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In this paper cardinality κ collections of open covers of a topological space satisfying various conditions are studied. When $\kappa = \omega$ some of the conditions are equivalent to the space being metrizable and the union of a compact set and a discrete set. For a metrizable space some of the conditions are equivalent to complete metrizability. If $\kappa \neq \omega$ then the relationship between some of the conditions and the existence of scales is examined.

1. Introduction and definitions.

1.1. An ordinal number is the set of all ordinals which precede it and a cardinal number is an ordinal which cannot be put in a one-to-one correspondence with any ordinal which precedes it. Throughout this paper ω will denote the set of all finite ordinals and κ will denote an infinite cardinal number.

If M is a set, x is a point, and $\mathscr{H}$ is a collection of sets, then the star of M with respect to $\mathscr{H}$, denoted $st(M, \mathscr{H})$ is the union of all members of $\mathscr{H}$ which meet M and $st(x, \mathscr{H}) = st(\{x\}, \mathscr{H})$. A sequence $\mathscr{G} = \mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of a topological space S is called a development for S iff for each $x \in S$ and open set U containing x there is an n such that $st(x, \mathscr{G}_n) \subseteqq U$. Moreover, a development is monotonic iff $\mathscr{G}_{n+1} \subseteqq \mathscr{G}_n$ for all n. A space which admits a development is called a developable space and a regular-T_1 developable space is called a Moore space. A development $\mathscr{G}$ for a Moore space is star complete (see [16]) provided that if $\{M_0, M_1, M_2, \cdots\}$ is a sequence of closed sets such that for each n, $M_{n+1} \subseteqq M_n \subseteqq st(x, \mathscr{G}_n)$ for some $x \in S$ then $\bigcap M_n \neq \varnothing$. A Moore space having a star complete development is said to be star complete. A Moore space S is Moore-closed (see [5] and [6]) iff S is closed in each Moore space in which S is embedded.

A space S is a $w\Delta$-space (see [3]) iff there exists a sequence $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \cdots$ of open covers of S such that for each $x \in S$, if $x_n \in st(x, \mathscr{B}_n)$ then the sequence $\{x_0, x_1, x_2, \cdots\}$ has a cluster point. A space S has a G_δ^*-diagonal (see [10]) provided there is a sequence $\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of S such that if x and y are distinct points of S, there is an n such that $y \notin \overline{st(x, \mathscr{G}_n)}$.

A nonempty subset M of a topological space S is called discrete iff for each $x \in M$ there is an open set U such that $U \cap M = \{x\}$. A collection of sets is discrete if the closures of the sets are mutually exclusive and the union of any subcollection of these closures is closed. A topological space is κ-collectionwise Hausdorff iff for every closed discrete subset M having cardinality at most κ, there is a pairwise disjoint collection of open sets covering M, no member of which contains more than one point of M. A space which is κ-collectionwise Hausdorff for all cardinals κ is called collectionwise Hausdorff. Regular-T_1 spaces are ω-collectionwise Hausdorff.

1.2. For a topological space S, $\mathscr{D}(S)$ denotes the collection of all closed discrete subsets of S and $\mathscr{D}^*(S)$ denotes the collection of all infinite closed, discrete subsets of S. Consider the following conditions on a cardinality κ collection $\mathscr{G} = \{\mathscr{G}_\alpha : \alpha \in \kappa\}$ of open covers of S.

Condition A(κ). For each $D \in \mathscr{D}(S)$ and open set U containing D there exists an $\alpha \in \kappa$ such that $st(D, \mathscr{G}_\alpha) \subseteq U$.

Condition wA(κ). For each $D \in \mathscr{D}^*(S)$ and open set U containing D there exists an infinite subset D' of D and an $\alpha \in \kappa$ such that $st(D', \mathscr{G}_\alpha) \subseteq U$.

Condition B(κ). For each $D \in \mathscr{D}(S)$ there is an $\alpha \in \kappa$ such that if x and y are distinct points of D, then $st(x, \mathscr{G}_\alpha) \cap st(y, \mathscr{G}_\alpha) = \varnothing$.

Condition wB(κ). For each $D \in \mathscr{D}^*(S)$ there exists an infinite subset D' of D and an $\alpha \in \kappa$ such that if x and y are distinct points of D' then $st(x, \mathscr{G}_\alpha) \cap st(y, \mathscr{G}_\alpha) = \varnothing$.

Condition C(κ). For each $D \in \mathscr{D}(S)$ and $E \in \mathscr{D}(S)$ with $D \cap E = \varnothing$ there exists an $\alpha \in \kappa$ such that $st(D, \mathscr{G}_\alpha) \cap st(E, \mathscr{G}_\alpha) = \varnothing$.

Condition wC(κ). For each $D \in \mathscr{D}^*(S)$ and $E \in \mathscr{D}(S)$ with $D \cap E = \varnothing$ there exists an infinite subset D' of D and an $\alpha \in \kappa$ such that $st(D', \mathscr{G}_\alpha) \cap st(E, \mathscr{G}_\alpha) = \varnothing$.

Condition wwC(κ). For each $D \in \mathscr{D}^*(S)$ and $E \in \mathscr{D}^*(S)$ with $D \cap E = \varnothing$ there exist infinite subsets D' and E' of D and E respectively and an $\alpha \in \kappa$ such that $st(D', \mathscr{G}_\alpha) \cap st(E', \mathscr{G}_\alpha) = \varnothing$.

A space is said to satisfy one of the conditions above if it admits a collection of open covers which satisfies the condition. For the case where $\kappa = \omega$ reference to ω is dropped whence by condition A is meant condition A(ω) and so forth.

2. Moore spaces.

2.1. If S is a developable space which has a collection of open covers satisfying a condition defined in 1.2 for $\kappa = \omega$ then S has a monotonic development satisfying that condition.

THEOREM 2.2. *For a regular T_1-space S containing no infinite open and closed discrete subset the following statements are equivalent.*

(1) *S is a compact metric space.*

(2) *S is a Moore space and every infinite subset of S has a limit point.*

(3) *S satisfies condition* A.

(4) *S satisfies condition* B.

(5) *S satisfies condition* C.

(6) *S is developable and satisfies condition* wA .

(7) *S is developable and satisfies condition* wC.

(8) *S is Moore-closed and satisfies condition* wB.

(9) *S is Moore-closed and satisfies condition* wwC.

Proof. That (1) is equivalent of (2) is well known and that (1) implies each of the conditions (3) through (9) is immediate.

(3) $\rightarrow$ (2). A sequence of covers satisfying condition A is a development; thus S is a Moore space. Let $\mathscr{G}$ be a monotonic development for S satisfying condition A. Suppose $M = \{x_0, x_1, x_2, \cdots\}$ is an infinite set of limit points of S and M fails to have a limit point. Since S is ω-collectionwise Hausdorff there is a pairwise disjoint collection of open sets $U_0, U_1, U_2, \cdots$ with $x_0 \in U_0$, $x_1 \in U_1, \cdots$. For each n, let $y_n \in st(x_n, \mathscr{G}_n) \cap U_n - \{x_n\}$. Then $S - \text{cl}\{y_0, y_1, y_2, \cdots\}$ is an open set containing M but there does not exist an n such that $st(M, \mathscr{G}_n) \subseteqq S - \text{cl}\{y_0, y_1, y_2, \cdots\}$. Thus M must have a limit point.

(4) $\rightarrow$ (2). It follows immediately from the definitions that a space satisfying condition B is a $w\Delta$ space with a G_δ^*-diagonal and hence is developable. Let $\mathscr{G}$ be a monotonic development satisfying condition B and let M and $\{y_0, y_1, y_2, \cdots\}$ be as in the proof of (3) $\rightarrow$ (2). Then either $\{y_0, y_1, y_2, \cdots\}$ is closed in which case $M' = M \cup \{y_0, y_1, \cdots\}$ is a set for which condition B fails or $\{y_0, y_1, \cdots\}$ has a limit point y in which case $M' = M \cup \{y\}$ is a set for which condition B fails.

(5) $\rightarrow$ (2). The proof is similar to (4) $\rightarrow$ (2).

(6) $\rightarrow$ (2). The same proof as (3) $\rightarrow$ (2).

(7) $\rightarrow$ (2). Similar to (4) $\rightarrow$ (2).

(8) $\rightarrow$ (1). If $\mathscr{G}$ is a monotonic development for S satisfying condition wB then it is a star complete development. For if not, there exists a sequence $\{M_0, M_1, M_2, \cdots\}$ of closed sets such that $M_{n+1} \subseteqq M_n$ and a sequence $\{x_0, x_1, x_2, \cdots\}$ of points of S such that $M_n \subseteqq st(x_n, \mathscr{G}_n)$ for all n and $\cap M_n = \varnothing$. Without loss of generality it may be assumed that $M_n - M_{n+1} \neq \varnothing$ for all n. Let $y_n \in M_n - M_{n+1}$. Then $\{y_0, y_1, y_2, \cdots\} \in \mathscr{D}^*(S)$. Condition wB fails to hold for this set. A star complete Moore-closed space is compact ([6, Theorem 1.5]).

(9) $\rightarrow$ (1). Similar to (8) $\rightarrow$ (1).

2.3. Since a Moore space which is the union of a compact set and a discrete set is paracompact the next theorem follows immediately from the proof of Theorem 2.2. See Theorem 1.6 of [6] for a related result.

THEOREM 2.4. *For a regular T_1-space S the following statements are equivalent.*

(1) *S is a Moore space which is the union of a compact set and a discrete set.*

(2) *S is a metric space and the set of all limit points of S is compact.*

(3) *S satisfies condition* A.

(4) *S satisfies condition* B.

(5) *S satisfies condition* C.

(6) *S is developable and satisfies condition* wA.

(7) *S is developable and satisfies condition* wC.

2.3. The next theorem shows that Moore-closed was needed as part of the hypothesis in Theorem 2.2.

THEOREM 2.5. *For a metrizable space S the following statements are equivalent.*

(1) *S has a complete metric.*

(2) *S satisfies condition* wB.

(3) *S satisfies condition* wwC.

Proof. By the proof of Theorem 2.2 if S satisfies condition wB or condition wwC then S is star complete and hence complete Moore and thus has a complete metric by the result of Roberts [15].

(1)$\rightarrow$(2). Suppose S has a complete metric d. Let $B(x, \varepsilon) = \{y \in S: d(x, y) < \varepsilon\}$.

For each n let $\mathscr{G}_n = \{B(x, 1/2^{n+1}): n \in \omega\}$. Let M be a countably infinite, closed, discrete subset of S. For each

$$n,\ A_n = \{x \in M: st(st(x, \mathscr{G}_n), \mathscr{G}_n) \cap M \quad \text{is finite}\}\ .$$

Suppose A_n is finite for each n. There is a point $x_0 \in M - A_0$. Then let $x_{n+1} \in st(st(x_n, \mathscr{G}_n), \mathscr{G}_n) \cap M - A_{n+1} - \{x_0, x_1, \cdots, x_n\}$. The sequence $\{x_0, x_1, x_2, \cdots\}$ is Cauchy and hence converges to a point y. Thus y is a limit point of M which is impossible. Thus, there is a k such that A_k is infinite. Let $\{a_0, a_1, a_2, \cdots\}$ be the points of A_k. There is a least positive integer n_1 such that $a_{n_1} \notin st(st(a_0, \mathscr{G}_k), \mathscr{G}_k)$. There is a least positive integer n_2 such that $a_{n_2} \notin st(st(a_0, a_{n_1}, \mathscr{G}_k), \mathscr{G}_k)$. This process may be continued. The set $A = \{a_0, a_{n_1}, a_{n_2}, \cdots\}$ has the required property.

The proof of (1) $\to$ (3) is similar.

3. First countable spaces.

3.1. An interesting generalization of development due to E. E. Grace [1], is the concept of a quasi-development. A sequence $\mathscr{G} = \mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \cdots$ of collections of open subsets of a topological space S is called a quasi-development for S provided for each point p of S and open set U containing p there is an n such that $st(p, \mathscr{G}_n) \neq \varnothing$ and $st(p, \mathscr{G}_n) \subseteq U$. The conditions defined in 1.2 can be modified to conditions on quasi-developments by requiring that the collection $\mathscr{G}_n$ in the definition cover the closed discrete sets. The following theorem results from these modifications.

THEOREM 3.2. *For a regular T_1 space S containing no infinite, discrete, closed and open subset, the following statements are equivalent.*

(1) *S is compact metric.*
(2) *S has a quasi-development satisfying condition* A.
(3) *S has a quasi-development satisfying condition* B.
(4) *S has a quasi-development satisfying condition* C.
(5) *S has a quasi-development satisfying condition* wA.
(6) *S has a quasi-development satisfying condition* wC.

Proof. That (1) implies each of statements (2) through (6) is immediate.

By modifying the arguments of Theorem 2.2 only slightly, it can be shown that each of statements (2) through (6) implies that S is countably compact. By a result of Wicke and Worrell [17, Theorem 2.10] countably compact quasi-developable spaces are compact.

3.3. Another generalization of the results of §2 is obtained in the following fashion. Using the notation of Hodel in [11] let (S, τ) be a regular T_1 space, and let $g: S \times \omega \to \tau$ be a function such that $x \in g(x, n)$ for all $x \in S$ and $n \in \omega$ and such that $g(x, n+1) \subseteq g(x, n)$ for all n. If D is a nonempty set then $g^*(D, n) = \bigcup\{g(x, n): x \in D\}$. If C is one of the conditions defined in §1.2 and $st(x, \mathscr{G}_n)$ and $st(D, \mathscr{G}_n)$ are replaced by $g(x, n)$ and $g^*(D, n)$, respectively, the resulting statement defines g to have condition C. The next theorems follow from the corresponding proofs in §2.

THEOREM 3.4. *For a regular T_1 space S containing no infinite, open and closed subset, the following statements are equivalent.*

(1) *S is a first countable, countably compact space.*

(2) *There is a function g for S which satisfies condition* A.

(3) *There is a function g for S which satisfies condition* B.

(4) *There is a function g for S which satisfies condition* C.

(5) *S is first countable and there is a function g for S which satisfies condition* wA.

(6) *S is first countable and there is a function g for S which satisfies condition* wC.

THEOREM 3.5. *For a first countable, regular T_1 space S the following statements are equivalent*:

(1) *The set of all limit points of S is countably compact.*

(2) *Then is a function g for S which satisfies condition* A.

(3) *There is a function g for S which satisfies condition* B.

(4) *There is a function g for S which satisfies condition* C.

(5) *There is a function g for S which satisfies condition* wA.

(6) *There is a function g for S which satisfies condition* wC.

4. Uncountable collections.

4.1. If each definition in §1.2 is viewed as a cardinal function, a natural question is what is the minimum cardinal κ such that a space S admits a cardinality κ collection of open covers satisfying that condition? Also one might ask if a space S admits a cardinality κ collection of open covers satisfying a condition in 1.2 what does this imply about S? In this section some partial answers to these two questions are given.

A topological space S is said to have property $D(\kappa)$ iff for each closed discrete subset M of S with cardinality at most κ there is a collection $\mathscr{H}$ of mutually exclusive open sets such that (1) $\mathscr{H}$ covers M and each member of $\mathscr{H}$ contains only one point of M, and (2) if N is a set covered by $\mathscr{H}$ such that each member of $\mathscr{H}$ contains only one point of N then N has no limit point. A space which has property $D(\omega)$ is said to have property D (see [13, page 69]). For an infinite cardinal κ a space having the property that each of its subsets of cardinality κ has a limit point will be called κ-compact.

THEOREM 4.2. *If κ is an infinite cardinal, and S is a regular T_1 space which has property $D(\kappa)$, which satisfies at least one of the conditions $A(\kappa)$, $B(\kappa)$, or $C(\kappa)$, and which contains no infinite discrete, open and closed subset of cardinality κ, then S is κ-compact.*

Proof. Suppose the theorem is false. Then there exists a closed, discrete subset M of limit points of S. Let $\mathscr{H}$ be an open cover of S satisfying (1) and (2) of the definition of property $D(\kappa)$. Proceeding as in the proof of Theorem 3.2 will yield a contradiction.

4.3. Notation and definitions from set theory not stated here may be found in [12]. The usual axioms for set theory, the Zermelo-Fraenkel axioms including the axiom of choice, will be denoted by ZFC, the continuum hypothesis will be denoted by CH, and Martin's axiom will be denoted by MA. If α is a limit ordinal, the cofinality of α denoted $\mathrm{cf}(\alpha)$, is the least ordinal β such that there is a function f from β into α such that $\sup\{f(x): x \in \beta\} = \alpha$.

The set ${}^{\omega}\omega$ of all functions from ω to ω has two natural orders. If f and g are functions from ω to ω then $f < g$ iff $f(n) < g(n)$ for each $n \in \omega$, and $f <^* g$ iff there is an m such that $f(n) < g(n)$ for all $n > m$. A subset $\mathscr{S}$ of ${}^{\omega}\omega$ is called a scale provided for each $f \in {}^{\omega}\omega$ there is a $g \in \mathscr{S}$ such that $f <^* g$. A subset $\mathscr{S}$ of ${}^{\omega}\omega$ is called a dominating family iff for each $f \in {}^{\omega}\omega$ there is a $g \in \mathscr{S}$ such that $f < g$. If there is a scale with cardinality κ there is a dominating family of cardinality κ. See [8] for results on the existence of scales. Among the results there, Hechler shows that MA implies all scales have cardinality c and for each cardinal κ such that $\omega_1 \leqq \mathrm{cf}(\kappa) \leqq \kappa \leqq c$, it is consistent with ZFC that there exists a scale whose cardinality is κ.

Remark 4.4. Whether the converse to Theorem 4.2 is true, even for metric spaces, depends on the type of set theory assumed. The case for ω_1-compactness is of particular interest, since in metric spaces ω_1-compact, Lindelof, separable, and second countable are equivalent and important.

Theorem 4.5. CH *implies that if S is a metric space which has no uncountable, discrete, closed and open subset, the following are equivalent.*

(1) *S is ω_1-compact.*

(2) *S satisfies condition* $\mathrm{A}(\omega_1)$.

(3) *S satisfies condition* $\mathrm{B}(\omega_1)$.

(4) *S satisfies condition* $\mathrm{C}(\omega_1)$.

Proof. An argument similar to the one used in the proof of Theorem 2.2 will establish each of the implications (2) $\rightarrow$ (1), (3) $\rightarrow$ (1) and (4) $\rightarrow$ (1). That (1) implies each of the statements (2), (3), and (4) follows from the fact that an ω_1-compact metric space is second countable.

Example 4.6. If $\kappa < cf(c)$, there is a subspace S of the real line such that if $\mathscr{G} = \{\mathscr{G}_\alpha: \alpha \in \kappa\}$ is a collection of open covers of S, there is a $D \in \mathscr{D}(S)$ such that if $\beta \in \kappa$, there is a member of $\mathscr{G}_\beta$ containing more than one point of D. Thus S does not satisfy condition A(κ) or condition B(κ). A modification of the argument which follows shows that S does not satisfy condition C(κ) either. There is a subset S of the set R of real numbers such that both S and $R - S$ have cardinality c and, moreover, both S and $R - S$ intersect every uncountable, closed subset of R [4]. Suppose there is a collection $\mathscr{G} = \{\mathscr{G}_\alpha : \alpha \in \kappa\}$ of open covers of S contrary to the claim. For each point $t \in R - S$ there is a sequence $\{t_0, t_1, t_2, \cdots\}$ of points of S which converges to t. The set of terms of this sequence is discrete and closed in the subspace topology on S. For each $\alpha \in \kappa$, let T_α be the set of all points t belonging to $R - S$ such that no member of $\mathscr{G}_\alpha$ contains more that one point of the sequence $\{t_0, t_1, t_2, \cdots\}$. Since $cf(c) > \kappa$ and $\bigcup_{\alpha \in \kappa} T_\alpha = R - S$, for some $\alpha \in \kappa$, T_α has cardinality c. The closure in R of T_α contains a point p of S. The point p belongs to some member V of $\mathscr{G}_\alpha$. There is a set U open in R such that $V = U \cap S$. Moreover, $U \cap T_\alpha \neq \varnothing$. If $t \in U \cap T_\alpha$, U contains a tail of the sequence $\{t_0, t_1, t_2, \cdots\}$ associated with t, but then so does V and this is a contradiction.

4.7. In what follows the space Y will denote the set to which a point x belongs iff x is a nonnegative integer or for nonnegative integers n and k, $x = n - 1/(k + 2)$. The topology on Y is the subspace topology Y inherits as a closed subset of the set of real numbers with the usual topology.

LEMMA 4.8. *If S is a metric space and the set of all limit points of S is not compact, then S includes a closed subspace which is homeomorphic to the space Y.*

LEMMA 4.9. *If κ is an infinite cardinal; there is a cardinality κ collection of open covers of Y which satisfies at least one of conditions* A(κ), B(κ) *or* C(κ) iff *there is a dominating family of cardinality κ.*

Proof. If $\mathscr{G} = \{\mathscr{G}_\alpha : \alpha \in \kappa\}$ is a collection of open covers of Y, define f_α as follows. For each n, let $f_\alpha(n) = \inf\{i : n - 1/(i + 2) \in st(n, \mathscr{G}_\alpha)\}$. The set $\{f_\alpha : \alpha \in \kappa\}$ forms a dominating family provided the collection of covers $\{\mathscr{G}_\alpha : \alpha \in \kappa\}$ satisfies condition A(κ) or B(κ) or C(κ).

Conversely, if n and k are nonnegative integers and $f \in {}^\omega\omega$, let $U(n, k) = \{n\} \cup \{n - 1/(i + 2) : i \geqq k\}$ and $\mathscr{G}_f = \{\{n - 1/(k + 2)\} : n, k \in \omega\} \cup \{U(n, f(n)) : n \in \omega\}$. If $\mathscr{S}$ is a dominating family of cardinality κ, then $\{G_f : f \in \mathscr{S}\}$ is a cardinality κ collection of open covers satisfying conditions A(κ), B(κ) and C(κ).

THEOREM 4.10. *If κ is an infinite cardinal and S is a σ-compact metric space whose set of limit points is not compact, the following statements are equivalent.*

(1) *There is a dominating family of cardinality κ.*

(2) *S satisfies condition* $\mathrm{A}(\kappa)$.

(3) *S satisfies condition* $\mathrm{B}(\kappa)$.

(4) *S satisfies condition* $\mathrm{C}(\kappa)$.

Proof. Suppose there is a dominating family $\mathscr{S}$ having cardinality κ. There exists an increasing sequence $\{F_0, F_1, F_2, \cdots\}$ of compact sets whose union is S. For each pair of nonnegative integers n and k, let $\mathscr{G}_n^k$ be a cover of F_n by open balls, centered at a point of F_n, with radius less than $1/2^{k+1}$. For each $g \in \mathscr{S}$ define $\mathscr{G}_g = \{\mathscr{G}_n^{g(n)}: n \in \omega\}$. *If $D \in \mathscr{D}(S)$* there is a function $f \in {}^\omega\omega$ such that if x and y are distinct points of D, at least one of which belongs to D, then $d(x, y) > 1/2^{f(n)}$. It is then easily seen that $\mathscr{G} = \{\mathscr{G}_g: g \in \mathscr{S}\}$ satisfies each of conditions $\mathrm{A}(\kappa)$, $\mathrm{B}(\kappa)$, and $\mathrm{C}(\kappa)$.

To prove the implications $(4) \to (1)$, $(3) \to (1)$, $(2) \to (1)$, note that S includes a closed subspace homeomorphic to Y. Each of conditions $\mathrm{A}(\kappa)$, $B(\kappa)$ and $\mathrm{C}(\kappa)$ is hereditary on closed subsets. Lemma 4.9 gives the desired result.

REMARK 4.11. Example 4.6 shows that Theorem 4.10 does not hold in general for all metric spaces. The next result improves 4.10 slightly, but the collections of sets are no longer covers.

THEOREM 4.12. *If there is a dominating family with cardinality ω_1 and S is an ω_1-compact metric space which is the union of ω_1, compact sets, there is a collection $\mathscr{G}$ of each type below.*

(1) *$\mathscr{G} = \{\mathscr{G}_\alpha: \alpha \in \omega_1\}$ is a collection of sets of open subsets of S having the property that if $D \in \mathscr{D}(S)$ and U is an open set including D, there is an $\alpha \in \omega_1$ such that $\mathscr{G}_\alpha$ covers D and $st(D, \mathscr{G}_\alpha) \subseteq U$.*

(2) *$\mathscr{G} = \{\mathscr{G}_\alpha: \alpha \in \omega_1\}$ is a collection of sets of open subsets of S having the property that if $D \in \mathscr{D}(S)$, there is an $\alpha \in \omega_1$ such that $\mathscr{G}_\alpha$ covers D and if x and y are distinct points of D, then $st(x, \mathscr{G}_\alpha) \cap st(y, \mathscr{G}_\alpha) = \varnothing$.*

(3) *$\mathscr{G} = \{\mathscr{G}_\alpha: \alpha \in \omega_1\}$ is a collection of sets of open subsets of S having the property that if D and E are pairwise disjoint sets, $D \in \mathscr{D}(S)$, and $E \in \mathscr{D}(S)$ there is an $\alpha \in \omega_1$ such that $\mathscr{G}_\alpha$ covers $D \cup E$ and $st(D, \mathscr{G}_\alpha) \cap st(E, \mathscr{G}_\alpha) = \varnothing$.*

Proof. There is a collection $\{F_\alpha: \alpha \in \omega_1\}$ of compact subsets of S whose union is S. For each $\alpha \in \omega_1$ the collection $\{F_\beta: \beta \in \alpha\}$ is a countable collection of compact sets and if $D \in \mathscr{D}(S)$ there is an

$\alpha \in \omega_1$ such that $\{F_\beta: \beta \in \alpha\}$ covers D. The construction in 4.10 applied to this collection yields a cardinality ω_1 collection of open covers of $\cup \{F_\beta: \beta \in \alpha\}$. Then the family of all collections for all $\beta \in \omega_1$ satisfies each of conditions (1), (2), and (3) and has cardinality ω_1.

REMARK 4.13. If there is a dominating family of cardinality ω_1, then the irrationals—indeed, every metric space which is the continuous image of the irrationals—is ω_1-compact and is the union of a cardinality ω_1 collection of compact sets (see [9]). If CH is false then the space of Example 4.6 is not the union of a cardinality ω_1 collection of compact sets.

5. An application for the set of real numbers.

5.1. A scale $\mathscr{S}$ which is order isomorphic to the ordinal α is called an α-scale. Hausdorff in [7] showed that CH implies there is an ω_1-scale. The set of rational real numbers is denoted Q, the set of irrational real numbers is denoted P, and the set of real numbers is denoted R. $\mathscr{F}$ will denote the set $\mathscr{D}(Q) \cap \mathscr{D}(R)$.

THEOREM 5.2. *If there is a dominating family of cardinality ω_1, and $\omega_1 < cf(c)$, then every subcollection $\mathscr{H}$ of $\mathscr{F}$ with cardinality c has a subcollection $\mathscr{H}'$ with cardinality c, and such that $\cup \mathscr{H}'$ has no irrational limit point.*

Proof. It follows with the aid of Theorem 4.10 there is a collection $\mathscr{G} = \{\mathscr{G}_\alpha: \alpha \in \omega_1\}$ of open covers of R satisfying condition $B(\omega_1)$ and such that for each α, G_α is countable and locally finite. For $\alpha \in \omega_1$ let $\mathscr{G}_\alpha = \{V_n: n \in \omega\}$. For each n, $x_0^n, x_1^n, x_2^n, \cdots$ are the points of the set $Q \cap V_n$. There is a dominating family $\mathscr{S}$ with cardinality ω_1. For $f \in \mathscr{S}$ define $D_f^\alpha = \bigcup_{n \in \omega} \{x_t^n: t \leqq f(n)\}$. The set $\mathscr{C} = \{D_f^\alpha: \alpha \in \omega_1 \text{ and } f \in \mathscr{S}\}$ has cardinality at most ω_1. Each D_f^α is a closed discrete subset of R. Moreover, $\mathscr{C}$ has the property that if $F \in \mathscr{F}$ there is an $\alpha \in \omega_1$ and an $f \in \mathscr{S}$ such that $F \subseteqq D_f^\alpha$. Thus since $\mathscr{H}$ has cardinality c and $cf(c) > \omega_1$ there is an $f \in \mathscr{S}$ and an $\alpha \in \omega_1$ such that D_f^α includes c members of $\mathscr{H}$. Let $\mathscr{H}'$ be those members of $\mathscr{H}$ which are contained in D_f^α.

THEOREM 5.3. *Assuming* CH, *there is a subcollection $\mathscr{H}$ of $\mathscr{F}$ having cardinality c and such that if $\mathscr{H}'$ is any subcollection of $\mathscr{H}$ with cardinality c, then $\cup \mathscr{H}'$ has an irrational limit point.*

Proof. For each nonnegative integer n, let $\{x_0^n, x_1^n, x_2^n, \cdots\}$ denote the rational numbers in the interval $(n, n + 1)$. CH implies there is an ω_1-scale $\mathscr{S}$. For each $f \in \mathscr{S}$ let $D_f = \{x_t^n : t \leqq f(n) \text{ and } n \in \omega\}$. Let $\mathscr{H} = \{D_f : f \in \mathscr{S}\}$. For each subset $\mathscr{H}'$ of $\mathscr{H}$ having cardinality c, $\mathscr{S}' = \{f \in \mathscr{S} : D_f \in \mathscr{H}'\}$ is cofinal in $\mathscr{S}$. Hence $\cup \mathscr{H}'$ is dense in R.

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