have a rational section $f'_i/f_0$ of $\mathcal{O}_c(D_0 + D_\infty)$ with principal part $1/f_0 X(f_0)$. Define $f'_\infty$ by the identity $(*).$ Reversing the above argument, one can readily check that the required extension $\tilde{f}$ is given by $(f_0 + f'_0 \delta, f_\infty + f'_\infty \delta)$. Thus the claim implies the proposition.

To prove the claim, assume that $f_i + f'_i \delta$ satisfies condition $(**)$. Then near supp($D_i$) we have $f_i + f'_i \delta = (\alpha_0 + \alpha_1 \delta)(\beta_0 + \beta_1 \delta)^d = \alpha_0 \beta_0^d + \alpha_1 \beta_1^d \delta$. If this is a regular function on $C_i$, then $\alpha_0 \beta_0^d + \alpha_1 \beta_1^d \delta$ is a rational function on $C_i$.
\[(\alpha_1 \beta_0^e + d \beta_0^{-1} \beta_1 \alpha_0) \delta,\] where the \(\alpha\)'s and \(\beta\)'s are regular and \(\alpha_0\) is a unit. So \(\beta_0 f'_i/f_i = (\alpha_1 \beta_0 + d \beta_0 \alpha_0) / \alpha_0\) is regular. As \(D_i = (\beta_0 = 0)\), this means that \(\text{div}(f'_i/f_i) + D_i \geq 0\) along \(D_i\). Conversely, assume that this is true. By the assumption on \(f\), we have \(f_i = \alpha_0 \beta_0^e\), where \(\alpha_0\) is unit and \(D_i = (\beta_0 = 0)\) near \(D_i\). Take \(\alpha_i = 0\) and \(\beta_i = 1/d(\beta_0 f'_i/f_i)\). Thus \(\beta_i\) is regular and condition (***) is satisfied for \(\alpha = \alpha_0 + \alpha_i \delta\) and \(\beta = \beta_0 + \beta_i \delta\) near \(D_i\).

Away from \(D_i\) on \(C_r\supp(X)\), \(\text{div}(f'_i/f_i) + D_i \geq 0 \iff f'_i\) is regular because \(f_i\) is a unit there. On the other hand, away from \(\supp(X)\), \(f_i + f'_i \delta\) is a section of \(\mathscr{X}_x = f'_i\) is regular. This completes the proof of the claim. \(\square\)

Immediately we get

**Corollary.** Assume that \(H^1(C, \mathscr{O}_C(D_0 + D_\infty))\) is not zero and \(\text{char}(k) \not\mid \deg(f)\). Then, for all but a finite number of points \(c\) of \(C\), the required extension \(\tilde{f}\) does not exist if \(X\) is the Schiffer variation \((1/t)(d/dt)\) at the point \(c\).

**Proof.** The rational differential \(df\) is not zero. In characteristic zero, this is true because \(f\) is not constant. In characteristic \(p\), it follows because \(p\) does not divide the degree of \(f\).

Thus for all but a finite number of points \(c\) of \(C\), \(X(f) = (1/t)(d/dt)\) is a principal part at \(c\) of order one. As \(H^1(C, \mathscr{O}_C(D_0 + D_\infty)) \neq 0\), by the previous remark, we know that almost of these principal parts are not zero. Hence our corollary follows from the proposition. \(\square\)

Now we will be done as soon as we finish the

**Proof of the theorem.** Let \(g: \tilde{C} \to M\) be a family of smooth complete curves where \(M\) is an irreducible variety. We will also assume that, for each point \(m\) of \(M\), \(g\) gives a universal deformation of the fiber \(g^{-1}(m)\).

For some positive integer \(e\), let \(\text{Div} \to M\) be the variety of effective relative divisors on \(\tilde{C}/M\) of degree \(e\). Let \(X\) be the subset \([(D_0, D_\infty) | dD_0 \sim dD_\infty]\) of \(\text{Div} \times_M \text{Div}\).

By the see-saw principle, \(X\) is a closed subset. Consider the conditions \(D_0 \neq D_\infty\) and \(H^1(C, \mathscr{O}_C(D_0 + D_\infty)) \neq 0\). The first condition defines an open subset of \(X\) and the second condition defines a closed subset of \(X\) by upper-semicontinuity. Let \(Y\) be the subvariety of \(X\) defined by these two conditions. We have a tautological morphism \(\sigma: Y \to M\). The precise statement of the theorem is that \(\sigma\) omits an open dense subset of \(M\).

Assume otherwise. Then by Sard's theorem (char \(k = 0\)) we may find a locally closed smooth subvariety \(M'\) of \(Y\), which is etale over
M. Replacing $M$ by $M'$ and $\tilde{C}$ by $\tilde{C} \times_M M'$, we may assume that we have effective relative divisors $\tilde{D}_0$ and $\tilde{D}_\infty$ on $C$, whose fibers over $M$ satisfy the above two conditions. We may find effective relative divisors $\tilde{E}$, $\tilde{D}_0$ and $\tilde{D}_\infty$ on $\tilde{C}$ such that $\tilde{D}_0$ and $\tilde{D}_\infty$ have no common component and $\tilde{E} + \tilde{D}_i = \tilde{D}_i$. Further replacing $M$ by an open subset we may assume that $\tilde{D}_0$ and $\tilde{D}_\infty$ are disjoint.

By construction we have a rational function $\tilde{f}$ on $\tilde{C}$ with zero divisor $d\tilde{D}_0$ and polar divisor $d\tilde{D}_\infty$. By the first condition $\tilde{D}_0$ and $\tilde{D}_\infty$ are relative divisors of positive degree and consequently $\tilde{f}$ is not constant. We may now specialize our attention to a particular fiber $C$ of $g$. By the corollary and the universality of the deformation, $H^1(C, \mathcal{O}_C(\tilde{D}_0 + \tilde{D}_\infty)|_C) = 0$. Hence its quotient $H^1(C, \mathcal{O}_C(\tilde{D}_0 + \tilde{D}_\infty)|_C) = 0$. As this violates the second condition, the theorem follows by contradiction.

\[ \square \]

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Received January 30, 1980. Research supported by NSF Grant #7900965.

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