MAXIMAL GROUPS IN SANDWICH SEMIGROUPS OF BINARY RELATIONS

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A sandwich semigroup is given as follows. Let \( R \) be an arbitrary but fixed binary relation on a finite set \( X \). For relations \( A \) and \( B \) on \( X \) we say \( (a, b) \in A \ast B \) (the product of \( A \) and \( B \)) if there are \( c \) and \( d \) in \( X \) such that \( (a, c) \in A \), \( (c, d) \in R \) and \( (d, b) \in B \). This semigroup is denoted \( B_X(R) \). In this paper we study maximal groups in \( B_X(R) \) for various classes of \( R \).

Sandwich semigroups of binary relations were introduced in [2]. These semigroups arise naturally in automata theory, and their role in automata theory is studied in [3]. Montague and Plemmons [5] have shown that given a finite group \( G \) there is some set \( X \) such that \( G \) is a maximal group in \( B_X \), the usual semigroup of binary relations. We show there are classes of \( R \) for which this result holds and others for which it does not hold.

If \( R \) is a relation and \( E \) is a nonzero idempotent in \( B_X(R) \), then we write \( G_E(R) \) for the maximal group determined by \( E \) and call \( E \) an \( R \)-idempotent. In § 1 we give a class of relations for which \( G_E(R) \) is trivial for any relation \( R \) in this class and any \( R \)-idempotent \( E \). In § 2 we produce a class of relations for which the Montague-Plemmons result holds. That is, any finite group \( G \) arises as a maximal group for some \( X \) and some relation \( R \) in this class. Finally, in § 3 we show there is a class of relations for which some but not all finite groups arise.

Throughout we use Boolean matrix representation for relations. That is, if \( R \) is a relation over \( X \) where \(|X| = n\), then \( R \) is represented by an \( n \times n \) matrix where the \((i, j)\) entry is a 1 if \((x_i, x_j)\) is in \( R \) and 0 otherwise. These matrices are multiplied using Boolean arithmetic.

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1. \( B_X(R) \) containing only trivial groups. Let \( \Gamma \) be the collection of (nonzero) matrices with the property that all nonzero columns are the same. For \( R \) in \( \Gamma \) it is easy to see that if the \((i, j)\) entry of \( R \) is zero then either row \( i \) or column \( j \) of \( R \) is zero. The following theorem characterizes \( R \)-idempotents for any \( R \) in \( \Gamma \) and shows that \( G_E(R) \) is trivial for any \( R \) in \( \Gamma \) and any \( R \)-idempotent \( E \).
THEOREM 1. Let $R$ be in $\Gamma$. Then

(i) $A$ is an $R$-idempotent if and only if all nonzero rows of $A$ are the same and for some $i$ and $j$ such that the $(i, j)$ entry of $R$ is nonzero we have the $(j, i)$ entry of $A$ is nonzero.

(ii) If $E$ is an $R$-idempotent, then $G_E(R)$ is trivial.

Proof. Throughout the proof let $a_{ij}$ ($r_{ij}$) denote the $(i, j)$ entry of the matrix $A(R)$.

(i) Assume $A$ is an $R$-idempotent. $AR$ has zero columns where $R$ does and since all nonzero columns of $R$ are alike, all nonzero columns of $AR$ are alike. Let

$$\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n
\end{pmatrix}$$

denote the nonzero columns of $AR$. Writing out the product $ARA$ we see that for each $i$ such that $b_i = 1$ we have a nonzero row of $A$ and each nonzero row is identical.

Assume for each $k$ and $m$ such that $r_{km} = 1$ we have $a_{mk} = 0$. Clearly, if column $j$ of $R$ is zero, then column $j$ of $AR$ is zero. We show if column $j$ of $R$ is nonzero, then row $j$ of $A$ is zero. These two statements imply $(AR)A = 0$, a contradiction. Let column $j$ of $R$ be nonzero and denote by $b_{ij}$ the $(j, i)$ entry of $AR$. Then for any $i$

$$b_{ij} = \sum_{k=1}^{n} a_{jk}r_{ki} = \begin{cases} 
\sum_{k=1}^{n} a_{jk}r_{ki} & \text{if column } i \text{ of } R \text{ is nonzero} \\
0 & \text{otherwise}
\end{cases}$$

(hence $r_{ki} = r_{kj}$)

$$= 0 \quad \text{in either case by the assumption.}$$

Thus row $j$ of $AR$ is zero which implies row $j$ of $(AR)A = A$ is zero.

Conversely, assume $r_{ij} = 1$ and $a_{ji} = 1$. If row $k$ of $A$ is nonzero, then $a_{ki} = 1$. From $a_{ki} = r_{ij} = a_{ji} = 1$ we have the $(k, i)$ entry of $ARA$ is 1 and so row $k$ of $ARA$ is nonzero. Since $a_{ki} = 1$, row $k$ of $AR$ is row $i$ of $R$ and so the $(k, j)$ entry of $AR$ is nonzero. Furthermore, since $a_{ji} = 1$ we have row $k$ of $ARA$ is row $j$ of $A$. But all rows of $A$ are the same so row $k$ of $ARA$ is row $k$ of $A$. If row $k$ of $A$ is zero, then row $k$ of $ARA$ is zero. Hence we have $ARA = A$ and $A$ is an $R$-idempotent.

(ii) Let $E$ be an $R$-idempotent and $A$ be in $G_E(R)$. Throughout the remainder of the proof we use the following:
$e_{ij}$ denotes the $(i, j)$ entry of $E$,
$b_{ij}$ denotes the $(i, j)$ entry of $AR$,
$c_{ij}$ denotes the $(i, j)$ entry of $ARE$.

We show $a_{ij} = e_{ij}$ for any $i$ and $j$.

Let $e_{ij} = 0$. Then, by the remark preceding the theorem, either row $i$ or column $j$ of $E$ is zero. If row $i$ is zero, then row $i$ of $ERA = A$ is zero and so $a_{ij} = 0$. If column $j$ is zero, then column $j$ of $ARE = A$ is zero and so $a_{ij} = 0$.

Let $e_{ij} = 1$. We show $a_{ij} = 1$. Assume not, that is assume $a_{ij} = 0$. We first show row $i$ and column $j$ of $A$ are zero. We have

$$c_{ij} = \sum_{k=1}^{n} b_{ik} e_{kj}.$$ 

Since all nonzero columns of $E$ are alike, then for any nonzero columns $n$ and $j$ of $E$ it follows that $c_{ij} = c_{im}$. But $ARE = A$ implies $c_{ij} = a_{ij} = 0$ and so row $i$ of $A$ is zero. Similarly column $j$ of $A$ is zero.

We now show $A = 0$, a contradiction. If row $k$ of $E$ is zero, then $ERA = A$ implies row $k$ of $A$ is zero. If row $k$ of $E$ is nonzero, then $e_{kj} = 1$ since $e_{ij} = 1$. By the above we know column $j$ of $A$ is zero, so $a_{kj} = 0$. Thus we have $e_{kj} = 1$ and $a_{kj} = 0$. Using the above arguments, this implies row $k$ of $A$ is zero.

2. $B_{X}(R)$ containing all finite groups. Let $\Gamma$ be any class of matrices such that for every positive integer $n$ the matrix

$$\begin{pmatrix} I_n & A \\ B & C \end{pmatrix}$$

is in $\Gamma$ where $I_n$ is the $n \times n$ identity matrix, $A$ is an arbitrary $n \times k$ matrix, $B$ is an arbitrary $k \times n$ matrix and $C$ is an arbitrary $k \times k$ matrix.

**Theorem 2.** If $G$ is a finite group, then $G$ is a maximal group in $B_{X}(R)$ for some nonidentity matrix $R$ in $\Gamma$ and some $X$.

**Proof.** From Montague and Plemmons [5] we know there is an $X'$ such that $G$ is isomorphic to $G_{X'}(I)$ where $E'$ is an idempotent in $B_{X'}(I)$ ($I$ is the identity relation). Let $X'$ have $n$ elements and

$$R = \begin{pmatrix} I_n & A \\ B & C \end{pmatrix}$$

where $R$ is $k \times k$ with $k$ greater than $n$ and $A$, $B$ and $C$ are arbitrary. The matrix $E$ where
is an $R$-idempotent. Let $A$ be in $G_E(R)$ where

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}. $$

Then, $A*E = A = E*A$ gives $Q = R = S = 0$ and $PE' = E'P = P$. Let $B$ be the $R$-inverse of $A$ in $G_E(R)$. Then

$$B = \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix}$$

and $B*A = E = A*B$ give $PP' = E = P'P$ and so $P$ is in $G_{E'}$. Thus the map $\theta$ from $G_E(I)$ to $G_E(R)$ given by

$$\theta(P) = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$$

is an isomorphism.

We remark here that the $R$ and $X$ of the theorem are not unique. In fact $G$ is in $B_X(R)$ for all $X$ containing at least $n$ elements. Also, if $R$ is as in the theorem and $R' = PRQ$ where $P$ and $Q$ are invertible, then the map $\theta$ from $B_X(R)$ onto $B_X(R')$ given by $\theta(A) = QAP$ is an isomorphism.

The following theorem shows the symmetric groups arise in $B_X(R)$ where $R$ is a permutation.

**Theorem 3.** Let $R$ be a permutation in $B_X(I)$ for some arbitrary but fixed $X$ where $X$ has $n$ elements. Then $R'$, the inverse of $R$ in $B_X(I)$, is an $R$-idempotent and $G_{E'}(R)$ is isomorphic to $S_n$, the symmetric group on $n$ elements.

**Proof.** It is clear that $R'$ is an $R$-idempotent, and for all $A$ in $B_X(R)$ we have $A*R' = R'*A = A$. It remains to be shown that only permutations have an $R$-inverse with respect to $R'$. If $A$ is a permutation, then $AR$ and $RA$ are permutations and $(R'A'R')(RA) = (AR)(R'A'R') = R'$ where $A'$ is the $I$-inverse of $A$. Thus, $R'A'R'$ is the $R$-inverse of $A'$.

Conversely, assume for some $A$ we have a $B$ such that $A*B = B*A = R'$. If $A$ is not a permutation, then either $xA = \emptyset$ for some $x$ in $X$ or for some $x$ and $y$ in $X$ with $x \neq y$ we have $xA = yA$. In the former case we have $\emptyset = x(A*B) = xR'$. In the latter case since $R$ is a permutation, we have $x(A*B) = y(A*B)$ and so $x(R') = y(R')$ for $x \neq y$. Neither case is tenable and so $A$ must be a permutation.
We show in the next section that there is a class of matrices such that some groups are not in $B_X(R)$ for any $R$ in this class.

The question now arises, "Do we always have either all groups or only trivial groups?" This is answered negatively in the next section.

3. $B_X(R)$ containing only some groups. In this section we look at a class of matrices for which some, but not all, groups appear in $B_X(R)$ for $R$ in this class. We show that for any $R$ in this class the maximal groups in $B_X(R)$ are a special type.

Consider the class $\Gamma$ of matrices having the block form

$$\begin{pmatrix} I_k & A \\ 0 & 0 \end{pmatrix}$$

where $I_k$ is the $k \times k$ identity matrix and $A$ is a $k \times n$ matrix whose $(1, 1)$ entry is a 1 and all other entries are 0. We will establish our results for matrices in this class and show the results also hold for matrices of the forms

$$\begin{pmatrix} I_k & A \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_k & 0 \\ A & 0 \end{pmatrix}$$

where $A$ has exactly one nonzero entry. Throughout this section all sandwich matrices $R$ will be in $\Gamma$.

**Theorem 4.** The following are necessary and sufficient for $E$ to be an $R$-idempotent.

(i) Assume row $j$ has a 1 in the $(j, 1)$ position. If row $j$ also has a 1 in positions $P_1, \ldots, P_m$, then row $j$ is the sum of rows 1, $k + 1$ and rows $P_1, \ldots, P_m$. Otherwise it is just the sum of rows 1 and $k + 1$.

(ii) Assume row $j$ has a 0 in the $(j, 1)$ position. If row $j$ also has a 1 in positions $P_1, \ldots, P_m$, then row $j$ is the sum of rows $P_1, \ldots, P_m$. If there are no such rows $p_i$, then row $j$ is zero.

**Proof.** Let $ERE = E$. Since rows $k + 1$ through $n$ of $R$ are zero, then columns $k + 1$ through $n$ of $E$ do not affect the product $ER$. Thus, we consider entries in columns 1 through $k$ of $E$.

(i) If row $j$ has a 1 in the $(j, 1)$ position, then $\{x_i, x_{k+1}\}$ is in $x_jER$. Thus $\{x_i, x_{k+1}\}E$ is in $x_jERE = x_jE$ and rows 1 nad $k + 1$ are in row $j$. That is, row $j$ has 1's at least where rows 1 and $k + 1$ have 1's. If row $j$ has a 1 in the $(j, p_i)$ position for $p_i$ in $\{2, \ldots, k\}$, then $x_{p_i}$ is in $x_jER$ and $xE_{p_i}$ is in $x_jERE = x_jE$ and row $p_i$ is contained in row $j$. Clearly if the $(j, p_i)$ entry is 0, then $x_{p_i}$ is not in
$x_iERE$ and hence row $p_i$ is not in $x_iERE = x_iE$. Thus, $x_iE = x_iERE = \{x_1, x_{p_1}, \ldots, x_{p_m}, x_{k+1}\}E$ where the $(j, p_i)$ entries are nonzero, and the result follows.

(ii) From the proof of (i) we see $x_jE = x_jERE = \{x_{p_1}, \ldots, x_{p_m}\}E$ where the $(j, p_i)$ entry is a 1, and the result follows.

Conversely, consider row $j$ of $E$. We show $x_jE = x_jERE$. If row $j$ has a 1 in the $(j, 1)$ position and in the $(j, p_i), \ldots, (j, p_m)$ positions for $p_i$ in $\{2, \ldots, k\}$, then $x_jERE = \{x_1, x_{p_1}, \ldots, x_{p_m}, x_{k+1}\}E = \{x_1, x_{p_1}, \ldots, x_{p_m}, x_{k+1}\}E$. By hypothesis, row $j$ is the sum of rows 1, $p_1, \ldots, p_m, k + 1$ and $x_jE = \{x_1, x_{p_1}, \ldots, x_{p_m}, x_{k+1}\}E$. If row $j$ has a 0 in the $(j, 1)$ position, then the proof is similar except we exclude $x_1$ and $x_{k+1}$.

**Example 1.** If $n = 7$ and $k = 4$, then the matrix

$$E = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

is an $R$-idempotent, but the matrix

$$F = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

is not an $R$-idempotent.

We now look at elements in $G_E(R)$.

**Theorem 5.** Let $A$ be in $G_E(R)$.

(i) Row $m$ of $A$ is zero if and only if row $m$ of $E$ is zero.

(ii) Rows $j$ and $m$ of $A$ are equal if and only if rows $j$ and $m$ of $E$ are equal.

(iii) Row $m$ of $A$ is the sum of a subset of the rows 1 through $k + 1$ of $E$.

(iv) Row $j$ of $A$ is the sum of rows $p_1, \ldots, p_i$ of $A$ if and only if row $j$ of $E$ is the sum of rows $p_1, \ldots, p_i$ of $E$. 
Proof. (i) and (ii) follow directly from $ARA' = E$ and $ERA = A$ where $A'$ denotes the $R$-inverse of $A$.

(iii) From $ARE = A$ we have row $m$ of $A$ is

$$
\begin{pmatrix}
\sum a_{mi}e_{1i} + \sum a_{mj}e_{ji} + \cdots + a_{mk}e_{ki} + a_{ml}e_{k+1,i} \\
\sum a_{m1}e_{1i} + \sum a_{m2}e_{ji} + \cdots + a_{mk}e_{ki} + a_{ml}e_{k+1,i} \\
\sum a_{m1}e_{2i} + a_{m2}e_{ji} + \cdots + a_{mk}e_{ki} + a_{ml}e_{k+1,i} \\
\vdots \\
\sum a_{m1}e_{ni} + a_{m2}e_{ji} + \cdots + a_{mk}e_{ki} + a_{ml}e_{k+1,i}
\end{pmatrix}
$$

where $a_{ij}(e_{ij})$ is the $(i, j)$ entry of $A(E)$. If $a_{mp_1}, \ldots, a_{mp_j} = 0$ and $a_{mp_{j+1}}, \ldots, a_{mp_k} = 1$ where $p_i$ is in $\{1, \ldots, k\}$, then row $m$ of $A$ is

$$
(e_{p_{j+1,1}} + e_{p_{j+2,1}} + \cdots + e_{p_{j+1,n}} + e_{p_{j+2,n}} + \cdots + e_{p_{q,n}})
$$

which is the sum of rows $p_{j+1}, \ldots, p_q$ of $E$. If $a_{m1} = 1$, then we also have $e_{k+1,} t$ in each entry where $t$ runs from 1 to $n$.

(iv) If row $j$ of $A$ is the sum of rows $p_i, \ldots, p_t$ of $A$ and if $A'$ denotes the $R$-inverse of $A$ we have

$$
x_jE = x_jARA' = \{x_{p_i}, \ldots, x_{p_t}\}ARA' = \{x_{p_i}, \ldots, x_{p_t}\}E.
$$

The converse is similar.

Thus, for example, if $X$ has 7 elements and $k = 4$ and row $m$ of $A$ is $(1\ 0\ 1\ 1\ 0\ 0\ 1)$, then this row is the sum of rows 1, 3, 4 and 5 of $E$.

We remark here that this theorem is also valid if $R$ has the form

$$
\begin{pmatrix}
I_k & A \\
0 & 0
\end{pmatrix}
$$

where $A$ has exactly one nonzero entry, say the $(i, j)$ entry where $j \geq k + 1$ is nonzero. For in the above proof we use row $i$ where we previously used row 1 and column $j$ where we used column $k + 1$. Similarly, by using the word "column" where we used "row" the result also holds for any $R$ of the form

$$
\begin{pmatrix}
I_k & 0 \\
A & 0
\end{pmatrix}
$$

where $A$ has exactly one nonzero entry.

The goal now is to show how to construct an arbitrary $A$ in $G_E(R)$ and thereby show only certain groups arise in $B_X(R)$. From Theorems 4 and 5 (iv) we see that we need only show the construction of the first $k + 1$ rows of $A$. The remaining rows are determined by their pattern in $E$. That is, if row $m$ of $E$, for $m > k + 1$, is the of rows $p_i, \ldots, p_t$ or $E$ where $p_i$ is between $q$ and $k + 1$ inclusive, then row $m$ of $A$ is the sum of rows $p_i, \ldots, p_t$ of $A$. We make the following definitions which are illustrated in Example 2.
DEFINITION 1. Let $S$ be a sum of a subset of the first $k+1$ rows of $A$, but $S$ is not one of the first $k+1$ rows of $A$ (and may not even be any row of $A$). Then $S$ is called a row associated with $A$. If any row of $A$ or row associated with $A$ is the sum of rows $p_i, \ldots, p_t$, then each $p_i$ is called a summand. $S$ is the maximal sum of rows $p_i, \ldots, p_t$ if every one of the first $k+1$ rows contained in $A$ is a $p_i$. We also refer to $S$ as a maximal row associated with $A$.

DEFINITION 2. Each row $m$ of $A$ is the sum of a subset of the first $k+1$ rows of $A$ and some of the associated rows of $A$. Let row $m$ be listed as a summand only if it is not the sum of rows distinct (not necessarily different) from itself. Then we say the sum is maximal if all rows contained in row $m$ and all maximal rows associated with $A$ contained in row $m$ are listed as summands. If row $m$ is the maximal sum of $N$ rows we write $S_m(A) = N$ and say row $m$ has order $N$.

When we say row $m$ of $A$ is a sum of $N$ rows of $A$, we mean each summand is either one of the first $k+1$ rows of $A$ or a row associated with $A$.

We now make the following classification of the nonzero rows of $A$ and the rows associated with $A$.

DEFINITION 3. If every summand of row $m$ is identical to row $m$, then row $m$ is called an independent row. If at least one summand of row $m$ is proper and if row $m$ is not the sum of its proper summands, then it is called fixed. If at least one summand of row $m$ is proper and if row $m$ is the sum of its proper summands, then it is called dependent.

By this definition rows associated with $A$ are dependent. Thus, when we refer to a dependent row, it may or may not be in $A$.

EXAMPLE 2. Let $A$ be given below where $k = 8$.

$$
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
9 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
$$
\( S_i(A) = 1 \) for \( i = 3, 4, 5, 7 \) and 8 and \( S_t(A) = 4 \) (sum of rows 1, 2, 3 and 4), \( S_s(A) = 2 \) (sum of rows 3 and 4) and \( S_{10}(A) = 2 \) (sum of rows 4 and 5). We also have row 6 is the sum of rows 6, 7 and 8 and \( S \) where \( S \) is the sum of rows 7 and 8 and so \( S_6(A) = 4 \). Row 9 is the sum of rows 1 through 5 and \( S_1 \) and \( S_2 \) where \( S_1 \) is the sum of rows 3 and 5, \( S_2 \) is the sum of rows 4 and 5 and \( S_3 \) is the sum of rows 2, 3, 4 and 5. Therefore, \( S_9(A) = 8 \). Note that \((1 1 1 1 1 0 0 0 0 0)\) considered as the sum of rows 1 and 5 of \( A \) is associated with \( A \), but would not be a maximal row associated with \( A \) unless we considered it as the sum of rows 1, 2, 3, 4, 5 and 9 of \( A \). Rows 3, 4, 5, 7 and 8 are independent, rows 1, 6 and 9 are fixed, and rows 2 and 10 are dependent.

The following sequence of propositions will enable us to construct an arbitrary element in \( G_E(R) \) for an \( R \)-idempotent \( E \). Throughout we let \( A \) be in \( G_E(R) \).

**Proposition 1.**

(i) Row \( m \) of \( E \) is independent if and only if row \( m \) of \( A \) is independent.

(ii) Row \( m \) of \( E \) is fixed if and only if row \( m \) of \( A \) is fixed.

(iii) Row \( m \) of \( E \) is dependent if and only if row \( m \) of \( A \) is dependent.

**Proof.** We prove the “if” part of (i), (ii) and (iii) and the “only if” parts must follow.

(i) Let row \( m \) of \( E \) be the maximal sum of rows \( p_1, \cdots, p_i \) of \( E \). Each of these rows will be identical to row \( m \). Thus, by Theorem 6 (ii) and (iv) row \( m \) of \( A \) is the maximal sum of rows \( p_1, \cdots, p_i \), all just like row \( m \) of \( A \) and row \( m \) of \( A \) is independent.

(ii) Let row \( m \) of \( E \) be the maximal sum of rows \( p_1, \cdots, p_i \), where either \( m \) is a \( p_i \) or some row \( p_i \) is identical to row \( m \). Apply Theorem 6 (ii) and (iv) to show row \( m \) of \( A \) is the maximal sum of rows \( p_1, \cdots, p_i \) of \( A \) where either \( m \) is a \( p_i \) or some row \( p_i \) is identical to row \( m \). Thus, row \( m \) of \( A \) is fixed.

(iii) As above, apply the definition of dependent row along with Theorem 6 (ii) and (iv).

**Proposition 2.** \( S_m(E) = N \) if and only if \( S_m(A) = N \).

**Proof.** Assume \( A \neq E \) or there is nothing to prove. Assume \( S_m(E) = N \) and row \( m \) of \( E \) is the maximal sum of rows \( p_1, \cdots, p_N \) of \( E \). Assume rows \( p_1, \cdots, p_j \) are in \( E \) (as usual \( p_i \) is between 1 and \( k + 1 \) inclusive) and rows \( p_{j+1}, \cdots, p_N \) are maximal associated
with $E$. Thus, row $m$ of $E$ is the sum of rows $p_i, \ldots, p_j$ of $E$ (not maximal unless $j = N$), and so row $m$ of $A$ is the sum of rows $p_i, \ldots, p_j$ of $A$.

Assume row $p_q$ is one of the dependent rows associated with $E$ and is the sum of rows $p_{z_1}, \ldots, p_{z_t}$ of $E$ where $p_{z_i}$ is between 1 and $j$ inclusive. Then the sum of rows $p_{z_1}, \ldots, p_{z_t}$ of $A$ is associated with $A$. For if it were one of the first $k + 1$ rows of $A$, say row $q$, then by Theorem 6(ii) row $q$ of $E$ would be the sum of rows $p_{z_1}, \ldots, p_{z_t}$ of $E$. But this sum is not a row of $E$. Similarly, for each row $p_i$ associated with $E$, we get a corresponding row $p_i$ associated with $A$. Furthermore, each is maximal in $A$ since it was in $E$. Thus $S_m(A)$ is greater than or equal to $N$. If $S_m(A)$ is strictly greater than $N$, then either there is another row in $A$ in the sum of row $m$ or another row associated with $A$ in the sum. In the former case, we contradict Theorem 5(ii), in the latter case this associated row of $A$ will give rise to another associated row of $E$ contradicting the fact that the sum was maximal.

Conversely assume $S_m(A) = N$ and $S_m(E) = M \neq N$. But by the above $S_m(E) = M$ implies $S_m(A) = M$ and we have a contradiction.

**Proposition 3.** Given the fixed and independent rows of $A$ we can determine the dependent rows of $A$.

*Proof.* The dependent rows of $A$ will be in the same positions as the dependent rows of $E$. Let row $m$ of $E$ be dependent and the maximal sum of rows $p_i, \ldots, p_j$ of $E$ where rows $p_i, \ldots, p_j$ are dependent. By the definition of maximal sum, every summand of any row $p_i$ for $i$ between 1 and $j$ inclusive will be one of the rows $p_{z_1}, \ldots, p_{z_t}$ and by the definition of dependent row, each summand is proper. Thus, dependent rows are redundant in a maximal sum, and row $m$ of $E$ is the sum of rows $p_{z_1}, \ldots, p_{z_t}$ of $E$ where each $p_i$ is independent or fixed. By Theorem 5(ii) and Proposition 3 row $m$ of $A$ is the sum of rows $p_{i+1}, \ldots, p_i$ of $A$ which will be fixed or independent as they are in $E$.

From Theorem 5(ii) and Propositions 1 and 2 we have the following proposition.

**Proposition 4.** Row $m$ of $A$ has the same number and types of summands as row $m$ of $E$.

Proposition 4 is useful in constructing the independent and fixed rows of $A$. Recall, each independent row of $E$ is a row of $E$. That is, it cannot be associated with $E$. By Theorem 5(ii) and Proposi-
tion 1 each of these rows must be an independent row of $A$. Similarly, each fixed row of $E$ must be some fixed row of $A$. The following definitions help us apply Proposition 4.

**Definition 4.** If an independent row is a summand of a fixed row, it is called *Type 1*. Otherwise it is *Type 2*.

Propositions 1 and 4 now give the following.

**Proposition 5.** Row $m$ of $E$ is independent of *Type 1* (*Type 2*) if and only if row $m$ of $A$ is independent of *Type 1* (*Type 2*).

**Definition 5.** A fixed row of $A$ is called a *maximal fixed row* (MFR) if it is not the summand of any fixed row different from itself. An MFR together with its summands is called a *maximal fixed block* (MFB). MFRs (or MFBs) with the same number and types of summands are said to be in the same class. We define a *sub-MFR* (sub-MFB) to be any MFR (MFB) within an MFR (MFB). A fixed row is a *minimal fixed row* (mFR) if it does not contain any fixed summands. An mFR together with its summands is called a *minimal fixed block* (mFB).

We remark that a fixed row may be both an MFR and an mFR. Every MFB is either an mFB or contains an mFB.

**Example 3.** Let

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

Then

$$H = \begin{pmatrix}
\left[ \begin{array}{c}
C \\
B \\
D \\
E \\
F \\
\end{array} \right]
\end{pmatrix}
$$

$H$ is an MFB with $B$, $D$ and $F$ as sub-MFBs. $B$ and $D$ are in the same class. $C$ and $E$ are sub-MFBs of $B$ and $D$ respectively and are mFBs. $F$ is also an mFB.

Proposition 4 now gives us the following:
PROPOSITION 6. Row \( m \) of \( E \) is an MFR with an associated MFB in class \( \Gamma \) if and only if row \( m \) of \( A \) is an MFR with associated MFB in class \( \Gamma' \).

We now give the construction of the first \( k + 1 \) rows of \( A \).

**Step 1.** If any rows of \( E \) are zero, then the corresponding rows in \( A \) are zero.

**Step 2.** Distinct independent rows of Type 2 in \( E \) are permuted observing Theorem 5 (ii).

**Step 3.** MFBs of the same class in \( E \) are permuted to form MFBs of this class in \( A \). We must observe Propositions 1 and 2. That is, subblocks may need to be permuted within an MFB.

**Step 4.** If within an MFB there are independent rows of Type 2 (thus, they are actually independent rows of Type 1 in \( E \)), then they may be permuted.

**Step 5.** Repeat Steps 3 and 4 with sub-MFBs. That is, sub-MFBs of the same MFB and of the same class may be permuted and within them, independent rows of Type 2 may be permuted.

**Step 6.** Repeat Step 4 until mFBs have been permuted and their independent rows of Type 2 have been permuted.

**Step 7.** Calculate the dependent rows by the fixed and independent rows and the pattern of \( E \) (as in the proof of Proposition 3).

**Theorem 6.** \( A \) is in \( G_E(R) \) if and only if \( A \) is constructed as above.

**Proof.** If \( A \) is in \( G_E(R) \), then Propositions 1 through 6 show that is \( A \) constructed as above. Conversely, let \( A \) be constructed as above. We must show \( A*E = A = E*A \) and the existence of an inverse. We first show \( A*E = E*A = A \).

**Case 1.** Row \( m \) of \( A \) is independent or fixed. Then it is some row of \( E \), say row \( p \). Thus, \( x_mA = x_pE \) and \( x_mA*E = x_pE*E = x_pE = x_mA \). Assume row \( m \) of \( E \) has ones in the \( p_i \), \( i \) positions for \( p_i \) between 1 and \( k \) inclusive. Row \( m \) is the sum of rows \( p_i \), \( i \) if the \( (m, 1) \) position is a zero and so \( x_mE = x_mE \). It is the sum of rows \( p_i \), \( i \), \( k + 1 \) if the \( (m, 1) \) position is a 1. In
the former case, row $m$ of $A$ is the sum of rows $p_1, \ldots, p_t$ of $A$ and $x_mE^*A = x_mE = \{x_{p_1}, \ldots, x_{p_t}\}A = x_mA$. In the latter case row $m$ of $A$ is the sum of rows $p_1, \ldots, p_t$, $k + 1$ of $A$ and $x_mE^*A = \{x_{p_1}, \ldots, x_{p_t}, x_{k+1}\}A = x_mA$.

**Case 2.** Row $m$ of $A$ is dependent. Then row $m$ of $E$ is dependent. Assume row $m$ of $E$ is the sum of rows $p_1, \ldots, p_t$ of $E$ where row $p_t$ is fixed or independent. Thus, row $m$ of $A$ is the sum of rows $p_1, \ldots, p_t$ of $A$ where row $p_t$ is fixed or independent in $A$. Thus, from Case 1, for each $p_t$ we have $x_{p_t}A*E = x_{p_t}A = x_{p_t}E*A$. Now, $x_mE^*E = \{x_{p_1}, \ldots, x_{p_t}\}A*E = x_{p_1}A*E + x_{p_2}A*E + \ldots + x_{p_t}A*E = x_{p_1}A + x_{p_2}A + \ldots + x_{p_t}A = \{x_{p_1}, \ldots, x_{p_t}\}A = x_mE$. Similarly $x_mE^*A = x_mA$.

We now construct a $B$ by the above rules and show $B$ is an $R$-inverse of $A$.

**Step 1.** If row $m$ of $E$ is zero, then row $m$ of $B$ is zero.

**Step 2.** Independent rows of Type 2. Assume rows $p_1, \ldots, p_t$ of $E$ are distinct independent rows of Type 2. Let $\theta$ be the permutation on $p_1, \ldots, p_t$ where row $p_t$ of $E$ is row $\theta(p_t)$ of $A$. Let these independent rows be permuted in $B$ by $\theta^{-1}$. That is, row $\theta(p_t)$ of $E$ is row $p_t$ of $B$.

**Step 3.** MFBs of the same class. Permute these in $B$ following the same scheme above for independent rows of Type 2.

**Step 4.** Independent rows of Type 2 within an MFB. Let MFBs $B_1, \ldots, B_t$ be of the same class and let each $B_i$ have distinct independent rows $b_{i1}, b_{i2}, \ldots, b_{iu}$ of Type 2. Assume $\theta$ permutes the blocks as they are permuted in $A$ (similar to $\theta$ in Step 2). Then in $A$, block $B_i$ occupies the position $\theta(B_i)$ occupies in $E$ and in $B$, block $\theta(B_i)$ occupies the position block $B_i$ does in $E$. If rows $b_{i1}, \ldots, b_{it}$ of block $B_i$ have been permuted in $A$, then apply the same permutation to the corresponding rows in block $\theta(B_i)$ of $B$.

**Step 5.** Sub-MFRs. These are formed in $B$ following the same scheme as for independent rows in Step 4.

**Step 6.** Continue as in Steps 4 and 5 for independent rows of Type 2 within sub-MFBs and for sub-MFBs within the sub-MFBs until the process terminates with mFBs.

**Step 7.** Dependent rows. These are determined by independent and fixed rows.
Thus we have a $B$ such that $B \ast E = B = E \ast B$. Let the independent rows of Type 2 in $A$ and $B$ be as in Step 2 above. Then for each $i$, $x_{\theta(p_i)}(A \ast B) = x_{p_i}(B \ast E) = x_{p_i}(B) = x_{\theta(p_i)}(E)$. Similarly for each $i$, $x_{\theta(p_i)}(B \ast A) = x_{\theta(p_i)}(E \ast A) = x_{p_i}(A) = x_{\theta(p_i)}(E)$. Thus, for any independent row, say $x_m$, of Type 2 we have $x_m(A \ast B) = x_m(E) = x_m(B \ast A)$. Similar proofs give the same result for MFRs. Now consider independent rows of Type 2 within an MFB as in Step 4. By the construction, if row $m$ of $E$ is row $p$ of $A$, then row $p$ of $E$ is row $m$ of $B$ where row $m$ is in $B_i$ and row $p$ is in $\theta(B_i)$. This implies $x_m(E) = x_p(A)$ and $x_p(E) = x_m(B)$ and for each row $m$ in $B_i$ we have $x_m(E) = x_p(A) = x_p(E \ast A) = x_m(B \ast A)$. Similarly, if row $m$ of $E$ is row $q$ of $B$, then row $q$ of $E$ is row $m$ of $A$ and $x_m(E) = x_q(B) = x_q(E \ast B) = x_m(A \ast B)$. Thus, for these rows $x_m(A \ast B) = x_m(E) = x_m(B \ast A)$. Sub-MFRs satisfy $x_m(A \ast B) = x_m(E) = x_m(B \ast A)$ by the same type of proof. We now show the result for dependent rows. Let row $m$ of $E$ be dependent. Then it is the sum of rows $p_1, \ldots, p_t$ of $E$ which are fixed or independent, and rows $m$ of $A$ and $B$ are the sums of rows $p_1, \ldots, p_t$ of $A$ and $B$ respectively. Since $x_m(A \ast B) = x_m(E) = x_m(B \ast A)$ for row $m$ fixed or independent, we have $x_m(E) = \{x_{p_1}, \ldots, x_{p_t}\}E = \{x_{p_1}\}E + \cdots + \{x_{p_t}\}E = \{x_{p_1}\}A \ast B + \cdots + \{x_{p_t}\}A \ast B = \{x_{p_1}, \ldots, x_{p_t}\}A \ast B = x_m(A \ast B)$.

**Corollary 1.** $G_E(R)$ is trivial if and only if

(i) No two distinct independent rows of Type 2 are in $E$.
(ii) No independent rows of Type 1 can be permuted.
(iii) No two fixed rows of $E$ are in the same class.

**Corollary 2.** $G_E(R)$ is nontrivial if and only if it contains a nontrivial subgroup isomorphic to a permutation group.

**Proof.** Assume $G_E(R)$ is nontrivial. Then at least one of the three statements of Corollary 1 must be false. Assume (i) is false and let $p_1, \ldots, p_t$ be the distinct independent rows of Type 2. Let $A$ be the set of all $A$ in $G_E(R)$ formed by permuting rows $p_1, \ldots, p_t$ of $E$ and leaving all other rows of $E$ stationary. $A$ is a subgroup of $G_E(R)$ isomorphic to the permutation group on $\{p_1, \ldots, p_t\}$. A similar proof establishes the result if we assume (ii) or (iii) is false.

The converse is clear.

If for each $N_i$ in $\{N_1, \ldots, N_p\}$ there are $n_i$ identical independent rows of Type 2 and also if for each $C_k$ in the set $\{C_1, \ldots, C_s\}$ there are $c_k$ MFBs of class $C_k$ where $c_k$ is greater than 1, then $G_E(R)$ contains a subgroup isomorphic to $G = P_p \times P_{c_1} \times P_{c_2} \times \cdots \times P_{c_s}$ where $P_T$ is the permutation group on the set of $T$ elements. As in the proof of Theorem 6 let $\mathcal{A}$ in $G_E(R)$ be the set of all $A$ such
that the independent rows of Type 1 are fixed. Then $A \simeq G$. Thus we have the following

**COROLLARY 3.** If $E$ contains no independent rows of Type 1 that can be permuted or if no MFBs are of the same class, then $G_E(R)$ is isomorphic to a direct product of permutation groups.

**EXAMPLE 4.** Let $k = 6$ and

$$E = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Rows 1, 2, 3, 7 and 8 are independent of Type 2; but since rows 1, 7 and 8 are alike and 2 and 3 are alike, we only get one permutation from these. Row 4 is fixed and rows 5 and 6 are independent of Type 1. Thus, $G_E(R) = \{E, A\}$ where

$$A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

**EXAMPLE 5.** Let $k = 8$ and

$$E = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$
Row 1 is independent of Type 2. Row 2 is an MFR with rows 2, 3 and 4 as summands and so \( S_2(E) = 3 \). Row 5 is an MFR with rows 5, 6, 7, 8 and 9 as summands and so \( S_5(E) = 5 \). Rows 6 through 9 form a sub-MFB of row 5. From the above we see no permutations can be formed and \( G_E(R) \) is trivial.

**EXAMPLE 6.** Let \( k = 16 \) and \( R \) be \( 18 \times 18 \). Let

\[
B_1 = \begin{bmatrix} S_1 \\ B_1_2 \\ S_2 \end{bmatrix}, \quad E = \begin{bmatrix} I_1 \\ I_2 \\ s_1 \\ I_3 \\ I_4 \end{bmatrix}
\]

\[
\begin{bmatrix}
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
\end{bmatrix}
\]

\[
B_1 \text{ and } B_2 \text{ are MFBs of the same class and can be permuted. } S_1 \text{ and } S_2 \text{ are sub-MFBs of } B_1, s_1 \text{ is a sub-MFB of } S_2. \text{ Similarly, } S_3 \text{ and } S_4 \text{ are sub-MFBs of } B_2 \text{ and } s_2 \text{ is a sub-MFB of } S_3. \text{ Note } s_1 \text{ and } s_2 \text{ are mFBs and } I_1 \text{ through } I_5 \text{ are independent of Type 1. } S_1 \text{ and } S_2 \text{ (and } S_3 \text{ and } S_4) \text{ are not of the same class. The pairs } (I_1, I_2), (I_3, I_4), (I_5, I_6) \text{ and } (I_7, I_8) \text{ are independent of Type 2 within blocks } S_1, s_1, s_2 \text{ and } S_4 \text{ respectively and can be permuted within these blocks. Observe, if we permute } B_1 \text{ and } B_2, \text{ then we must permute } S_1 \text{ and } S_2 \text{ and } S_3 \text{ and } S_4 \text{ within the blocks. Thus we can describe } G_E(R) \text{ as follows. If we do not permute } B_1 \text{ and } B_2, \text{ then we have 16 elements of this form—one for each of the possible permutations of the pairs of independent rows. If we do permute } B_1 \text{ and } B_2, \text{ then we again have 16 elements. Thus, } G_E(R) \text{ has 32 elements. The first 16 elements described form the subgroup } K = S_2 \times S_2 \times S_2 \times S_2.
where $S_2$ is the symmetric group on the set of two elements. For example the element
\[
\left( \begin{array}{cc}
1 & 2 \\
2 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & 2 \\
2 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & 2 \\
2 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & 2 \\
2 & 1
\end{array} \right)
\]
in $K$ corresponds to the element $A$ in $G_e(R)$ with rows $I_5$ and $I_4$ and $I_6$ and $I_5$ interchanged. Rows $I_1$, $I_2$, $I_7$ and $I_8$ are not permuted. We can consider elements of $G_e(R)$ as 5-tuples $(A, B, C, D, E)$ where each entry is a permutation of 1, 2. $A$ represents the permutation of $B_1$ and $B_2$, $B, C, D$ and $E$ represent the permutations of the pairs $(I_1, I_2), (I_3, I_4), (I_5, I_6)$ and $(I_7, I_8)$ respectively. Consider the elements where $A$ is the identity to be of Type 1, and those where $A$ represents the permutation of $B_1$ and $B_2$ to be of Type 2. Let $X = (A, B, C, D, E)$ and $Y = (A', B', C', D', E')$ be elements of $G_e(R)$. The multiplication in $G_e(R)$ is given by
\[
XY = \begin{cases} 
(AA', BB', CC', DD', EE') & \text{if } X \text{ and } Y \text{ are both Type 1} \\
(AA', BB', CD', DC', EB') & \text{if either } X \text{ or } Y \text{ is Type 2}.
\end{cases}
\]
We remark that the above theorems and propositions are also valid if $R$ has the form
\[
\left( \begin{array}{cc}
I_k & A \\
0 & 0
\end{array} \right) \quad \text{or} \quad \left( \begin{array}{cc}
I_k & 0 \\
A & 0
\end{array} \right)
\]
where $A$ has exactly one nonzero entry. The proofs would be as indicated in the remarks following Theorem 5.

It is not known if there is a way to determine the maximal groups in $B_e(R)$ for any given $R$. It would be interesting to find properties of the relation $R$ that determine the maximal groups.

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