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**ON THE TRANSFORMATION OF FOURIER COEFFICIENTS
OF CERTAIN CLASSES OF FUNCTIONS**

KENNETH F. ANDERSEN

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Suppose $f(x) \in L^1(0, \pi)$ and let $a = \{a_\nu\}, (b = \{b_\nu\})$ denote the Fourier cosine (sine) coefficients of f extended to $(-\pi, \pi)$ as an even (odd) function, that is

$$a_0 = \frac{2}{\pi} \int_0^\pi f(x) dx, \quad a_\nu = \frac{2}{\pi} \int_0^\pi f(x) \cos \nu x dx, \\
 b_\nu = \frac{2}{\pi} \int_0^\pi f(x) \sin \nu x dx. \quad \nu = 1, 2, \dots$$

The sequence transformations T and T' are defined by

$$(Ta)_0 = a_0, \quad (Ta)_\nu = \frac{1}{\nu} \sum_{j=1}^\nu a_j, \quad (T'a)_\nu = \sum_{j=\nu}^\infty (a_j/j), \quad \nu = 1, 2, \dots$$

The purpose of this note is to characterize those rearrangement invariant function spaces $L^p(0, \pi)$ which are left invariant by the operators T and T' acting on Fourier coefficients of functions in these spaces. Our results include and improve some results of Hardy, Bellman and Alshynbaeva.

G. H. Hardy [5] proved that if $f \in L^p(0, \pi)$ for some $p, 1 \leq p < \infty$, then $Ta = \{(Ta)_\nu\}$ is the sequence of Fourier cosine coefficients of a function also in $L^p(0, \pi)$; R. Bellman [2] proved the analogous theorem for T' except that now $1 < p \leq \infty$. Recently E. Alshynbaeva [1] gave necessary and sufficient conditions on an Orlicz space $L_{M\phi}$ in order that $L_{M\phi}$ may replace the L^p space in the results of Hardy and Bellman, thus answering a question of P. L. Ul'yanov. The analogues for the sequences $\{b_\nu\}$ were also studied.

We denote by f^* the nonnegative, nonincreasing function on $(0, \pi)$ which is equi-measurable with f , that is, for all $\lambda > 0$

$$|\{x \in (0, \pi) : |f(x)| > \lambda\}| = |\{x \in (0, \pi) : f^*(x) > \lambda\}|.$$

We suppose throughout that σ is a function norm defined on the measurable functions on $(0, \pi)$ which is rearrangement invariant in the sense that $\sigma(f) = \sigma(f^*)$. The associate of σ , denoted σ' , is then also rearrangement invariant and is given by

$$(1) \quad \sigma'(f) = \sup \left\{ \left| \int_0^\pi f(x)g(x) dx \right| : \sigma(g) \leq 1 \right\} \\
 = \sup \left\{ \int_0^\pi f^*(x)g^*(x) dx : \sigma(g) \leq 1 \right\}.$$

The upper and lower Boyd indices α, β of the Banach space $L^\sigma(0, \pi) = \{f: \sigma(f) < \infty\}$ are defined in [4] and satisfy $0 \leq \beta \leq \alpha \leq 1$. For the Lorentz spaces $L^{p,q}(0, \pi)$ and in particular for the Lebesgue spaces $L^p(0, \pi)$, the indices α, β are both equal to p^{-1} . Indices for the Orlicz spaces are computed in [3]. It is well known that $L^\infty(0, \pi) \subseteq L^\sigma(0, \pi) \subseteq L^1(0, \pi)$ for every σ and it is not difficult to see that $L^p(0, \pi) \subseteq L^\sigma(0, \pi) \subseteq L^q(0, \pi)$ whenever $p^{-1} < \beta, \alpha < q^{-1}$.

We shall state and prove our theorems only for the case of cosine coefficients a ; for the case of sine coefficients b the statements of the theorems are the same with b replacing a and sine, replacing cosine throughout while the proofs are similar.

Concerning the sequence $\{a_n\}$ and the transformations T and T' we have the following theorems.

THEOREM 1. *The following statements are equivalent.*

(a) *For every $f \in L^\sigma(0, \pi)$ with Fourier cosine coefficients $a = \{a_n\}$, Ta is the sequence of Fourier cosine coefficients of a function in $L^\sigma(0, \pi)$.*

(b) *The lower index β of $L^\sigma(0, \pi)$ satisfies $\beta > 0$.*

THEOREM 2. *The following statements are equivalent.*

(a) *For every $f \in L^\sigma(0, \pi)$ with Fourier cosine coefficients $a = \{a_n\}$, $T'a$ is the sequence of Fourier cosine coefficients of a function in $L^\sigma(0, \pi)$.*

(b) *The upper index α of $L^\sigma(0, \pi)$ satisfies $\alpha < 1$.*

Since $\alpha = \beta = p^{-1}$ for the space L^p , Theorems 1 and 2 yield the results of Hardy and Bellman cited above. It is well known, and in any event follows easily from the formulae for α, β in [3], that for the Orlicz space $L_{M\Phi}$, the lower index β satisfies $\beta > 0$ if and only if Φ satisfies the Δ_2 condition, i.e., $\Phi(2t) \leq M\Phi(t)$, $t \geq t_0$; the upper index α satisfies $\alpha < 1$ if and only if the Young's function Ψ complementary to Φ satisfies the Δ_2 condition. Hence Theorems 1 and 2 yield Alshynbaeva's Theorems 1 and 2 with a sharpening of the necessity part of his Theorem 2 in that we do not have to assume $|t \log t| \leq c\Phi(t)$, $t \geq t_0 > 0$.

We shall require the following lemma relating to the operators P and P' defined for $0 < x < \pi$ by

$$(Pf)(x) = \cot(x/2) \int_0^x f(t) dt, \quad (P'f)(x) = \int_x^\pi f(t) \cot(t/2) dt.$$

LEMMA 1. *The following are equivalent.*

(a) *$Pf \in L^\sigma(0, \pi)$ for every $f \in L^\sigma(0, \pi)$.*

- (b) *There is a constant c such that $\sigma(Pf) \leq c\sigma(f)$, for all $f \in L^\sigma(0, \pi)$.*
- (c) *The upper index α of $L^\sigma(0, \pi)$ satisfies $\alpha < 1$.*
- (d) *The lower index β' of $L^{\sigma'}(0, \pi)$ satisfies $\beta' > 0$.*
- (e) *There is a constant c such that $\sigma'(P'f) \leq c\sigma'(f)$, for all $f \in L^{\sigma'}(0, \pi)$.*
- (f) *$P'f \in L^{\sigma'}(0, \pi)$ for every $f \in L^\sigma(0, \pi)$.*

Proof of Lemma 1. Let $(P_1f)(x) = \left(\int_0^x f(t)dt\right)/x$. There are positive constants c, c_1, c_2 such that for all $f \geq 0$

$$c_1(Pf)(x) \leq (P_1f)(x) \leq c_2\left((Pf)(x) + \int_0^\pi f(t)dt\right) \leq c_2((Pf)(x) + c\sigma(f))$$

and since $f \in L^\sigma(0, \pi)$ if and only if $|f| \in L^\sigma(0, \pi)$ it follows that (a) is equivalent to the corresponding statement with P replaced by P_1 ; similarly P_1 may replace P in (b). Analogously, $(P'_1f)(x) = \int_x^\pi (f(t)/t)dt$ may replace P' in statements (e) and (f). Thus, it suffices to prove the lemma with P replaced by P_1 and P' replaced by P'_1 throughout. For this, the chain of implications (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) follows in turn from Lorentz [7, p. 486], Boyd [4, p. 1253], Boyd [4, Lemma 5] and Boyd [4, p. 1253]; (e) clearly implies (f), while if (f) holds and $f \in L^\sigma(0, \pi)$, $g \in L^{\sigma'}(0, \pi)$ with $f \geq 0, g \geq 0$ then Fubini's theorem shows that

$$\int_0^\pi g(x)(P_1f)(x)dx = \int_0^\pi f(t)(P'_1g)(t)dt \leq \sigma(f)\sigma(P'_1g) < \infty$$

so $P_1f \in L^\sigma$ (see Lorentz [7, p. 484]) and (a) holds. This proves the lemma.

LEMMA 2. *If $a = \{a_\nu\}$ is the sequence of Fourier cosine coefficients of $f \in L^\sigma(0, \pi)$ then $c = \{c_\nu\}$, $c_0 = 0$, $c_\nu = a_\nu/\nu$, $\nu = 1, 2, \dots$ is the sequence of Fourier cosine coefficients of a function $F \in L^\sigma(0, \pi)$.*

Proof of Lemma 2. Let $K(t) = -\log|2 \sin(t/2)|$, $|t| < \pi$. According to [8, p. 180], c is the sequence of Fourier cosine coefficients of

$$F(x) = \frac{1}{\pi} \int_{-\pi}^\pi f(x+t)K(t)dt, \quad 0 < x < \pi.$$

Now for any t , $|t| < \pi$ we set $f_t(x) = f(x+t)$ and observe that since f is even on $(-\pi, \pi)$, for all $\lambda > 0$

$$\begin{aligned}
 |\{x \in (0, \pi): |f(x)| > \lambda\}| &= \frac{1}{2} |\{x \in (-\pi, \pi): |f(x)| > \lambda\}| \\
 &= \frac{1}{2} |\{x \in (-\pi, \pi): |f_t(x)| > \lambda\}| \\
 &\geq \frac{1}{2} |\{x \in (0, \pi): |f_t(x)| > \lambda\}|
 \end{aligned}$$

so that $f_t(x)$ considered as a function on $0 < x < \pi$ satisfies $(f_t)^*(x) \leq f^*(x/2)$ and it then follows from (1) that $\sigma(f_t) \leq 2\sigma(f)$. Hence, if $g \in L^\sigma(0, \pi)$ with $g \geq 0$

$$\begin{aligned}
 \int_0^\pi |F(x)|g(x)dx &\leq \int_{-\pi}^\pi |K(t)|dt \int_0^\pi |f(x+t)|g(x)dx \\
 &\leq \int_{-\pi}^\pi |K(t)|\sigma(f_t)\sigma'(g)dt \leq 2\sigma(f)\sigma'(g) \int_{-\pi}^\pi |K(t)|dt
 \end{aligned}$$

so that upon taking the supremum over $g \in L^\sigma(0, \pi)$ with $g(x) \geq 0$, $\sigma'(g) \leq 1$ it follows that $\sigma(F) \leq 2\sigma(f) \int_{-\pi}^\pi |K(t)|dt < \infty$. Thus $F \in L^\sigma(0, \pi)$ and the lemma is proved.

Since $L^\sigma(0, \pi)$ contains all the constant functions, we may assume without loss of generality that $a_0 = 0$ in the proofs of Theorem 1 and 2.

Proof of Theorem 1. As Hardy [5] has shown, Ta is the sequence of Fourier cosine coefficients of $g(x) = (P'f(x) + F(x))/2$, where F is given by Lemma 2. Thus, if (a) holds, Lemma 2 shows that we must have $P'f \in L^\sigma(0, \pi)$ whenever $f \in L^\sigma(0, \pi)$ and then Lemma 1 shows that $\beta > 0$ so (b) holds. Conversely, if (b) holds, Lemma 1 shows that $P'f \in L^\sigma(0, \pi)$ while Lemma 2 shows that $F \in L^\sigma(0, \pi)$ so that $g \in L^\sigma(0, \pi)$ and (a) holds. This proves the theorem.

Proof of Theorem 2. Suppose first that (a) holds and $f \in L^\sigma(0, \pi)$. Let δ be such that $\int_0^\delta f^*(x)dx = \frac{1}{2} \int_0^\pi |f(x)|dx$, $0 < \delta < \pi$, and set

$$g(x) = \begin{cases} f^*(x) & \text{if } 0 < x < \delta \\ -f^*(x) & \text{if } \delta < x < \pi. \end{cases}$$

Clearly $g \in L^\sigma(0, \pi)$, $g(x)$ is nonnegative, nonincreasing on $(0, \delta)$, and $\int_0^\pi g(x)dx = 0$. Let $a^* = \{a_j^*\}$ denote the Fourier cosine coefficients of $g(x)$. Since $g \in L^\sigma(0, \pi)$ and (a) holds, it follows that $(T'a^*)_1 = \sum_{j=1}^\infty (a_j^*)/j$ converges, and according to Loo [6, p. 273]

$$(T'a^*)_1 = \lim_{N \rightarrow \infty} \frac{1}{\pi} \int_0^\pi (1 - \cos Nx)(Pg)(x)dx.$$

But then since $(Pg)(x)$ is integrable on (δ, π) the Riemann Lebesgue

lemma guarantees the existence of

$$\lim_{N \rightarrow \infty} \int_0^{\delta} (1 - \cos Nx)(Pg)(x) dx .$$

Now, $(Pg)(x)$ nonincreasing on $(0, \delta)$ shows

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_0^{\delta} (1 - \cos Nx)(Pg)(x) dx &\geq \lim_{N \rightarrow \infty} \sum_{k=1}^{\lfloor \frac{N\delta}{2\pi} \rfloor} (Pg) \left(\frac{2k\pi}{N} \right) \frac{1}{\pi} \int_{2(k-1)\pi/N}^{2k\pi/N} (1 - \cos Nx) dx \\ &= \frac{1}{\pi} \lim_{N \rightarrow \infty} \frac{2\pi}{N} \sum_{k=1}^{\lfloor \frac{N\delta}{2\pi} \rfloor} (Pg) \left(\frac{2k\pi}{N} \right) \\ &= \frac{1}{\pi} \int_0^{\delta} (Pg)(x) dx \\ &= \frac{2}{\pi} \int_0^{\delta} g(t) \log \left| \frac{\sin(\delta/2)}{\sin(t/2)} \right| dt . \end{aligned}$$

It follows that $|g(t)| \log^+(1/t)$ is integrable on $(0, \pi)$ and hence [6, p. 273] $T'a^*$ is the sequence of Fourier cosine coefficients of $H(x) = ((Pg)(x) + G(x))/2$ where G is the function associated by Lemma 2 to the sequence a^* . Since $G \in L^\sigma(0, \pi)$ for any σ , and $H \in L^\sigma(0, \pi)$ by hypothesis, it follows that $Pg \in L^\sigma(0, \pi)$. Now, since $|(Pf)(x)| \leq (P|g|)(x)$ it follows that $Pf \in L^\sigma(0, \pi)$ whenever $f \in L^\sigma(0, \pi)$ so then Lemma 1 shows $\alpha < 1$. Thus (a) implies (b).

Conversely, suppose (b) holds. There is a number $p > 1$ such that $\alpha < p^{-1}$ so $L^\sigma(0, \pi) \subset L^p(0, \pi)$ and hence if $f \in L^\sigma(0, \pi)$ Hölder's inequality shows $\int_0^\pi |f(t)| \log^+(1/t) dt < \infty$. According to Loo [6, p. 273-274] $T'a$ is then the sequence of Fourier cosine coefficients of $h(x) = (Pf(x) + F(x))/2$ where F is the function of Lemma 2. Now Lemma 1 shows that $Pf \in L^\sigma(0, \pi)$ and hence $h \in L^\sigma(0, \pi)$ so (a) holds. The theorem is proved.

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