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Functions of translation type were introduced by H. Reiter and studied by the author in some detail. In this paper we introduce a class of Banach spaces of functions on a locally compact group, including the spaces $\mathscr{M}^p(G)$ of Liu-van Rooij-Wang. Necessary and sufficient conditions are given under which these spaces can be characterized by functions of translation type. As application it is shown that a subclass of these spaces, including Wiener's algebra, satisfies a certain minimality property. Furthermore, we obtain a generalization of a theorem on Fourier transforms, due to Edwards-Hewitt-Ritter, in a very simple manner, whereas the original proof took several pages.

Our notation follows that of [13]. G always denotes an arbitrary locally compact group with left Haar measure dx. For a measurable set $M \subset G$ let $|M|_G$ be the left Haar measure of M. Δ_G denotes the Haar modulus of G.

We write $\mathscr{C}(G)$ for the linear space of all continuous, complex-valued functions on G, $\mathscr{C}^\circ(G)$ for the Banach space of all continuous functions vanishing at infinity with the norm $\|f\|_\infty = \sup_{x \in G}|f(x)|$ and $\mathscr{K}(G)$ for the subspace of all functions with compact support (supp). $L^1_{\text{loc}}(G)$ denotes the space of all locally integrable functions on G. As usual measurable functions coinciding locally almost everywhere (l.a.e.) shall be identified. The spaces $L^p(G)$ have the usual meaning. The left [right] translation operators $L_y[R_y]$ are defined by $L_y f(x) := f(y^{-1}x)[R_y f(x) := f(xy)]$. We call two functions w_1 and w_2 equivalent (l.a.e.), $w_1 \sim w_2$(l.a.e.) if there exist constants $C_1, C_2 > 0$ such that $C_1 w_1(x) \leqq w_2(x) \leqq C_2 w_1(x)$(l.a.e.). Φ_A shall denote the characteristic function of the set A. Given a measurable, locally essentially bounded function w on G, $(L^1_w(G), \|\ \|_{1,w})$ denotes the space of all measurable functions f on G such that fw is in $L^1(G)$ with the norm $\|f\|_{1,w} := \|fw\|_1$. This space is a Banach convolution algebra if and only if $w \sim w_1$ l.a.e. for some $w_1 \in L^1_{\text{loc}}(G)$ satisfying $0 < w_1 < \infty$ and $w_1(xy) \leqq C w_1(x) w_1(y)$ for all $x, y \in G$ (cf. [9]). Moreover, $L^1_w(G)$ possesses bounded approximate left units since $\mathscr{K}(G)$ is dense in $L^1_w(G)$.

Throughout this paper we shall always consider Banach spaces $(B(G), \|\ \|_B)$ of measurable functions, continuously embedded into $L^1_{\text{loc}}(G)$ and satisfying the following conditions:

(B1) $B(G)$ is left invariant, i.e., L_y defines a continuous linear operator on $B(G)$ for every $y \in G$.

(B2) $B(G)$ is a solid Banach space of [continuous] functions, i.e., $f \in B(G)$ and $|g(x)| \leqq |f(x)|$ l.a.e. for any measurable [continuous] function g implies $g \in B(G)$ and $\|g\|_B \leqq \|f\|_B$.

(B3) $B(G)$ is a Banach convolution module over some Beurling algebra $L^1_w(G)$, i.e., $h \in L^1_w(G)$ and $f \in B(G)$ implies $h*f \in B(G)$ and $\|h*f\|_B \leqq \|h\|_{1,w}\|f\|_B$.

REMARK 0.1. (i) Each Banach function space satisfying (B1) – (B3) contains $\mathscr{K}(G)$ as a subspace (cf. [6]).

(ii) If $\mathscr{K}(G)$ is dense in $B(G)$ then the map $y \to L_y f$ is continuous for each $y \in G$ and $f \in B(G)$ (cf. [6]).

(iii) The continuity of the map $y \to L_y f$ is equivalent to the fact that $B(G)$ satisfies (B3) with $w(y) := \max(1, \|L_y\|_B)$ and that $L^1_w(G)*B(G)$ is dense in $B(G)$ (cf. [7]). Thus, if especially $\|L_y\|_B = 1$ for all $y \in G$, $B(G)$ is an essential Banach convolution module over $L^1(G)$.

(iv) If there is a constant C such that $\|L_y\|_B \leqq C$ holds for all $y \in G$ one can choose an equivalent norm $\|\ \|'_B$ on $B(G)$ such that $\|L_y\|'_B = 1$ for all $y \in G$, in particular we have $\|L_y f\|'_B = \|f\|'_B$ for all $f \in B(G)$.

I. Functions of translation type. Basic results. Functions of translation type were introduced by H. Reiter in [14] to show certain functorial properties of the space $\mathfrak{S}(G)$ of Schwartz-Bruhat functions. Their properties are discussed in some detail in [2] and [3]. Let us shortly recall the definition and derive some results that will be used in the proof of our main result.

DEFINITION 1.1. A continuous real-valued function $\omega: G \to \boldsymbol{R}$, $\omega \geqq 0$, is called a function of [left] translation type if it satisfies the following condition (V) [(V_L)]:

There is an open, compactly generated subgroup $G(\omega) < G$ with the property that for each compact set $K \subset G(\omega)$ there exists a constant $C_K > 0$ such that $\omega(xa) \leqq C_K\omega(x)[\omega(a^{-1}x) \leqq C_K\omega(x)]$ holds for all $x \in G$ and each $a \in K$.

By $V(G)[V_L(G)]$ we shall denote the cone of all functions of [left] translation type. Furthermore, we put $V^B(G) := V(G) \cap B(G)$ and we write $V^p(G)$ and $V^{1,w}(G)$ if $B(G) = L^p(G)$ or $L^1_w(G)$ (analogously $V^p_L(G)$ and $V^{1,w}_L(G)$). Moreover, we denote $V^B(G) \cap V^B_L(G)$ by $V^B_T(G)$.

REMARK 1.1. (i) Each $\omega \in V^B(G)$ has a representation of the form $\omega = \sum_{n \geqq 1} L_{y_n}\omega_n$ with $y_n \in G$ and $\omega_n \in V^B(G)$, $\omega_n(x) \neq 0 \Leftrightarrow x \in G(\omega)$.

(ii) If G is connected one has to choose $G(\omega) = G$.

(iii) A discrete group G is not of interest for the considerations of this paper, since in this case we may choose $G(\omega) = \{e\}$ and hence $V_T^B(G) = B(G)$.

LEMMA 1.1. *Let $\omega \in L^1(G)$, $\omega \geqq 0$, be given such that ω satisfies* (V) $[(V_L)]$. *If $k \in \mathscr{K}_+(G)$ such that* $\operatorname{supp} k \subset G(\omega)$ *then $\omega \sim \omega * k$ and $\omega * k \in V^1(G)[V_L^1(G)]$.*

Proof. We shall only consider the case (V) since the other case is even easier.

It is obvious that $\omega * k$ is a continuous, positive function in $L^1(G)$. If we denote $K{:} = \operatorname{supp} k$, we get:

$$\begin{aligned}\omega * k(x) &= \int_K \Delta_G(y^{-1})\omega(xy^{-1})k(y)dy \leqq C_{K^{-1}} \cdot \int_K \Delta_G(y^{-1})k(y)dy \cdot \omega(x) \\ &= C_1\omega(x)\end{aligned}$$

and

$$\begin{aligned}\omega(x) &= C' \cdot \int_K \Delta_G(y^{-1})k(y)\omega(x)dy \leqq C' \cdot C_K \cdot \int_K \Delta_G(y^{-1})\omega(xy^{-1})k(y)dy \\ &= C_2 \cdot \omega * k(x)\end{aligned}$$

and hence $\omega * k \in V^1(G)$.

LEMMA 1.2. *Let ρ be a locally essentially bounded function on G satisfying $\rho \geqq 0$ and $\rho(xy) \leqq \rho(x)\rho(y)$. Then for every function $\omega_1 \in V_T^1(G)$ we can find some $\omega \in V_T^1(G)$ such that $\omega \cdot \rho \leqq \omega_1$ holds.*

Proof. Let $V \subset G$ be a symmetric, compact neighbourhood of e and let G' be the open subgroup of G generated by V. Then there exists a constant $p > 1$ such that $|K^n|_G \leqq |K|_G \cdot p^n$, $n \in N$ (cf. [11]). Moreover, the existence of a constant $q > 0$ satisfying

$$\sup\{\rho(x) \mid x \in V^n\} \leqq q^n$$

follows from the assumptions imposed on ρ. If we choose now a constant $a > \max\{p, q\}$ the function $g(x){:} = \sum_{n=1}^{\infty} a^{-n}\Phi_{V^n \setminus V^{n-1}}(x)$ satisfies $g(y^{-1}x) \leqq ag(x)$ as well as $g(xy) \leqq ag(x)$ for all $x \in G$ and each $y \in V$ and hence the properties (V) and (V_L) are satisfied for $G(g) = G'$. Moreover, it is obvious that $g(x)\rho(x) \leqq 1$ holds for all $x \in G$. Using Lemma 1.1 we get for any $k \in \mathscr{K}_+(G)$ with $\operatorname{supp} k \subset G'$: $g * k \sim g$ and $g * k \in V_T^1(G)$. If we choose now $\omega_1 \in V_T^1(G)$ and put $\omega{:} = \omega_1 \cdot (g * k)$ we derive $\omega \cdot \rho \leqq \omega_1 \cdot (g * k) \cdot \rho \leqq C' \cdot \omega_1 \cdot g \cdot \rho \leqq C' \cdot \omega_1$. Since $\omega \in V_T^1(G)$ the proof is complete.

COROLLARY 1.3. *Let G be an arbitrary locally compact group.*

Then we have: $V_T^B(G) \neq \{0\}$.

Proof. Due to Lemma 1.2 there exists a nontrivial function $\omega \in V^{1,w}(G)$. Hence, by Lemma 1.1 $\omega * k \in V^B(G)$ for each $k \in \mathscr{K}_+(G) \subset B(G)$ with $\operatorname{supp} k \subset G(\omega)$.

LEMMA 1.4. *If* $\mathscr{K}(G)$ *is dense in* $B(G)$ *then* $V^B(G)$ *is dense in* $B_+(G)$.

Proof. For every $f \in B_+(G)$ there exists some $k \in \mathscr{K}_+(G)$ such that $\|f - k\|_B < \varepsilon/2$. Moreover, choose some $\omega \in V^B(G)$ with $\|\omega\|_B < \varepsilon/2$ and $\operatorname{supp} k \subset \operatorname{supp} \omega$. Then it is easily seen that $\omega' := \omega + k \in V^B(G)$ and hence $\|f - \omega'\|_B < \varepsilon$.

REMARK 1.2. (i) The left version of Lemma 1.2 will be used in a crucial way in order to prove our main theorem.

(ii) $V^{1,w}(G)$ is dense in $(L^1_w)_+(G)$.

(iii) If $\mathscr{K}(G)$ is dense in $B(G)$ and $\|L_y\|_B = 1$ for all $y \in G$ then $V^B(G)$ is contained in $\mathscr{C}^\circ(G)$. Since the proof is essentially the same as the proof of Proposition 1 of [2] it is omitted.

II. The main results. Let us begin with the following definition:

DEFINITION 2.1. (i) $\mathfrak{V}^B(G) := \{f \in \mathscr{C}(G) \mid \text{there exists some } \omega_f \in V^B(G) \text{ with } |f| \leqq \omega_f\}$.

(ii) Let $g \in \mathscr{K}(G)$, $g \neq 0$, and put $f^g(x) := \|(L_x g) \cdot f\|_\infty$. Then we define: $\mathscr{M}_g^B(G) := \{f \in \mathscr{C}(G) \mid f^g \in B(G)\}$.

(iii) $B^g(G) := \{f^g \mid f \in \mathscr{M}_g^B(G)\}$.

If $B = L^p$ we shall write $\mathfrak{V}^p(G)$ and $\mathscr{M}_g^p(G)$. If $B = L^1_w$ we write $\mathfrak{V}^{1,w}(G)$ and $\mathscr{M}_g^{1,w}(G)$.

THEOREM 2.1. $\mathscr{M}_g^B(G)$ *is a Banach space with the norm* $\|f\|_{(B)} := \|f^g\|_B$ *that is continuously embedded into* $B(G)$ *and satisfies* (B1)–(B3) *with respect to each Banach algebra* $L^1_w(G)$ *that acts on* $B(G)$.

Proof. It is obvious that $\| \quad \|_{(B)}$ is a norm and that $\mathscr{M}_g^B(G)$ is a Banach space with this norm, continuously embedded into $B(G)$ and satisfying (B1) and (B2). To show (B3) let $h \in L^1_w(G)$ and $f \in \mathscr{M}_g^B(G)$ be given. Then

$$(h*f)^g(x) = \|(L_x g)\cdot(h*f)\|_\infty \leqq \int_G |h(y)|\,\|(L_{y^{-1}x} g)\cdot f\|_\infty dy$$

$$= \int |h(y)|\, L_y(f^g)(x)\,dy = |h| * f^g(x)$$

holds and hence $\|h*f\|_{(B)} \leqq \|h\|_{1,W}\|f\|_{(B)}$.

Moreover, we shall use the following uniform partition of unity that was introduced by H. G. Feichtinger in [8] in a more general form. Let $V = V^{-1}$ be an open, relatively compact subset of G. Then there exists a subset $Y = \{y_i\}_{i\in I} \subset G$ such that

$$(*) \qquad G = \bigcup_{i\in I} y_i V \quad \text{and} \quad \sup_{x\in G} |\{i \mid x \in y_i V\}| \leqq C < \infty$$

holds. Furthermore, there exists a bounded partition of unity $(\varphi_i)_{i\in I} \subset \mathscr{C}^\circ(G)$, i.e.,

$$\sum_{i\in I} \varphi_i(x) = 1 \quad \text{and} \quad \sup_{i\in I} \|\varphi_i\|_\infty \leqq C' < \infty ,$$

such that $\operatorname{supp} \varphi_i \subset y_i V$, $i \in I$.

REMARK 2.1. (i) The spaces $\mathfrak{B}^p(G)$ where introduced in [2] and the spaces $\mathscr{M}_g^p(G)$ in [12]. In [12] the independence of the spaces $\mathscr{M}_g^p(G)$ of the function g was claimed but the argument used there fails, if no additional properties—related to commutativity—are imposed on G. In Theorem 2.4 we shall give necessary and sufficient conditions for the independence of g. In the case $B(G) = L^p(G)$ it turns out that the spaces $\mathscr{M}_g^p(G)$ are in fact independent of the choice of the function g for arbitrary G. Moreover, this theorem shows that $\mathfrak{B}^B(G) = \mathscr{M}_g^B(G)$ if and only if these conditions are satisfied. Furthermore there will be given an example of a Banach space satisfying (B1)–(B3) such that $\mathscr{M}_g^B(G)$ is not independent of g.

(ii) In [2] it was shown that $\mathfrak{B}^1(G) = W^1(G)$ holds, where $W^1(G)$ denotes Wiener's algebra as introduced by H. G. Feichtinger in [5] and, using the main result of [5], $\mathfrak{B}^p(G) = \mathscr{M}_g^p(G)$ was derived. Our main theorem gives a direct proof of this result, as well. It seems to be worth noticing that it is possible to give a characterization of the Banach space $\mathscr{M}_g^B(G)$ in terms of functions of translation type without the direct use of a norm.

(iii) The spaces $\mathscr{M}_g^p(G)$ were considered by various authors using other but equivalent definitions (cf. [1], [4], [5], [10], [15]). Using the terminology of these authors we have successively: $\mathscr{M}_g^p(G) = l^p(\mathscr{C})$, $\mathscr{M}_g^p(G) = \mathfrak{T}_p(G) \cap \mathscr{C}(G)$, $\mathscr{M}_g^1(G) = W^1(G)$, $\mathscr{M}_g^p(\boldsymbol{R}) = (\mathscr{C}^\circ, l^p)$ and $\mathscr{M}_g^p = (\mathscr{C}^\circ, l^p)$. In all these cases except in [5] G is supposed to be abelian.

LEMMA 2.2. *Let $V \subset G$ and $\{y_i\}_{i\in I} \subset G$ be given such that* $(*)$ *holds and let $(\varphi_i)_{i\in I}$ be a partition of unity corresponding to the covering $(y_i V)$. Then each $\omega \in V^1(G)$ satisfies*

$$\sum_{i \in I} \|\varphi_i \omega\|_\infty \leqq C \cdot \sum_{i \in I} \omega(y_i) < \infty .$$

Proof. $\|\varphi_i\omega\|_\infty = \sup_{x \in V} |\varphi_i(y_i x)\omega(y_i x)| \leqq \|\varphi_i\|_\infty C_V \omega(y_i) = C \cdot \omega(y_i)$ and therefore $\sum_{i \in I} \|\varphi_i\omega\|_\infty \leqq C \cdot \sum_{i \in I} \omega(y_i)$. Furthermore $\sup_{x \in G} |\{i \mid x \in y_i V\}| \leqq C'$ implies $\sum_{i \in I} \Phi_{y_i V} \leqq C'$ and hence

$$\sum_{i \in I} \int_{y_i V} f(x)dx \leqq C' \cdot \int_G f(x)dx$$

for each $f \in L^1_+(G)$.

From this relation we derive:

$$\sum_{i \in I} \omega(y_i) \leqq C_V \cdot \sum_{i \in I} \int_{y_i V} \omega(x)dx \leqq C_V C' \cdot \int_G \omega(x)dx < \infty .$$

LEMMA 2.3. *For all $g \in \mathscr{K}(G)$, $g \neq 0$, we have $\mathfrak{B}^B(G) \subset \mathscr{M}_g^B(G)$.*

Proof. Let $f \in \mathfrak{B}^B(G)$ with $|f| \leqq \omega$ and $\omega \in V^B(G)$. Then $\omega^g(x) = \|(L_x g) \cdot \omega\|_\infty = \sup\{|g(y)\omega(xy)|, y \in \operatorname{supp} g\} \leqq C_K \|g\|_\infty \omega(x)$ and $\|f\|_{(B)} \leqq \|\omega\|_{(B)} \leqq C\|\omega\|_B < \infty$ since $B(G)$ and $\mathscr{M}_g^B(G)$ satisfy (B2).

Now we are able to prove our main theorem.

THEOREM 2.4. *The following conditions are equivalent.*

(1) *$R_a B^g \subset B$ and $\rho(a) := \sup\{\|(R_a f^g\|_B / \|f^g\|_B), f \in \mathscr{M}_g^B(G)\}$ is locally bounded for all $g \in \mathscr{K}(G)$, $g \neq 0$.*

(2) *$\mathfrak{B}^B(G) = \mathscr{M}_g^B(G)$ for all $g \in \mathscr{K}(G)$, $g \neq 0$.*

(3) *$\mathscr{M}_g^B(G)$ is independent of the choice of the function $g \in \mathscr{K}(G)$, $g \neq 0$, and different functions g yield equivalent norms.*

Proof. (1) $\Rightarrow$ (2) We have to show that to every $f \in \mathscr{M}_g^B(G)$ there exists some $\omega_f \in V^B(G)$ with $|f| \leqq C \cdot \omega_f$, $C < \infty$.

Since ρ is locally bounded, strictly positive and satisfies $\rho(xy) \leqq \rho(x)\rho(y)$ we can choose functions $\omega, \omega_1 \in V^1_T(G)$ such that $\omega \cdot \rho \leqq \omega_1$ holds (Lemma 1.2). If we now put:

$$\omega_f(x) := \|(L_x\omega) \cdot f\|_\infty$$

we obtain:

(i) $|f(x)| = C \cdot |L_x\omega(x) f(x)| \leqq C \cdot \omega_f(x)$.

(ii) ω_f is continuous and $\omega_f \geqq 0$.

(iii) $\omega_f(xa) = \|(L_{xa}\omega) \cdot f\|_\infty = \sup_{y \in G} |\omega(a^{-1}x^{-1}y) f(y)|$
$\leqq \sup_{y \in G} |\omega(x^{-1}y) f(y)| \cdot C_K = C_K \cdot \omega_f(x)$ for all $x \in G$.

(iv) Without loss of generality we can assume that $\operatorname{supp} g$ contains an open, symmetric neighbourhood V of e satisfying $g(v) \cdot C' \geqq 1$ for all $v \in V$. Now choose a set $\{y_i\}_{i \in I} \subset G$ such that $(*)$ is fullfilled and a corresponding partition of unity. Then we have:

$$\begin{aligned}\omega_f(x) &= \Big\| \Big(L_x \Big(\sum_{i\in I} \varphi_i \omega \Big) \Big) \cdot f \Big\|_\infty \leqq \sum_{i\in I} \| (L_x(\varphi_i\omega) \cdot f \|_\infty \\ &\leqq C' \cdot \sum_{i\in I} \| (L_x(\varphi_i\omega)) \cdot L_{xy_i} g \cdot f \|_\infty \leqq C' \sum_{i\in I} \| \varphi_i\omega \|_\infty \| (L_{xy_i} g) \cdot f \|_\infty \\ &= C' \sum_{i\in I} \| \varphi_i \omega \|_\infty (R_{y_i}(f^g))(x) .\end{aligned}$$

Now Lemma 1.2 and Lemma 2.2 yield:

$$\begin{aligned}\| \omega_f \|_B &\leqq C' \sum_{i\in I} \| \varphi_i\omega \|_\infty \| R_{y_i}(f^g) \|_B \leqq C'' \sum_{i\in I} \omega(y_i)\rho(y_i) \| f^g \|_B \\ &\leqq C'' \sum_{i\in I} \omega_1(y_i) \| f \|_{(B)} < \infty .\end{aligned}$$

Hence $\omega_f \in V^B(G)$ and $|f| \leqq \omega_f$.

(2) $\Rightarrow$ (3) is a simple application of the closed graph theorem.

(3) $\Rightarrow$ (1) Let $g \in \mathscr{K}(G)$ be given and $K \subset G$ be compact. Then there exists $h \in \mathscr{K}_+(G)$ such that $\sup_{a\in K} |L_a g| \leqq h$ and $\| f^h \|_B \leqq C \cdot \| f^g \|_B$ for all $f \in \mathscr{M}_g^B(G) = \mathscr{M}_h^B(G)$, $C < \infty$. Thus we obtain $\sup_{a\in K} \| R_a(f^g) \|_B \leqq C \cdot \| f^g \|_B$ for all $f \in \mathscr{M}_g^B(G)$ and hence ρ is locally bounded.

From now on we shall always assume that the conditions of the above theorem are satisfied and therefore we shall write $\mathscr{M}^B(G)$ instead of $\mathscr{M}_g^B(G)$.

COROLLARY 2.5. *$\mathfrak{B}^B(G)$ is a Banach space with the norm $\| \;\; \|_{(B)}$ and satisfies the conditions* (B1)–(B3).

REMARK 2.2. The following statements are easily derived from Theorem 2.4.

(i) If $h \in \mathfrak{B}_T^1(G)$ then we have $\mathscr{M}^B(G) = \{f \in \mathscr{C}(G) \mid f^h \in B(G)\}$.

(ii) If $\mathscr{M}^{B_2} \subset B_1(G) \subset B_2(G)$ then $\mathscr{M}^{B_2}(G) = \mathscr{M}^{B_1}(G)$.

(iii) The spaces $l^p(L^q)$ of measurable functions being locally in L^q and globally in l^p (cf. [1], [10], [15]) satisfy of course the conditions of our theorem. Especially $l^p(\mathscr{C}^0) = \mathscr{M}^p$ holds (cf. [1]). It is easy to see that also $\mathscr{M}^p = \mathscr{M}^{l^p(l^q)}$ holds.

At the end of this section we shall introduce a new family of Banach spaces satisfying (B1)–(B3) that will give us an example of a Banach space that does not satisfy the conditions of Theorem 2.3.

For a continuous function f on G and a closed subgroup $H < G$ f/H shall denote the restriction from f to H. For simplicity we write $\| f/H \|_1$ instead of $\| f/H \|_{L^1(H)}$.

DEFINITION 2.2. Let $1 \leqq p \leqq \infty$. Then we define

$$E^p(G, H) := \left\{ f \in L^p(G) \cap \mathscr{C}^\circ(G) \mid \sup_{x \in G} \| (L_x f)/H \|_1 < \infty \right\} .$$

THEOREM 2.6. *$E^p(G, H)$ is a Banach space with the norm*

$$\| f \|_{E,p} := \| f \|_p + \| f \|_\infty + \sup_{x \in G} \| (L_x f)/H \|_1 ,$$

satisfying (B1)–(B3) *with respect to* $L^1(G)$.

Proof. It is a matter of routine to check that $E^p(G, H)$ is a Banach space and satisfies (B1) and (B2). Moreover, even $\| L_y f \|_{E,p} = \| f \|_{E,p}$ holds. Now let $f \in L^1(G)$ and $g \in E^p(G, H)$ be given. Then we have

$$\begin{aligned} \| (L_x(f * g))/H \|_1 &\leqq \int_H \int_G | L_x f(y) | \, | g(y^{-1}h) | \, dy dh \\ &= \int_G | L_x f(y) | \int_H | g(y^{-1}h) | \, dh dy \leqq \| f \|_1 \cdot \sup_{y \in G} \| (L_y g)/H \|_1 \end{aligned}$$

and hence $\| f * g \|_{E,p} \leqq \| f \|_1 \cdot \| g \|_{E,p}$.

EXAMPLE. Let us consider the space $E^\infty(G, H)$, where G denotes the "$ax + b$"-group, i.e., $G = \{(a, b) \mid a, b \in \boldsymbol{R}, a > 0\}$ with multiplication $(a, b) \cdot (a', b') := (aa', ab' + b)$ and $H := \{(d^n, 0) \mid n \in \boldsymbol{Z}, d \text{ fixed}\}$, where d will be chosen in a appropriate way. Then H is a non-normal subgroup of H.

Now let $g, k \in \mathscr{K}_+(G)$ be given such that $g(e) \neq 0$ and $k(e) \neq 0$, and let $K_1 := \operatorname{supp} g$ and $K_2 := \operatorname{supp} k$. If furthermore, $c = (c_1, c_2)$, $c_2 \neq 0$, is given and $c_n := (d^n, 0) \cdot (c_1, c_2)$ then a simple computation shows that d can be chosen such that

$$(*) \quad K_2 c_n K_1^{-1} H \cap K_2 c_m K_1^{-1} H = \emptyset \quad \text{holds for all} \quad n, m \geqq 1, n \neq m .$$

Now define $f(x) := \sum_{n \geqq 1} (1/n) R_{c_n^{-1}} k(x)$. Then we have

$$f^g(x) = \| (L_x g) \cdot f \|_\infty = \sum_{n \geqq 1} \frac{1}{n} \| (L_x g) \cdot R_{c_n^{-1}} k \|_\infty$$

due to $(*)$ since $\operatorname{supp}(f^g) = \bigcup_{n \geqq 1} K_2 c_n K_1^{-1}$ and hence $f, f^g \in \mathscr{C}^\circ(G)$. Moreover $(*)$ implies

$$\begin{aligned} \| (L_x f)/H \|_1 &\leqq g(e)^{-1} \| (L_x(f^g))/H \|_1 \leqq g(e)^{-1} \sup_{\substack{n \geqq 1 \\ h \in H}} \frac{1}{n} \| (L_{x^{-1}h} g) \cdot (R_{c_n^{-1}} k) \|_\infty \\ &\leqq g(e)^{-1} \| g \|_\infty \| k \|_\infty \quad \text{for all} \quad x \in G . \end{aligned}$$

Therefore we have $f, f^g \in E^\infty(G, H)$. On the other hand we have:

$$\|(R_c(f^g))/H\|_1 \geq \sum_{n\geq 1} f^g(c_n) = \sum_{n\geq 1} \frac{1}{n} \|(L_{c_n}g)\cdot(R_{c_n^{-1}}k)\|_\infty$$
$$\geq \sum_{n\geq 1} \frac{1}{n} |g(e)|\,|k(e)| = \infty$$

since $\{(d^n, 0), n \geq 1\} \subset H \cap \operatorname{supp}(R_c(f^g))$ and $g(e), k(e) \neq 0$.

Thus f and f^g are functions in $E^\infty(G, H)$ such that $R_c(f^g)$ is not in $E^\infty(G, H)$ and hence the space $E^\infty(G, H)$ does not satisfy the conditions (1)–(3) of Theorem 2.4.

III. Applications.

THEOREM 3.1. *$\mathfrak{B}^{1,w}(G)$ is the minimal space in the family of all Banach spaces satisfying* (B1)–(B3) *with respect to* $L^1_w(G)$.

Proof. Let $f \in \mathfrak{B}^{1,w}(G)$ be given. By Remark 1.2 (resp. Lemma 1.4) $V^{1,w}(G)$ is dense in $(L^1_w)_+(G)$ and hence, given $\varepsilon > 0$, there exists $\omega \in V^{1,w}(G)$ such that $|f| \leq \omega$ and $\|\omega\|_{1,w} \leq \|f\|_{1,w} + \varepsilon$. Choosing any $k \in \mathscr{K}_+(G)$ such that $\operatorname{supp} k$ is contained in the component of the identity (which in turn is contained in $G(\omega)$), Lemma 1.1 yields $\omega \leq C_k\cdot(\omega * k)$ and C_k does not depend on ω. Thus we get

$$\|f\|_B \leq \|\omega\|_B \leq C_k\|k\|_B\|\omega\|_{1,w} \leq C'_k(\|f\|_{1,w} + \varepsilon) \text{ for all } \varepsilon > 0 .$$

Hence $\|f\|_B \leq C\cdot\|f\|_{1,w}$ for every $f \in \mathfrak{B}^{1,w}(G)$.

REMARK 3.1. This theorem reduces to Corollary 3 of [2] and to Theorem 4 of [5] in the case $w \equiv 1$. The case $w \equiv 1$ was also generalized in an other direction in [8].

For the rest of this section we shall assume that G is abelian and that $B(G)$ satisfies (B1)–(B3) with respect to $L^1(G)$.

THEOREM 3.2. *Let $f \in L^1(G)$ with compact support be given. If $\hat{f} \in B(G)$ then $\hat{f} \in \mathscr{M}^B(G)$ and there exists a constant M depending only on $B(G)$ and* $\operatorname{supp} f$ *such that* $\|\hat{f}\|_{(B)} \leq M\|\hat{f}\|_B$.

Proof. Take any function $k \in \mathscr{K}_+(G)$ such that $k = 1$ on $\operatorname{supp} f$ and $\hat{k} \in \mathscr{M}^1(\hat{G}) = \mathfrak{B}^1(\hat{G})$ (in fact for every function $k \in \mathscr{K}_+(G)$ with $\hat{k} \in L^1(\hat{G})$ one has $\hat{k} \in \mathfrak{B}^1(G)$, since $\mathfrak{B}^1(\hat{G})$ is a Segal algebra, cf. [2], [13]). Thus there is a function $\omega \in V^1(\hat{G})$ with $|\hat{k}| \leq \omega$ and from $f = f\cdot k$ we derive $|\hat{f}| \leq |\hat{f}|*\omega$. Since it is obvious that $|\hat{f}|*\omega \in V^B(\hat{G})$ we have $\hat{f} \in \mathfrak{B}^B(\hat{G})$. Moreover, $(\hat{f})^g \leq |\hat{f}|*\omega^g$ holds (see proof of Theorem 2.1) and hence $\|\hat{f}\|_{(B)} = \|(\hat{f})^g\|_B \leq \|\omega^g\|_1\|\hat{f}\|_B = M\|\hat{f}\|_B$ with $M = \|\omega^g\|_1 < \infty$ since $\omega^g \sim \omega$ (see proof of Lemma 2.3).

REMARK 3.2. (i) This theorem generalizes earlier results of F. Holland (cf. [10]), Edwards—Hewitt—Ritter (cf. [4]) and J. Stewart (cf. [15]). Besides, our proof seems to be the simplest one. For arbitrary abelian G this theorem was first established by Edwards—Hewitt—Ritter in the case $B(G) = L^p(G), 2 \leqq p < \infty$, in order to prove their main results on multipliers. But their proof, using the theory of entire functions, runs over several pages. The proofs of Holland ($G = \boldsymbol{R}$) and Stewart (G abelian) use Hölder's inequality together with other results from Harmonic Analysis.

(ii) Let f satisfy the hypothesis of Theorem 3.2 with $B = l^p(L^q)$ (cf. [1]). Under these assumptions $\hat{f} \in l^p(L^1)$ is stated in Lemma 2 of [1]. Using Theorem 3.2 together with Remark 2.2 (iii) we get $\hat{f} \in \mathscr{M}^p(G) = l^p(\mathscr{C})$. This improves part of this lemma, since $l^p(\mathscr{C}) \subsetneqq l^p(L^1)$ (except for discrete G).

Added in proof. In the paper "Banach convolution algebras of Wiener's type" (to appear in Proc. Conf. "Functions, Series, Operators", Budapest) H. G. Feichtinger introduced so called Banach spaces of Wiener's type $W(B, C)$. The construction involves a certain compact set $Q \subset G$. In order to show the independence of Q the author assumes C to be right invariant. Since it can be shown that for the special case $B = \mathscr{C}^\circ(G)$ this independence is equivalent to the independence of the spaces $\mathscr{M}_g^C(G)$ of the function g, it follows from our example that one cannot omit a condition related to right invariance of C. Right invariance of C also implies $\mathfrak{B}^C(G) = \mathscr{M}^C(G) = W(\mathscr{C}^\circ(G), C)$. In fact even for arbitrary spaces $W(B, C)$ our condition (i) of Theorem 2.4, if modified in the obvious way, is necessary and sufficient for the independence of Q. However, a characterization of the spaces $W(B, C)$ in the sense of Theorem 2.4 by functions of translation type is impossible for general B.

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