ABSOLUTE CONVERGENCE FIELDS OF SOME TRIANGULAR MATRIX METHODS

Arun Kumar and D. P. Sahu
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Recently Das [2] has obtained results on the comparison of the absolute convergence fields between the Nörlund matrix and its product with the Cesàro matrix. In the present paper a similar investigation for the Riesz matrix \((\overline{N}, p_n)\) matrix) is made.

1. Let \(A = (a_{n,k})\) be an infinite lower triangular matrix, that is \(a_{n,k} = 0,\) if \(k > n,\) transforming sequence \(s = \{s_n\}\) into the sequence \(A(s)\) defined by

\[ A(s) = \{A_n(s)\} = \left\{ \sum_{k=0}^{n} a_{n,k} s_k \right\}. \]

The sequence \(s\) is said to be absolutely summable \(A\) or summable \(|I|,\) if the transformed sequence \(A(s)\) is of bounded variation, that is if \(\sum_{n=1}^{\infty} |A_n(s) - A_{n-1}(s)| < \infty.\) The absolute convergence field of \(A,\) denoted by \(|A|,\) is the set of all sequences which are summable \(|A|,\) The matrix \(A\) is said to be absolute conservative if \(|I| \subseteq |A|,\) where \(I\) is the identity matrix.

Let \(\{p_n\}\) be a sequence of constants, real or complex, such that \(P_n = \sum_{k=0}^{n} p_k \neq 0.\) When \(a_{n,k} = (p_{n-k})/P_n,\) \(A\) is called the \((N, p_n)\) matrix and for \(a_{n,k} = p_k/P_n,\) \(A\) is called the \((\overline{N}, p_n)\) matrix. The \((\overline{N}, p_n)\) matrix for \(p_n > 0\) and \(P_n \to \infty\) is also denoted by the \((R, P_n, 1)\) matrix. When the sequence \(\{p_n\}\) is such that \(p_n = 1\) for all \(n,\) both the \((N, p_n)\) and the \((\overline{N}, p_n)\) matrices reduce to the \((C, 1)\) matrix.

For two matrix methods \(A\) and \(B, AB\) transform of \(s\) is defined by \(A(B(s)).\) In particular,

\[
\tilde{t}_m(p, q) = \frac{1}{P_m} \sum_{k=0}^{m} \frac{p_k}{Q_k} \sum_{n=0}^{k} q_n s_n,
\]

where \(\tilde{t}_n(p, q)\) denotes \((\overline{N}, p_n)(\overline{N}, q_n)\) transform of \(s.\)

Throughout the present paper we write \(P_n^{(1)} = \sum_{k=0}^{n} p_k,\) and for any sequence \(\{\theta_n\},\) \(\Delta n \theta_n = \Delta \theta_n = \theta_n - \theta_{n+1}\) and \(\theta_n = 0,\) if \(n < 0; K\) denotes a positive constant, not necessarily the same at each occurrence.

2. Concerning the relative inclusion of the absolute convergence fields of \((N, p_n)\) and \((N, p_n)(C, 1),\) the following is known (see Das [2], Theorem 2 and Theorem 5).

**Theorem A.** Let the sequence \(\{p_n\}\) be such that \(p_n > 0\) and
\[ \frac{p_{n+1}}{p_n} \leq \frac{p_n}{p_{n+1}} \leq 1. \quad \text{Then} \quad |N, P_n| \subseteq |(C, 1)(N, p_n)|. \]

Silverman [5] has shown that the \((N, p_n)\) matrix is not permutable with the \((C, 1)\) matrix unless the \((N, p_n)\) matrix is a CesÀ­ro matrix. However, it has been proved that Theorem A is true even if the \((C, 1)(N, p_n)\) is permuted ([2], Theorem 4).

It has been proved (see Prasad and Pati (4)) that the absolute Riesz summability \(|R, \lambda_n, r|\) implies the summability \(|R, \phi(\lambda_n), r|\), provided, roughly speaking, the \(\phi(x)\) is reasonable regular and does not increase more rapidly than a power of \(x\). But from Lemma 4 we see that

\[ (1.2) \quad |\bar{N}, P_n| \subseteq |\bar{N}, p_n| \]

if and only if \(p_nP_n^{(1)} = O((P_n)^2)\).

The following theorems which we prove in the present paper show that if we consider the product of \((C, 1)\) and \((\bar{N}, p_n)\) in place of \((\bar{N}, p_n)\) in (1.2) the relation (1.2) holds good for a fairly wider class of sequences \(\{p_n\}\).

**Theorem 1.** Let \(\{p_n\}\) be a nonnegative sequence. Then \(|\bar{N}, P_n| \subseteq |(C, 1)(\bar{N}, p_n)|\), if

\[ (1.3) \quad \frac{k \sum_{n=k+1}^{\infty} \frac{1}{n^2}}{P_k} \leq K, \quad k = 1, 2, \ldots. \]

**Theorem 2.** Let \(\{p_n\}\) be a nonnegative sequence. Then

\[ |\bar{N}, P_n| \subseteq |(\bar{N}, p_n)(C, 1)|. \]

The condition (1.3) seems to be quite less restrictive but it is not true even for all nonnegative sequences; for if we consider the sequence \(\{p_n\}\) such that \(P_n = \log 2\) and for \(n > 0\), \(p_n\) is chosen to be either 0 or 1 in such a way that \(\log(n + 2) \sim P_n\). It is easy to see that for this case (1.3) is not satisfied.

Concerning the inclusion relation between the absolute convergence fields of the \((\bar{N}, q_n)\) and the \((C, 1)(N, p_n)\) methods we prove the following.

**Theorem 3.** Suppose that \(\{p_n\}\) is nonnegative nonincreasing sequence and that \(\{q_n\}\) is positive and nondecreasing sequence. Then

\[ |\bar{N}, q_n| \subseteq |(C, 1)(N, p_n)|. \]

It is interesting to observe that the relation \(|\bar{N}, q_n| \subseteq |(N, p_n)(C, 1)|\) also holds good follows from Lemma 4. Since for non-
decreasing sequence \( \{q_n\}, Q_n \leq (n + 1)q_n \), we see that with \( \{q_n\} \) in place of \( \{p_n\} \) and \( q_n = 1 \) in Lemma 4, the hypotheses of Lemma 4 are satisfied. Hence

\[
|\tilde{N}, q_n| \leq |C, 1|.
\]

But for nonnegative nonincreasing sequence \( \{p_n\} \) it follows from Lemma 3 that \( (N, p_n) \) is absolutely regular. Hence \( |\tilde{N}, q_n| \leq |(N, p_n)(C, 1)| \).

2. For the proof of the theorems we need the following results. In what follows we write \( \alpha_n = p_{n+1}P_{n+1}/P_n \) and \( C_m = m + 1 - P_{m+1}^{(1)}P_{m+1} \).

**Lemma 1.** In order that any \( \{x_n\} \in |I| \) implies \( \{x_n\} \in |A| \), where \( A = (a_m, n) \), it is necessary and sufficient that \( \sum_{k=0}^{\infty} a_{n,k} \) converges for all \( n \) and

\[
\sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} (a_{n+1,k} - a_{n,k}) \right| \leq K, \quad m = 0, 1, 2, \ldots .
\]

**Lemma 1** is contained in ([6], Theorem 3).

**Lemma 2.** For \( m, n = 0, 1, 2, \ldots \)

(i) \[
\sum_{k=0}^{m} \left( \frac{P_{n+k}}{P_k} \right) P_{n+1}^{(1)} = P_m - \alpha_m ;
\]

(ii) \[
\sum_{k=0}^{m} \left( \frac{P_{n-k}}{q_k} \right) Q_k = P_n - P_{n-m-1} - P_{n-m-1}Q_m .
\]

The proof of Lemma 2 is direct. The following lemma is contained in [3].

**Lemma 3.** If \( \{p_n\} \) is nonnegative, nonincreasing, then for all \( k \geq 0 \) and \( 1 \leq a \leq b \leq \infty \),

\[
\sum_{n=a}^{b} P(n, k) = \sum_{n=a}^{b} \left( \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \leq 1 ,
\]

and, for any \( n > 0 \), \( P(n, k) \geq 0 \).

**Lemma 4.** Let \( q_n > 0 \) and \( p_n \neq 0 \). Then in order that \( |\tilde{N}, p_n| \leq |\tilde{N}, q_n| \), it is necessary and sufficient that \( P_n/P_n = O(Q_n/q_n) \).

The sufficiency part of the lemma in a less general form is due to Sunouchi [6]. The present form is due to Bosanquet ([1], p. 654).
3. **Proof of Theorem 1.** Let \( \bar{t}_n(P) \) denote the \((N, P_n)\) transform of \( \{s_n\} \). We have
\[
P_n s_n = P_n^{(1)} \tilde{t}_n(P) - P_{n-1}^{(1)} \tilde{t}_{n-1}(P) , \quad n = 0, 1, 2, \ldots ,
\]
so that
\[
\tilde{t}_n(1, p) = \frac{1}{n+1} \sum_{r=0}^{n} \frac{1}{P_r} \sum_{s=0}^{r} \frac{P_s}{P_r} \left( P_s^{(1)} \tilde{t}_r(P) - P_{s-1}^{(1)} \tilde{t}_{r-1}(P) \right)
\]
\[
\overset{(3.1)}{=} \frac{1}{n+1} \sum_{r=0}^{n} \frac{1}{P_r} \left\{ \sum_{s=r}^{n} \left( \frac{P_s}{P_r} \right) P_r^{(1)} \tilde{t}_r(P) + \alpha_s \tilde{t}_s(P) \right\}
\]
\[
\overset{(3.1)}{=} \frac{1}{n+1} \sum_{r=0}^{n} \left\{ \sum_{s=r}^{n} \left( \frac{P_s}{P_r} \right) P_r^{(1)} + \alpha_r \right\} \tilde{t}_r(P)
\]
\[
= \sum_{r=0}^{n} a_{n,r} \tilde{t}_r(P) ,
\]
say. Writing \( \beta_{n,m} = \sum_{r=0}^{m} a_{n,r} \) and observing that \( a_{n,r} = 0 \) for \( r > n \) we see that \( \beta_{n,m} = 1 \) for \( n \leq m \) and for \( n \geq m \)
\[
\beta_{n,m} = \frac{1}{n+1} \sum_{r=0}^{m} \left\{ \sum_{s=r}^{n} \left( \frac{P_s}{P_r} \right) P_r^{(1)} + \alpha_r \right\} .
\]

We first simplify \( \beta_{n,m} \) for \( n \geq m \). By virtue of the result (i) of Lemma 2, we have\(^1\)
\[
\overset{(3.2)}{=} \sum_{r=0}^{m} \sum_{s=0}^{n} \left( \frac{P_s}{P_r} \right) P_r^{(1)}
\]
\[
= \sum_{s=0}^{m} \left( \sum_{r=0}^{s} + \sum_{r=s+1}^{n} \right) \left( \frac{P_s}{P_r} \right) P_r^{(1)}
\]
\[
= \sum_{s=0}^{m} \frac{1}{P_s} \sum_{r=0}^{s} P_r^{(1)} \left( \frac{P_s}{P_r} \right) + \sum_{s=m+1}^{n} \frac{1}{P_s} \sum_{r=0}^{m} P_r^{(1)} \left( \frac{P_s}{P_r} \right)
\]
\[
= m + 1 + (P_m - \alpha_m) \sum_{s=m+1}^{n} 1/P_s - \sum_{s=0}^{m} \alpha_s / P_s
\]
so that
\[
\beta_{n,m} = \frac{m + 1}{n + 1} + \frac{1}{n + 1} (P_m - \alpha_m) \sum_{s=m+1}^{n} \frac{1}{P_s} .
\]

In order to prove the theorem, it is sufficient to show that the matrix \( (a_{n,r}) \) in (3.1) is absolutely conservative. From Lemma 1, we see that the matrix \( (a_{n,r}) \) will be absolutely conservative if
\[
\overset{(3.3)}{=} \sum_{m=0}^{\infty} |\beta_{n+1,m} - \beta_{n,m}| \leq K , \quad m = 0, 1, 2, \ldots ,
\]
since \( \beta_{n,m} = 1 \) for \( n \leq m \). From (3.2) we get
\[
\overset{(3.3)}{=} \sum_{m=0}^{\infty} |\beta_{n+1,m} - \beta_{n,m}| \leq K , \quad m = 0, 1, 2, \ldots ,
\]
\[
1 \text{ We assume here onwards } \sum_{b=a}^{a} 0 \text{ if } b < a.
\]
\[
\sum = \sum_{n=m}^{\infty} \left| \frac{m + 1}{(n + 1)(n + 2)} + \frac{1}{(n + 1)(n + 2)} (P_m - \alpha_m) \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n + 2)P_{n+1}} (P_m - \alpha_m) \right|.
\]

evidently,

\begin{equation}
\sum \leq \sum_{n=m}^{\infty} \frac{(m + 1)/(n + 1)(n + 2)}{R(m) + L(m)},
\end{equation}

where

\[
R(m) = P_m \sum_{n=m}^{\infty} \left| \frac{1}{(n + 1)(n + 2)} \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n + 2)P_{n+1}} \right|
\]

and \(L(m) = \alpha_m R(m)/P_m\).

We have

\[
R(m) \leq P_m \sum_{n=m}^{\infty} \left| \frac{1}{(n + 1)(n + 2)} \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{n - m}{P_{n+1}} \right|
\]

\[
+ P_m \sum_{n=m}^{\infty} \frac{m + 1}{(n + 1)(n + 2)P_{n+1}}
\]

\[
= P_m X(m) + P_m Y(m), \ \text{say}.
\]

In view of the fact that for nonnegative sequence \(\{p_n\}, \{P_n\}\) is nondecreasing, it follows that

\begin{equation}
P_m Y(m) \leq K.
\end{equation}

Observing that \(\sum_{n=m}^{\infty} 1/P_s > (n - m)/P_n\), we obtain

\[
X(m) = \sum_{n=m}^{\infty} \left\{ \frac{1}{(n + 1)(n + 2)} \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n + 2)P_{n+1}} \right\} + Y(m).
\]

We now prove that \(P_m X(m) \leq K\). For, we first estimate

\[
X^*(N, m) = \sum_{n=m}^{N} \left\{ \frac{1}{(n + 1)(n + 2)} \sum_{s=m+1}^{n} \frac{1}{P_s} - \frac{1}{(n + 2)P_{n+1}} \right\}.
\]

First changing the order of summation and then using that \(\sum_{n=1}^{N} 1(n + 1)n = 1/s - 1/(N + 1)\), we get

\begin{equation}
X^*(N, m) = \sum_{s=m+1}^{N} \frac{1}{(s + 1)P_s} - \frac{1}{N + 2} \sum_{s=m+1}^{N} \frac{1}{P_s} - \sum_{n=m+1}^{N+1} \frac{1}{(n + 1)P_n}.
\end{equation}

If \(P_n \to \infty\) as \(n \to \infty\), it follows from (3.6) that \(X^*(N, m) \to 0\) as \(N \to \infty\). If \(P_n \not\to \infty\), then, since \(\{P_n\}\) is nondecreasing, \(P_n \to a\) finite limit \(P\), say, as \(n \to \infty\); and in this case \(X^*(N, m) \to -1/P\). In view of this and (3.5) it follows that

\begin{equation}
P_m X(m) \leq K.
\end{equation}
From (3.5) and (3.7) we get

\[(3.8) \quad R(m) \leq K.\]

Now we estimate \(L(m)\). We have

\[L(m) \leq \alpha_n(X(m) + Y(m)) .\]

From the hypothesis (1.3) we see that \(\alpha_n\) is bounded whenever \(P_n \leq K\). Using this fact and the observations made just after (3.5), we see that \(\alpha_n X(m) \leq K\). That \(\alpha_n y(m) \leq K\) follows from the hypothesis (1.3). Thus \(L(m) \leq K\). This together with (3.8) and (3.4) yields \(\Sigma = O(1)\), since the first term in (3.4) is bounded. This proves Theorem 1.

**Proof of Theorem 2.** Following closely the proof for (3.1) we see that

\[\bar{t}_n(p, 1) = \frac{1}{P_n} \sum_{r=0}^{n} \left\{ \sum_{s=r}^{n} \frac{p_s P_{r}^{(1)}}{s + 1} \left(1 - \frac{1}{P_r}\right) + \frac{p_r P_{r}^{(1)}}{(r + 1) P_{r+1}} \right\} \bar{t}_r(P)\]

\[= \sum_{r=0}^{n} a_{n,r} \bar{t}_r(P) .\]

Thus, in this case

\[\beta_{n,m} = \frac{1}{P_n} \sum_{r=0}^{m} \left\{ \sum_{s=r}^{m} p_s P_{r}^{(1)} \left(1 - \frac{1}{P_r}\right) + \frac{p_r P_{r}^{(1)}}{(r + 1) P_{r+1}} \right\} \]

for \(n \geq m\) and \(\beta_{n,m} = 1\) for \(m \geq n\).

Using the technique, with the result (ii) in place of (1) of Lemma 2, for obtaining (3.2) we see that

\[\beta_{n,m} = \frac{P_m}{P_n} + (C_m/P_n) \sum_{s=m+1}^{n} P_s/(s + 1) .\]

Now we proceed to prove that for this case also (3.3) holds good. We have

\[\Sigma = \sum_{n=m}^{\infty} \left| \frac{P_{n+1}}{P_n P_{n+1}} + C_m \frac{P_{n+1}}{P_{n+1}} \left(1 - \frac{1}{P_n} \sum_{s=m+1}^{n} \frac{p_s}{s + 1} - \frac{1}{n + 2}\right) \right| \]

\[\leq C_m \sum_{n=m}^{\infty} \frac{P_{n+1}}{P_n} \left| \frac{1}{P_n} \sum_{s=m+1}^{n} \frac{p_s}{s + 1} - \frac{P_n - P_m}{(n + 2) P_n} \right| \]

\[+ C_m P_m \sum_{n=m}^{\infty} \frac{p_{n+1}}{(n + 2) P_n P_{n+1}} + P_m \sum_{n=m+1}^{\infty} \frac{P_{n+1}}{P_n P_{n+1}} \]

\[= \Sigma_1 + \Sigma_2 + \Sigma_3 .\]

To prove that \(\Sigma_1\) is bounded we first consider the following sum

\[\Sigma(N) = \sum_{n=m+1}^{N} \frac{P_{n+1}}{P_n} \left| \frac{1}{P_n} \sum_{s=m+1}^{n} \frac{p_s}{s + 1} - \frac{P_n - P_m}{(n + 2) P_n} \right| .\]
Observing that \((n + 2) \sum_{s=m+1}^{N} \frac{p_s}{s + 1} > P_n - P_m\) we see that the expression under the modulus sign in \(\Sigma(N)\) is nonnegative. Hence by a change of order of summation we get

\[
\Sigma(N) = \sum_{s=m+1}^{N} \frac{p_s}{s + 1} \sum_{s=m+1}^{N} \frac{p_{s+1}}{P_s P_{s+1}} - \sum_{n=m+1}^{N} \frac{p_{n+1}}{(n + 2)P_{n+1}} + P_m \sum_{s=m+1}^{N} \frac{p_{n+1}}{(n + 2)P_n P_{n+1}}.
\]

Since

\[
(3.9) \quad \sum_{s=m}^{N} \frac{p_{n+1}}{P_n P_{n+1}} = 1/P_s - 1/P_{N+1}
\]

and \(p_n/P_n \leq 1\), we have

\[
\Sigma(N) \leq \frac{K}{m + 1} + \frac{1}{P_{N+1}} \sum_{s=m}^{N} \frac{p_s}{s + 1} + \frac{P_m}{m + 3} \sum_{s=m+1}^{N} \frac{p_{n+1}}{P_s P_{n+1}}
\]

\[
(3.10) \quad \leq \frac{K}{m + 1} + \frac{1}{(m + 1)P_{N+1}} \sum_{s=0}^{N} p_s \leq \frac{K}{m + 1}.
\]

It is clear that the term of \(\Sigma_i\) for \(n = m\) is bounded. In view of this and (3.10) we get that \(\Sigma_1\) is bounded, since \(C_m \leq K\) by the fact that \(P_m^{(1)} \leq (m + 1)P_m\). That \(\Sigma_2\) and \(\Sigma_3\) are bounded follows from (3.9). Thus we get that \(\Sigma \leq K\) for all \(m\). This completes the proof of the theorem.

**Proof of Theorem 3.** It is interesting to observe from the result (1.4) that to prove Theorem 3, it is sufficient to show that \(|C, 1| \leq |(C, 1)(N, p_n)|\), which is just special case of Theorem 3 when \((N, q_n)\) is \((C, 1)\). But to prove this special case we require the same argument (except minor simplification of the method of the proof) as for the general case. In order to give a direct proof we consider the general case.

Let \(t_n(1, p)\) denote \((C, 1)(N, p_n)\) transform of the sequence \(\{s_n\}\). We have

\[
t_n(1, p) = \frac{1}{n + 1} \sum_{r=0}^{n} \left\{ \sum_{k=r}^{n} \left( \frac{A_r p_{k-r}}{q_r} \right) \frac{Q_r}{P_k} \right\} \bar{t}_r(q) = \sum_{r=0}^{n} a_n \bar{t}_r(q).
\]

So far the case

\[
\beta_{n,m} = \frac{1}{n + 1} \sum_{r=0}^{m} \sum_{k=r}^{n} \left( \frac{A_r p_{k-r}}{q_r} \right) \frac{Q_r}{P_k}.
\]

It is clear that \(\beta_{n,m} = 1\) if \(m \geq n\). Simplifying by using Lemma 2(ii) we see that for \(m \leq n\)
Now we prove that (3.3) is true for this case also. We have

\[\beta_{n,m} = 1 - \frac{1}{n + 1} \sum_{k=m+1}^{n} \left( \frac{P_{k-m-1}}{P_k} + \frac{Q_{m}p_{k-m-1}}{q_{m+1}P_k} \right).\]

Now we prove that (3.3) is true for this case also. We have

\[\Sigma = \sum_{n=m}^{\infty} \left( \frac{1}{n + 2} \left( \frac{1}{n + 1} \sum_{k=m+1}^{n} \frac{P_{k-m-1}}{P_k} - \frac{1}{n + 1(n + 2)} \sum_{k=m+1}^{n} \frac{P_{k-m-1}}{P_k} \right)\right)\]

\[\leq \lim_{M \to \infty} \sum_{n=m+1}^{M} \left( \frac{P_{n-m}}{(n + 2)P_{n+1}} - \frac{1}{n + 1(n + 2)} \sum_{k=m+1}^{n} \frac{P_{k-m-1}}{P_k} \right)\]

\[+ \frac{Q_{m+1}}{q_{m+1}} \sum_{n=m}^{\infty} \frac{P_{n-m}}{(n + 2)P_{n+1}} + \frac{Q_{m+1}}{q_{m+1}} \sum_{n=m}^{\infty} \frac{1}{(n + 1)(n + 2)} \sum_{k=m+1}^{n} \frac{P_{k-m-1}}{P_k}\]

\[= \lim_{M \to \infty} \Sigma'(M) + \Sigma'' + \Sigma''',\]

say. We first consider \(\Sigma'(M).\) Since for nondecreasing sequence \(\{p_n\}, \{P_{k-m-1}/P_k\}\) is nondecreasing in \(k\) for \(k > m,\) we get that the expression under the modulus sign in \(\Sigma'(M)\) is nonnegative. By a change of order of summation we obtain

\[\Sigma'(M) = \sum_{n=m+1}^{\infty} \frac{P_{n-m}}{(n + 2)P_{n+1}} - \sum_{k=m+1}^{\infty} \frac{P_{k-m-1}}{(k + 1)P_k} + \frac{1}{M + 2} \sum_{k=m+1}^{M} \frac{P_{k-m-1}}{P_k}\]

Hence

\[\Sigma'(M) = O(1).\]

To prove the boundedness of \(\Sigma''\) and \(\Sigma'''\) we first estimate the following sum. Observing that \((n - m + 1)p_{n-m} \leq P_{n-m} \leq P_{n+1}\) we see that for \(a = 1\) or \(2\)

\[\sum_{n=m}^{\infty} \frac{p_{n-m}}{(n + a)P_{n+1}} = \sum_{n=m}^{\infty} \frac{p_{n-m}}{(n + a)P_{n+1}} + \sum_{n=m+1}^{\infty} \frac{p_{n-m}}{(n + a)P_{n+1}}\]

\[\leq \frac{P_{m}}{(m + a)P_{m+1}} + \sum_{n=m+1}^{\infty} \frac{1}{(n - m)(n - m + 1)} \leq \frac{K}{m + 1}.\]

Since for nondecreasing sequence \(\{q_n\}, Q_n \leq (n + 1)q_n,\) we obtain from (3.13) that

\[\Sigma'' < \infty.\]

Applying the above reasoning after a change of order of summation, we see that

\[\Sigma''' = \frac{Q_{m+1}}{q_{m+1}} \sum_{k=m+1}^{\infty} \frac{P_{k-m-1}}{(k + 1)P_k} < \infty.\]
That $\Sigma$ is bounded follows when we use (3.12), (3.14) and (3.15) in (3.11). This completes the proof of Theorem 3.

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