

# Pacific Journal of Mathematics

**POINTWISE DOMINATION OF MATRICES AND COMPARISON  
OF  $\mathcal{J}_p$  NORMS**

BARRY SIMON

## POINTWISE DOMINATION OF MATRICES AND COMPARISON OF $\mathcal{S}_p$ NORMS

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**Let  $p$  be a real number in  $[1, \infty)$  which is not an even integer. Let  $N = 2[p/2] + 5$ . We give examples of  $N \times N$  matrices  $A$  and  $B$ , so that  $|a_{ij}| \leq b_{ij}$  but  $\text{Tr}([A^*A]^{p/2}) > \text{Tr}([B^*B]^{p/2})$ .**

Let  $A$  and  $B$  be  $N \times N$  matrices with

$$(1) \quad |a_{ij}| \leq b_{ij}.$$

If we define the  $p$  norm of a matrix by

$$(2) \quad \|A\|_p = \text{Tr}([A^*A]^{p/2})^{1/p}$$

then it is trivial that, if  $p$  is an even integer, then

$$(3) \quad \|A\|_p \leq \|B\|_p$$

when (1) holds. For one need only write out the trace explicitly in terms of matrix elements. In a more general context, we conjectured in [5] that (1) implies (3) whenever  $p \geq 2$ . The attractiveness of this conjecture is shown by the fact that I know of at least five people other than myself who have worked on proving it.

It was thus quite surprising that Peller [3] announced that (3) fails for some infinite matrices whenever  $p$  is not an even integer. In correspondence, Peller described his counterexample which relies on his beautiful but elaborate theory of  $\mathcal{S}_p$  Hankel operators (4) and on a paper of Boas (2). It follows from Peller's example that (3) must fail for some finite  $N$  but it is not clear for which  $N$ . Our purpose here is to give explicit  $N$  and to avoid the complications of Peller's  $\mathcal{S}_p$ -Hankel theory.

The idea of the construction is very simple. Boas [2] constructed polynomials  $f(z)$ ,  $g(z)$  with  $\int |f(e^{i\theta})|^p d\theta > \int |g(e^{i\theta})|^p d\theta$  even though the coefficients,  $a_n$ , of  $f$  and coefficients,  $b_n$ , of  $g$  obey  $|a_n| \leq b_n$ .  $a$  and  $b$  should be thought of as Fourier coefficients of  $f(e^{i\theta})$  and  $g(e^{i\theta})$ . It is obvious that for sufficiently large  $N$ ,  $\sum_{j=0}^{N-1} |f(e^{ij\theta_N})|^p \geq \sum_{j=0}^{N-1} |g(e^{ij\theta_N})|^p$  where  $\theta_N = 2\pi/N$ . Again  $f$  and  $g$  should be viewed as functions on  $Z_N$  and the coefficients of the polynomial (if  $N$  is larger than the degrees) as  $Z_N$ -Fourier components. But the functions on  $Z_N$  are naturally imbedded in  $N \times N$  matrices in such a way  $\|A\|_p^p$  is just  $\sum |f(e^{ij\theta_N})|^p$  and so that the order (1) is equivalent to the order on Fourier coefficients.

To be explicit, given  $N$  and  $c_0, \dots, c_{N-1}$  let  $A$  be the matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & \cdots & c_{N-3} & c_{N-2} \\ c_{N-2} & c_{N-1} & \cdots & c_{N-4} & c_{N-3} \\ \vdots & & & & \vdots \\ c_1 & c_2 & & & c_0 \end{pmatrix}$$

$z_N = \exp(i\theta_N)$  and let  $\varphi_j$  be the vector with components  $(1, z_N^j, z_N^{2j}, \dots, z_N^{(N-1)j})$ ;  $j = 0, \dots, N-1$  and observe that

$$(4) \quad A\varphi_j = f(j)\varphi_j$$

where

$$(5) \quad f(j) = \sum_{\ell=0}^{N-1} c_\ell \chi_\ell(j)$$

with

$$(6) \quad \chi_\ell(j) = z_N^{\ell j}.$$

We use (6) to define  $\chi_\ell$  for any integer  $\ell$  although, of course,  $\chi_\ell$  is periodic in  $\ell$  with period  $N$ .

Of course, we have just exploited the fact that if  $\sigma$  is the matrix which cyclicity permutes the coordinates by one component, then  $A\sigma = \sigma A$  (indeed  $A = \sum c_p \sigma^k$ ) and since  $\sigma^N = 1$ ,  $\sigma$  is naturally diagonalized in terms of the group  $Z_N$ . The  $\chi$ 's are just the characters of  $Z_N$ . (In Physicist's language, since  $A$  has periodic boundary conditions, one diagonalizes it in momentum space.)

Since the  $\varphi_j$  are orthogonal vectors,  $A$  is a normal operator. For such an operator  $\|A\|_p^p$  is just the sum of the  $p$ th powers of the eigenvalues, i.e.,

$$(7) \quad \|A\|_p^p = \sum_{j=0}^{N-1} |f(j)|^p.$$

We take

$$(8a) \quad k = \left[ \frac{1}{2}p \right] + 2$$

$$(8b) \quad N = 2k + 1 = 2 \left[ \frac{1}{2}p \right] + 5.$$

Motivated by Boas' example, we choose

$$(9) \quad c_0 = 1; \quad c_1 = r; \quad c_k = \lambda r^k; \quad c_\ell = 0, \quad \text{if } \ell \neq 0, 1, k$$

where  $r$  is sufficiently small and

$$(10) \quad \lambda = \left(\frac{1}{2}p - 1\right)\left(\frac{1}{2}p - 2\right) \cdots \left(\frac{1}{2}p - k + 1\right) / k! .$$

Notice that since  $p$  is not an even integer and since  $p/2 + 1 < k < p/2 + 2$ , we have that  $\lambda < 0$ . Let  $d_j = |c_j|$  and let  $B$  the corresponding matrix so (1) certainly holds.

We compute  $\|A\|_p^p$  using (7) and the binomial theorem which is certainly legitimate if  $r$  is sufficient small

$$\begin{aligned} |f(j)|^{p/2} &= \sum_{\ell=0}^{\infty} \binom{p/2}{\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} r^{\ell+m(k-1)} \lambda^m \chi_{\ell+m(k-1)}(j) \\ &= f_1(j) + f_2(j) + f_3(j) + 0(r^{2k+1}) \end{aligned}$$

where

$$\begin{aligned} f_1 &= \sum_{\ell=0}^{k-1} \binom{p/2}{\ell} r^{\ell} \chi_{\ell} \\ f_2 &= \sum_{\ell=k}^{2k-1} \left[ \binom{p/2}{\ell} + \lambda \binom{p/2}{\ell - k + 1} \binom{\ell - k + 1}{1} \right] r^{\ell} \chi_{\ell} \\ f_3 &= r^{2k} \chi_{2k} \left[ \binom{p/2}{2k} + \lambda(k + 1) \binom{p/2}{k + 1} + \lambda^2 \binom{p/2}{2} \right] . \end{aligned}$$

Because  $N = 2k + 1$ , the characters  $\chi_0, \dots, \chi_{2k}$  are orthogonal so squaring and summing:

$$\|A\|_p^p = \sum_{j=0}^{k-1} \binom{p/2}{j}^2 r^{2j} + r^{2k} \left[ \binom{p/2}{k} + \lambda \binom{p/2}{1} \right]^2 + 0(r^{2k+1}) .$$

The formula for  $\|B\|_p^p$  is identical, except  $\lambda$  is replaced by  $|\lambda| = -\lambda$ . But  $\lambda$  is exactly chosen so that

$$\binom{p/2}{k} - \lambda \binom{p/2}{1} = 0 .$$

Thus, for  $r$  small,  $\|A\|_p > \|B\|_p$ .

It was necessary to take  $N = 2k + 1$  rather than just  $k + 1$  to avoid cross terms between the  $r_0$  and  $r_{\ell}$  ( $\ell \leq 2k$ ) factors which have the wrong sign and only vanish because  $\chi_0$  and  $\chi_{\ell}$  are orthogonal for  $\ell \leq 2k$ .

We close this paper with a series of remarks:

(1) Peller constructs infinite matrices  $A, B$  which are matrices of compact operators on  $\ell_2$  with (1) holding,  $B \in \mathcal{S}_p$  and  $A \notin \mathcal{S}_p$ . It is easy to get such operators from our examples as follows: normalize  $A, B$  so that  $\|A\|_p > 1 > \|B\|_p \geq \|B\| \geq \|A\|$ . Let us view  $\ell_2$  as the tensor algebra over  $\mathbb{C}^N$ , i.e., as  $\mathbb{C} \oplus \mathbb{C}^N \oplus \mathbb{C}^{N^2} \oplus \dots$  and let  $\Gamma(A) = 1 \oplus A \oplus (A \otimes A) \oplus \dots$ . Then  $|\Gamma(A)_{ij}| \leq |\Gamma(B)_{ij}|$  and  $\Gamma(A), \Gamma(B)$  are compact,  $\Gamma(B) \in \mathcal{S}_p$  but  $\Gamma(A) \notin \mathcal{S}_p$ .

(2) Given any measure space,  $(M, \mu)$  with  $L^2(M, \mu)$  infinite dimensional, we cannot have that  $\|A\|_p \leq c\|B\|_p$  for some fixed  $c$  and all  $A, B$  with  $|(Af)(m)| \leq (B|f|)(m)$ . For one can always imbed  $C^N$  into  $L^2(M, \mu)$  in a way preserving  $\|A\|_p$  norms and order (map  $(a_1, \dots, a_n)$  into  $\sum a_i f_i(m)$  with  $f_i$  multiples of characteristic functions of disjoint sets). If  $\|A\|_p \leq c\|B\|_p$  held for  $L^2(M)$  it would hold for any  $C^N$ . But by taking tensor products of our example one can arrange that  $\|A\|_p/\|B\|_p$  is arbitrarily large. [It is interesting that this tensor product/operator theory version of Katznelson's remark (quoted in Bachelis [1]) is more natural than the function theoretic construction.]

(3) Let  $N(p)$  be the smallest  $N$  for which there exist matrices for which (1) holds but (3) fails. Clearly we have shown

$$N(p) \leq 2\left[\frac{1}{2}p\right] + 5$$

but equality is most unlikely for any  $p$ . Indeed for  $1 \leq p < 2$ , we have  $N(p) = 2$  since if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then  $\|B\|_p^p = 2^p$ ,  $\|A\|_p^p = 2(\sqrt{2})^p > \|B\|_p^p$  if  $p < 2$ . Moreover, we owe to S. Friedland the following simple argument showing that  $N(p) \geq 3$  if  $p > 2$ . If  $C, D$  are positive matrices with

$$(11) \quad |c_{ij}| \leq d_{ij}$$

then with  $\mu_j(\cdot)$  = singular values, we trivially have

$$\mu_1(C) \leq \mu_1(D) ; \quad \mu_1(C) + \mu_2(C) \leq \mu_1(D) + \mu_2(D)$$

(since for  $2 \times 2$  positive matrices  $\mu_1(C) + \mu_2(C) = \text{Tr}(C)$ ). By general rearrangement inequalities [5]

$$\text{Tr}(C^p) \leq \text{Tr}(D^p)$$

for any  $1 \leq p \leq \infty$ . Given  $A, B$  obeying (1) and applying this remark to  $C = A^*A, D = B^*B$ , we see that (3) holds for any  $p \geq 2$  if  $N = 2$ . It would be interesting to know the precise value of  $N(p)$ . Two natural guesses are  $[p/2] + 1$  and  $2[p/2]$ .

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